On the hydrodynamic closure of a transport-diffusion equation

M. Bisi[1], G. Spiga[1] and G. Toscani[2]

[1] *Dipartimento di Matematica, Università di Parma - Viale G. P. Usberti 53/A, 43100 Parma*
[2] *Dipartimento di Matematica, Università di Pavia - Via Ferrata 1, 27100 Pavia*

PACS 05.60.-k – Transport processes
PACS 05.20.Dd – Kinetic theory
PACS 02.30.Hq – Partial differential equations

Abstract. - In this paper we discuss the passage to hydrodynamics for a transport diffusion equation. It is shown that the self-similar solution of the diffusion equation can be fruitfully used to construct the Euler equations for the model, provided the initial density possesses sufficiently many moments. The results of the paper can be of interest in dissipative kinetic theory, where the role of the homogeneous cooling state in the passage to hydrodynamics has been shown only from a formal point of view.

Introduction. – This paper deals with some questions related to the passage to hydrodynamics for the transport–diffusion equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon}\, \Delta_v f. \tag{1}$$

In (1), $f = f(x, v, t)$ is a nonnegative function that represents the density of particles in position $x \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$ with velocity $v \in \mathbb{R}^d$, with $d \geq 1$. Here $v \cdot \nabla_x f$ is the usual transport operator, while $\Delta_v f$ is a linear diffusion operator which acts on the velocity variable v. The ε-parameter (Knudsen number) represents a measure of the mean free path, and has to be assumed small in fluid dynamic regimes.

Equations of type (1) are extensively used to model biological systems (cfr. the book of Okubo [14], where a variety of mathematical models of advection-diffusion in the ecological context are introduced and discussed). Equation (1) itself may be viewed as a special case of governing equations in important branches of Physics, such as the Caldeira–Legget semiclassical Fokker–Planck equation for reversible quantum systems [6], and the Landau–Fokker–Planck collision term for electron kinetics in plasmas and semiconductors [13]. Our main interest here, however, is related to the analogies that problem (1) has with the modeling of hydrodynamic equations for granular gases, described at a kinetic level by the dissipative Boltzmann equation. In this second case, the unknown is a time dependent density in phase space $F(x, v, t)$ satisfying a Boltzmann–Enskog equation for inelastic hard–spheres, which for the force–free case reads

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \frac{1}{\epsilon}\, C(F, F)(x, v, t). \tag{2}$$

In (2), $C(F, F)$ is the so–called granular collision operator, which describes the change in the density function due to creation and annihilation of particles in dissipative binary collisions [18]. Equations (1) and (2) present a lot of analogies. The (spatially homogeneous) *collision* operators $\Delta_v f$ and $C(F, F)$ are such that both local mass and momentum are conserved, while the energy varies with time, increasing in the diffusion case and decreasing in the granular case. In this last case, the loss of energy in the microscopic collision translates at a macroscopic level in the progressive cooling of the gas, a phenomenon which is responsible of most of the difficulties in extending methods of classical kinetic theory of ideal gases to granular ones. In the present case we get generation rather than dissipation of energy, but there are indeed other fields of non–conservative kinetic theory in which an essential quantity is produced "by collision", like for instance in modelling wealth distribution in open market economies [9, 16]. In this case, in fact, the mean wealth of a strong economy is known to increase with time.

A clear understanding of the new problems one has to deal with in the derivation of macroscopic equations in dissipative kinetic theory can be obtained through the use of the splitting method, very popular in the numerical approach to the Boltzmann equation [11, 15]. If at each time step we consider sequentially the transport and relaxation operators in the Boltzmann equation (2), during this short time interval we recover the evolution of the density from the joint action of the relaxation

$$\frac{\partial F}{\partial t} = \frac{1}{\epsilon}\, C(F,F)(x,v,t), \tag{3}$$

and transport

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = 0. \tag{4}$$

In classical kinetic theory, the energy is conserved in collisions, and the relaxation (3) pushes the solution towards the Maxwellian equilibrium with the same mass, momentum and energy of the initial datum. Then, if ϵ is sufficiently small, one can easily argue that the solution to (3) is *sufficiently close* to the Maxwellian, and this Maxwellian can be used into the transport step (4). When dissipation is present, solutions of the inelastic Boltzmann equation lose energy until all particles travel at the same speed, and the relaxation (3) pushes the solution towards the asymptotic state represented by a δ function concentrated in the mean velocity of the initial value. It is evident that, if $\epsilon << 1$, so that the solution to (3) is close to this *poor* asymptotic state, substitution into the transport step (4) does not lead to any correct behavior.

To circumvent this difficulty, one is led to a more precise study of the asymptotic behavior of the solution in the relaxation step. This is obtained by looking for exact self-similar solutions to (3) (homogeneous cooling states), with the aim to use this exact solution in the transport (4) [10].

The common assumption which has been at the basis of almost all papers on the matter is that there are only small spatial variations, so that the zero order approximation of the solution (and of any asymptotic expansion) is constituted by the homogeneous cooling state (see for instance Ref. [5] and the references therein). From a mathematical point of view, however, the possibility of using the homogeneous cooling state to close the transport equation (4) would require precise statements on the role of this exact solution, which has to be the intermediate solution of the homogeneous problem for a large *physical* class of initial densities. For a simplified model of the Boltzmann equation, the so–called Maxwellian model [2], recent results showed that this holds true [1, 3], while the same result has never been proven for the general model.

Since the role of the self-similar solution for the linear diffusion equation is well-understood, and the equation is linear, the analysis of the passage to hydrodynamics for equation (1) can be made rigorous under few assumptions on the initial data. Hopefully, the same methods could be fruitfully applied to obtain similar results for the dissipative Boltzmann equation.

¿From the mathematical point of view, the properties of the transport-diffusion equation (1) are widely studied, since the operator $L = -\Delta_v + v \cdot \nabla_x$ represents possibly the most important instance of application of hypoellipticity [12]. The corresponding evolution equation (1) is degenerate, but still presents some of the typical features of a parabolic equation; the word *ultraparabolic* is sometimes used for it [19]. While intensively studied from a theoretical point of view, however, the passage to hydrodynamics for equation (1) has to our knowledge never been tackled before.

For the sake of simplicity, we will consider in the sequel only the one-dimensional case ($d = 1$). This choice allows to present results in a clear and direct way. Similar results can be easily shown to hold, at the price of an increased level of difficulty in computations, when $d > 1$.

Exact solution and macroscopic moments. – Let us fix $d = 1$ in equation (1). Then $f = f(x, v, t)$ solves

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} = \frac{1}{\varepsilon}\,\frac{\partial^2 f}{\partial v^2}. \tag{5}$$

Due to its linearity, equation (5) can be solved explicitly. The direct method to solve it is reminiscent of the analogous one valid for the linear diffusion, and consists in resorting to Fourier transform. Let $f_0(x, v)$ be an initial distribution function, which we assume to be integrable in $\mathbb{R} \times \mathbb{R}$. Let us consider the Fourier transform of the distribution f, with respect to both x and v

$$\tilde{\tilde{f}}(k,\omega,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,v,t)\,\mathrm{e}^{-ikx}\,\mathrm{e}^{-i\omega v}\,dx\,dv\,.$$

The corresponding evolution equation for $\tilde{\tilde{f}}$ turns out to be

$$\frac{\partial \tilde{\tilde{f}}}{\partial t} - k\,\frac{\partial \tilde{\tilde{f}}}{\partial \omega} + \frac{\omega^2}{\varepsilon}\,\tilde{\tilde{f}} = 0\,, \tag{6}$$

and it can be easily verified that the solution of equation (6) takes the form

$$\tilde{\tilde{f}}(k,\omega,t) = \tilde{\tilde{f}}_0(k,\omega+kt)\,\exp\left(-\frac{\omega^2 t}{\varepsilon} - \frac{\omega k t^2}{\varepsilon} - \frac{k^2 t^3}{3\,\varepsilon}\right). \tag{7}$$

Hence, by inverse Fourier transforms we obtain

$$f(x,v,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} G(x,v,t;x',v')\,f_0(x',v')\,dx'\,dv' \tag{8}$$

where G indicates the Green function

$$\begin{aligned} G(x,v,t;x',v') = \frac{\sqrt{3}\,\varepsilon}{2\pi t^2}\exp\Big(-\frac{\varepsilon}{t^3}\Big[3(x-x')^2 \\ -3(x-x')(v+v')t + (v^2 + vv' + v'^2)t^2\Big]\Big), \end{aligned} \tag{9}$$

corresponding to initial datum $f_0(x, v) = \delta(x-x')\delta(v-v')$.

It is interesting to remark that the relaxation time of the spatial variable differs from the relaxation time of the velocity variable.

Since we know the explicit solution to equation (5), we may evaluate in exact form the major macroscopic velocity moments, that is the number density $n(x, t)$, the mean velocity $u(x, t)$, the temperature $T(x, t)$ and so on.

Given any regular test function $\varphi(v)$, we have

$$\int_{\mathbb{R}} \varphi(v) f(x,v,t)\,dv = \frac{\sqrt{3}\,\varepsilon}{2\pi t^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left\{-\varepsilon\left[\frac{3(x-x')^2}{t^3} - \frac{3(x-x')v'}{t^2} + \frac{v'^2}{t}\right]\right\}\left[\int_{\mathbb{R}} \varphi(v)\,\exp\left\{-\varepsilon\left[\frac{v^2}{t} + v\left(-\frac{3(x-x')}{t^2} + \frac{v'}{t}\right)\right]\right\} dv\right] f_0(x',v')\,dx'\,dv'\,. \tag{10}$$

Choosing $\varphi(v) = 1, v, v^2, v^3$ we obtain the moments of the distribution function up to the third order. Skipping all inessential computations and other intermediate details, we obtain

$$n(x,t) = \int_{\mathbb{R}} f(x,v,t)\,dv = \sqrt{\frac{3\,\varepsilon}{4\pi\,t^3}} \cdot \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left\{-\frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\,dx'\,dv'\,, \tag{11}$$

$$(nu)(x,t) = \int_{\mathbb{R}} v f(x,v,t)\,dv = \sqrt{\frac{3\,\varepsilon}{4\pi\,t^3}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{3(x-x')}{2t} - \frac{v'}{2}\right]\exp\left\{-\frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\,dx'\,dv', \tag{12}$$

$$\begin{aligned}(nT + nu^2)(x,t) &= \int_{\mathbb{R}} v^2\, f(x,v,t)\,dv \\ &= \frac{t}{2\,\varepsilon}\,n + \sqrt{\frac{3\,\varepsilon}{4\pi\,t^3}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{3(x-x')}{2t} - \frac{v'}{2}\right)^2 \\ &\cdot \exp\left\{-\frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\,dx'\,dv',\end{aligned} \tag{13}$$

$$\int_{\mathbb{R}} v^3 f(x,v,t)\,dv = \frac{3\,t}{2\,\varepsilon}\,n\,u + \sqrt{\frac{3\,\varepsilon}{4\pi\,t^3}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{3(x-x')}{2t} - \frac{v'}{2}\right)^3 \exp\left\{-\frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\,dx'\,dv'. \tag{14}$$

Self-similar solution and hydrodynamic equations. – While the knowledge of the solution to the kinetic equation (5) allows to recover all moments, the evolution equations for n, u, T do not in general constitute a closed system. In our case, this unpleasant property can be easily verified by multiplying equation (5) by 1, v, v^2, respectively, and integrating over the velocity variable v. By applying this procedure, and taking into account the conservation properties of the diffusion operator (conservation of mass and momentum), since

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} v^2 \frac{\partial^2 f}{\partial v^2}\,dv = \frac{2}{\varepsilon}\,n\,,$$

we can write the macroscopic equations for the transport-diffusion equation (5), that read

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial (nu)}{\partial x} = 0 \\ \dfrac{\partial (nu)}{\partial t} + \dfrac{\partial (nu^2 + nT)}{\partial x} = 0 \\ \dfrac{\partial (nu^2 + nT)}{\partial t} + \dfrac{\partial}{\partial x}\displaystyle\int_{\mathbb{R}} v^3\, f(x,v,t)\,dv = \dfrac{2}{\varepsilon}\,n\,. \end{cases} \tag{15}$$

This set of equations is exact, but not closed, because of the presence of the third order velocity moment in the evolution equation for the temperature. In dissipative kinetic theory, as briefly explained in the introduction, the typical procedure is to close the hydrodynamic equations by replacing higher order moments by the ones corresponding to the homogeneous cooling state [10]. We will adopt the same strategy here. As we shall see, since in our case the explicit solution is known, and the self-similar solution is the one corresponding to the linear diffusion, we can compare the exact third order moment with the one corresponding to the homogeneous similarity profile, and give explicit estimates in the limit $\varepsilon \to 0$.

In order to recover the self-similar profile, we consider the space homogeneous version of the kinetic equation (5), for which macroscopic equations (15) collapse immediately to $\dot{n} = 0\,,\ \ \dot{u} = 0\,,\ \ \dot{T} = \frac{2}{\varepsilon}$. Since in this case the second moment is growing with time, in analogy with dissipative kinetic theory we will call the self-similar profile the *homogeneous heating state*. Even if the evaluation of the self-similar solution is classic, we will briefly outline its derivation, since we want to relate it to the moments of the initial datum. Let n and u denote the number density and the mean velocity of f. Then, the distribution $\bar{f}(v) = f(v+u)/n$ has number density equal to 1 and mean velocity equal to 0, and it fulfils the same evolution equation. Since mass and momentum are conserved in time, for simplicity we can perform all required calculations using $\bar{f}$. At the end we will take into account that $f(v) = n\,\bar{f}(v-u)$.

Let us look for suitable time scales $\alpha(t)$ and $\tau(t)$ such that, if we put

$$\bar{f}(v,t) = \frac{1}{\alpha}\,g\Big(\frac{v}{\alpha},\tau\Big) = \frac{1}{\alpha}\,g(w,\tau)\,, \tag{16}$$

the temperature of the distribution g turns out to be constant. To this aim, the right scaling is $\alpha(t) = \sqrt{T(t)}$, and, if we put $\tau(t) = \log\sqrt{T(t)}$, we see that $g(w,\tau)$ satisfies the equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w}\left[\frac{\partial g}{\partial w} + w\,g\right]. \tag{17}$$

The so–called *homogeneous heating state* is the distribution function $f(v,t)$ corresponding to the stationary solution of (17), which is simply given by a Maxwellian distribution. Bearing in mind equality (16) and going back to f, the homogeneous heating state takes the form

$$f_{HHS}(v) = \frac{n}{\sqrt{2\pi T}}\, \mathrm{e}^{-\frac{(v-u)^2}{2T}}\,. \tag{18}$$

In order to close the exact hydrodynamic system (15), we have to evaluate the third order moment of the self-similar profile f_{HHS} in terms of n, u, T. By calculating the third moment of f_{HHS} by simple Gaussian–type integrals, we get simply $3nuT + nu^3$, and the corresponding closed (but approximated) set of macroscopic equations reads

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial (nu)}{\partial x} = 0 \\ \dfrac{\partial (nu)}{\partial t} + \dfrac{\partial (nu^2+nT)}{\partial x} = 0 \\ \dfrac{\partial (nu^2+nT)}{\partial t} + \dfrac{\partial [nu(3T+u^2)]}{\partial x} = \dfrac{2}{\varepsilon}\, n\,. \end{cases} \tag{19}$$

Error estimate. – The justification of the closure of moment equations through the set of equations (19) requires, in a hydrodynamic regime, in which the Knudsen number ε is a small parameter, a precise estimate on the difference between the exact third order velocity moment (14), and its approximate expression $3nuT + nu^3$. Let us set, for $k \in \mathbb{N}$,

$$\begin{aligned} M_k = {} & \sqrt{\frac{3}{4\pi\, t^3}} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{3(x-x')}{2t} - \frac{v'}{2}\right)^k \\ & \cdot \exp\left\{-\frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\, dx'\, dv'. \end{aligned} \tag{20}$$

Then, the velocity moments of the solution can be easily expressed in terms of M_k. More precisely we have

$$n = \sqrt{\varepsilon}\, M_0\,, \quad n\,u = \sqrt{\varepsilon}\, M_1\,, \quad nT + nu^2 = \frac{t}{2\,\varepsilon}\, n + \sqrt{\varepsilon}\, M_2\,,$$

$$\int_{\mathbb{R}^3} v^3\, f(v)\, dv = \frac{3\,t}{2\,\varepsilon}\, n\, u + \sqrt{\varepsilon}\, M_3\,.$$

In addition $3nuT + nu^3$ can be obtained as a combination of the previous formulas. In fact

$$3nuT + nu^3 = \frac{3\,t}{2\,\varepsilon}\, n\,u + \sqrt{\varepsilon}\left(3\,M_2 - 2\,\frac{M_1^2}{M_0}\right)\frac{M_1}{M_0}\,. \tag{21}$$

Therefore, in the difference $\Delta(x,t)$ between the exact third order moment and its approximate expression, the leading order ($O(1/\varepsilon)$) terms cancel out, and we obtain

$$\begin{aligned} \Delta(x,t) &= \int_{\mathbb{R}^3} v^3\, f(v)\, dv - \Big(3nuT + nu^3\Big) \\ &= \sqrt{\varepsilon}\left[M_3 - \left(3\,M_2 - 2\,\frac{M_1^2}{M_0}\right)\frac{M_1}{M_0}\right]. \end{aligned} \tag{22}$$

We note that the pseudo–moments M_k still contain an ε dependence. In order to evaluate in an explicit way the order of the overall ε-correction, it appears convenient to expand the moments M_k, $k = 0, \ldots, 3$, in terms of powers of ε. More precisely, let us define

$$\begin{aligned} \tilde{M}_k = {} & \sqrt{\frac{3}{4\pi\, t^3}} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{3(x-x')}{2t} - \frac{v'}{2}\right)^k \\ & \cdot \left\{1 - \frac{3\,\varepsilon}{4t^3}\Big[(x-x') - tv'\Big]^2\right\} f_0(x',v')\, dx'\, dv', \end{aligned} \tag{23}$$

and let us consider the difference $|M_k - \tilde{M}_k|$.

¿From now on, given a fixed positive constant $\delta < 1$, let $t \geq \delta$. For any fixed position x, let us denote with B_R the ball in $\mathbb{R}^2$ of radius R with center in the point $(x, v = 0)$,

$$B_R = \left\{(x',v') \in \mathbb{R}^2 : (x-x')^2 + (v')^2 \leq R^2\right\},$$

and correspondingly we define M_k^R and $\tilde{M}_k^R$ as the restrictions of the integrals (20) and (23) to the domain B_R. Since for $(x',v') \in B_R$ it holds

$$\frac{3}{4t^3}\Big[(x-x') - tv'\Big]^2 \leq \frac{3}{2\delta^3}\Big[(x-x')^2 + (v')^2\Big], \tag{24}$$

we have immediately from Taylor's formula

$$|M_k^R - \tilde{M}_k^R| \leq \varepsilon^2\, \frac{9}{8\,\delta^6}\, \mathcal{C}_0(x,k,\delta)\, R^4\,, \tag{25}$$

where

$$\begin{aligned} \mathcal{C}_0(x,k,\delta) = {} & \sqrt{\frac{3}{4\pi\, \delta^3}} \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{9(x-x')^2}{2\delta^2} + \frac{(v')^2}{2}\right]^{k/2} \\ & \cdot f_0(x',v')\, dx'\, dv'\,. \end{aligned}$$

On the other hand, in $\mathbb{R}^2 \backslash B_R$, we use the estimate $\mathrm{e}^{-\rho} - 1 + \rho \leq \rho$, which is valid for $\rho > 0$, to obtain after some algebra

$$|(M_k - M_k^R) - (\tilde{M}_k - \tilde{M}_k^R)| \leq \varepsilon\, \frac{3}{2\,\delta^3}\, \frac{1}{R^{2\alpha}}\, \Gamma_0(x,\alpha,k,\delta)\,. \tag{26}$$

In the above expression α is nonnegative and

$$\begin{aligned} \Gamma_0(x,\alpha,k,\delta) = {} & \sqrt{\frac{3}{4\pi\, \delta^3}} \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{9(x-x')^2}{2\delta^2} + \frac{(v')^2}{2}\right]^{k/2} \\ & \cdot \Big[(x-x')^2 + (v')^2\Big]^{1+\alpha} f_0(x',v')\, dx'\, dv'. \end{aligned}$$

We remark that Γ_0 is finite provided that the initial datum has bounded moments (with respect both to position and velocity) up to the order $k + 2 + 2\,\alpha$.

Thus, in conclusion,

$$|M_k - \tilde{M}_k| \leq \varepsilon^2 \frac{9}{8\delta^6} \mathcal{C}_0(x,k,\delta) R^4 + \varepsilon \frac{3}{2\delta^3} \Gamma_0(x,\alpha,k,\delta) \frac{1}{R^{2\alpha}}. \tag{27}$$

Optimizing over R the right-hand side, we get for $\alpha \neq 0$

$$|M_k - \tilde{M}_k| \leq C(x,\alpha,k,\delta)\, \varepsilon^{2(\alpha+1)/(\alpha+2)},$$

where

$$\begin{aligned} C(x,\alpha,k,\delta) = {} & \left\{\left(\frac{\alpha}{2}\right)^{\frac{2}{\alpha+2}} + \left(\frac{2}{\alpha}\right)^{\frac{\alpha}{\alpha+2}}\right\} \left(\frac{3}{2\delta^3}\right)^{\frac{2(\alpha+1)}{\alpha+2}} \\ & \cdot \left(\frac{\mathcal{C}_0(x,k,\delta)}{2}\right)^{\frac{\alpha}{\alpha+2}} \Big(\Gamma_0(x,\alpha,k,\delta)\Big)^{\frac{2}{\alpha+2}}. \end{aligned}$$

Hence

$$M_k = \tilde{M}_k^{(0)} + \varepsilon\, \tilde{M}_k^{(1)} + O\left(\varepsilon^{1+\beta}\right) , \tag{28}$$

where

$$\tilde{M}_k^{(0)} = \sqrt{\frac{3}{4\pi t^3}} \int_{\mathbb{R}}\!\int_{\mathbb{R}} \left(\frac{3(x-x')}{2t} - \frac{v'}{2} \right)^k f_0(x',v')\, dx' dv',$$

$$\tilde{M}_k^{(1)} = -\sqrt{\frac{3}{4\pi\, t^3}}\, \frac{3}{4t^3} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{3(x-x')}{2t} - \frac{v'}{2} \right)^k \cdot \Big[(x-x') - tv'\Big]^2 f_0(x',v')\, dx'\, dv' ,$$

and $\beta = \alpha/(\alpha+2)$ is positive and less than unity for any $\alpha > 0$.

By substituting expansions (28) with $0 \le k \le 3$ into equality (22), we get

$$\Delta(x,t) = \sqrt{\varepsilon} \left\{ \tilde{M}_3^{(0)} - 3 \frac{\tilde{M}_1^{(0)} \tilde{M}_2^{(0)}}{\tilde{M}_0^{(0)}} + 2 \frac{\left[\tilde{M}_1^{(0)}\right]^3}{\left[\tilde{M}_0^{(0)}\right]^2} + O(\varepsilon) \right\}. \tag{29}$$

In this leading order estimate the parameter α does not need to be positive, it may be taken equal to zero, so that the initial datum is required to have bounded moments only up to order 5. It is worth to remark that the $O(\sqrt{\varepsilon})$ may vanish (and accuracy improves) for special choices of the initial datum f_0, as shown below. A lengthy and tedious computation allows then to compute the contribution of order $\varepsilon^{3/2}$ in estimate (22) (omitted here for brevity), holding when f_0 has bounded moments up to some order $5+2\alpha$. We proved the following

Theorem *Let $f(x,v,t)$ be the solution to the transport diffusion equation (5), and let $f_{HHS}(x,v,t)$, defined in (18), be the corresponding homogeneous heating state, with the same moments of $f(x,v,t)$ up to the order two. Then if the initial density to equation (5) possesses enough space and velocity moments (at least of order 5), for all times $t \ge \delta > 0$ the estimate*

$$\left| \int_{\mathbb{R}} v^3\, f_{HHS}(x,v,t)\, dv - \int_{\mathbb{R}} v^3\, f(x,v,t)\, dv \right| \le C\sqrt{\varepsilon} \tag{30}$$

holds for the difference between third order velocity moments. The constant C only depends on the initial datum $f_0(x,v)$, and it is uniform in time.

Recalling that the relaxation step is represented by the heat equation, it follows easily by direct inspection that the error $\Delta(x,t)$ vanishes if we choose a Maxwellian velocity distribution with variance σ concentrated in $x=0$ as initial density

$$f_0(x,v) = \delta(x)\, \frac{1}{\sqrt{2\pi\,\sigma}}\, \exp\left(-\frac{v^2}{2\sigma} \right) , \tag{31}$$

including the limiting case $\sigma = 0$. In addition, it can be shown that there exist initial data (satisfying the assumptions of our Theorem) for which the error $\Delta(x,t)$ is actually $O(\sqrt{\varepsilon})$. Let us start from a two point mass density in velocity

$$f_0(x,v) = \delta(x)\, \Big[\alpha\, \delta(v-a) + (1-\alpha)\, \delta(v-b) \Big], \tag{32}$$

with $\alpha \in (0,1)$. Then, if we set

$$\mathcal{A} = \exp\left\{ -\frac{3\,\varepsilon}{4t^3} \left[x - t\,a\right]^2 \right\}, \quad \mathcal{B} = \exp\left\{ -\frac{3\,\varepsilon}{4t^3} \left[x - t\,b\right]^2 \right\},$$

the exact expression for the error can be cast as

$$\Delta(x,t) = \sqrt{\frac{3\,\varepsilon}{4\pi\, t^3}}\, \alpha(1-\alpha) \frac{(a-b)^3}{8}\, \frac{\alpha\,\mathcal{A} - (1-\alpha)\mathcal{B}}{\left[\alpha\,\mathcal{A} + (1-\alpha)\mathcal{B}\right]^2}\, \mathcal{A}\,\mathcal{B} ,$$

and it vanishes if and only if f_0 reduces to a unique Dirac delta, namely for the particular options $\alpha = 0$, $\alpha = 1$, or $a = b$. Consider finally a space homogeneous initial distribution

$$f_0(x,v) = \alpha\, \delta(v-a) + (1-\alpha)\, \delta(v-b) , \tag{33}$$

which of course does not satisfy the assumptions of the Theorem (spatial moments are unbounded). In this case, actually, the distribution function remains space homogeneous for all times and in practice the macroscopic equations (15) do not need any closure. Anyway, if we are interested in the difference $\Delta(x,t)$, easy computations provide

$$\Delta(x,t) = \alpha(1-\alpha)(1-2\alpha)(a-b)^3 .$$

The error remains constant ($O(1)$) for all $t > 0$, and it vanishes now not only in the degenerate cases pointed out for the previous example, but also in the case of balanced masses $\alpha = \frac{1}{2}$. In conclusion, an estimate like (30) is not guaranteed if we remove the assumption of finite moments in both variables x and v.

A remark on the Green function. – Using the linearity of equation (5), we calculated the exact solution by resorting to Fourier transform. A second method, based on self-similarity properties of equation (5), can be alternatively applied to obtain the same result. This method has been widely applied to study the intermediate asymptotics of both linear and nonlinear diffusion equations (cfr. [7] and the references therein). It can in fact be checked that, if we scale both space and velocity variables in equation (5), as

$$f(x,v,t) = \frac{1}{\alpha(t)\,\beta(t)}\, g\left(\frac{x}{\beta(t)}, \frac{v}{\alpha(t)}, \tau(t) \right) , \tag{34}$$

with $\alpha(t) = \sqrt{2t}$, $\beta(t) = \alpha^3(t)$, and $\tau(t) = 2\,\log(2t+1)$, the equation for the distribution $g(\tilde{x}, \tilde{v}, \tau)$ reads (omitting all tildas)

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial v} \left[\frac{1}{\varepsilon} \frac{\partial g}{\partial v} + v\, g \right] + \frac{\partial}{\partial x} \left[(3x - v)\, g \right] . \tag{35}$$

Equation (35) possesses a stationary solution

$$g_\infty = C\, \mathrm{e}^{-\,\Phi(x,v)} \quad \Phi(x,v) = \frac{\varepsilon}{2}\, v^2 + \frac{3}{2}\, \varepsilon \big(4x - v\big)^2, \tag{36}$$

where the constant C is chosen to identify the total mass. In case the total mass is taken equal to 1, $C = 2\sqrt{3}\varepsilon/\pi$. Going back to the original variables allows to obtain in this way the Green function $G(x, v, t; 0, 0)$, with G given by (9). The fact that the scaled equation (35) has a stationary solution, gives us in addition the possibility to study the possible relaxation. As usual, this can be done by resorting to an entropy argument. The role of entropy functionals in connection with the simulation of kinetic equations, and primarily the lattice Boltzmann equation, has been excellently described in [4, 8]. There, the importance of knowing entropy monotonicity in enforcing compliance of the method with macroscopic evolutionary constraints as well as in serving as a numerically stable computational tool for fluid flows and other dissipative systems out of equilibrium is enlightened. Within this inequality, significant successes in the simulation of complex hydrodynamic phenomena, ranging from slow flows in grossly irregular geometries to fully developed turbulence and to flows with dynamic phase transitions, have been obtained. In order to repeat analogous steps here, recalling that the candidate to be the entropy functional for equation (35) can be easily obtained looking at the form of the steady state [17], we consider the following H–functional

$$H(g) = \int_{\mathbb{R}} \int_{\mathbb{R}} g \, (\Phi(x, v) + \log g) \, dx \, dv \,. \qquad (37)$$

It can be easily verified that, in the class of all densities on $\mathbb{R} \times \mathbb{R}$ of unit mass, $H(g) \geq H(g_\infty)$, with equality if and only if $g = g_\infty$. Moreover, the time derivative of the H-functional is non positive, and

$$\frac{dH}{dt} = -I(g) = -\frac{1}{\varepsilon} \int_{\mathbb{R}}\int_{\mathbb{R}} g \left[\frac{\partial \Phi}{\partial v} + \frac{1}{g} \frac{\partial g}{\partial v} \right]^2 dx \, dv \leq 0 \,. \quad (38)$$

In addition the entropy production I vanishes if and only if $g \equiv g_\infty$. This suggests that the entropy decays in time, which would imply that the self-similar solution represents the intermediate asymptotics of any solution to the transport-diffusion equation (5). Unlike the classical heat equation, however, where the entropy production bounds from above the relative entropy, for the present model an inequality of the kind $H - H_\infty \leq c\, I(g)$ is not available, since the entropy production involves only gradients with respect to the velocity variable.

Conclusions. – In this paper, we showed that, provided the initial density for the transport-diffusion equation (5) has sufficiently many moments, the homogeneous heating state of the equation, namely the self-similar solution to the homogeneous problem with the same mass, momentum, and temperature of the solution, represents a good candidate to close the Euler equations for this non-conservative case. In [5] and in other papers by the same authors, the considered kinetic model reads as

$$\frac{\partial f}{\partial \tilde{t}} + \varepsilon \, v \, \frac{\partial f}{\partial x} = J(f, f),$$

hence their scaling (based on the assumption of small spatial gradients) does not coincide with the one of the present model, in which we are using a different time scale: $t = \varepsilon \, \tilde{t}$. Then, our Theorem (holding for $t \geq \delta$) implies that in [5] the closure by HCS turns out to be good only for $\tilde{t} \geq \delta/\varepsilon$ (namely only in the limit $\tilde{t} \to \infty$). In addition, due to its scaling properties with respect to time, position and velocity, equation (5) possesses a source-type solution (the Green function (9)). In analogy with the well-known result for the heat equation, it is tempting to assume that this source-type solution is the intermediate time-asymptotics of any other solution with the same mass. In general, results of this type for linear and nonlinear diffusion make use of entropy inequalities [7], which would be interesting to explore in this context, since the standard ones have been seen to fail.

* * *

Work performed in the frame of the activities sponsored by MIUR (Project "Kinetic and hydrodynamic equations of complex collisional systems"), by INDAM, by GNFM and by the Universities of Parma and Pavia.

REFERENCES

[1] Bisi M., Carrillo J.A. and Toscani G., *J. Statist. Phys.*, **124** (2006) 625.

[2] Bobylev A.V., Carrillo J.A. and Gamba I., *J. Statist. Phys.*, **98** (2000) 743.

[3] Bobylev A.V., Cercignani C. and Toscani G., *J. Statist. Phys.*, **111** (2003) 403.

[4] Boffi V., Gorban A.A., Karlin I.V. and Succi S., *Phys. Rev. Lett.*, **81** (1998) 6.

[5] Brey J.J., Dufty J.W., Kim C.S. and Santos A., *Phys. Rev. E*, **58** (1998) 4638.

[6] Caldeira A.O. and Legget A.J., *Ann. Phys.*, **149** (1983) 374.

[7] Carrillo J.A., Jüngel A., Markowich P.A., Toscani G. and Unterreiter A., *Monatsh. Math.*, **133** (2001) 1.

[8] Chen H., Karlin I.V. and Succi S., *Rev. Mod. Phys.*, **74** (2002) 1203.

[9] Cordier S., Pareschi L. and Toscani G., *J. Statist. Phys.*, **120** (2005) 253.

[10] Dufty J.W., Kinetic theory and Hydrodynamics for rapid granular flows–A perspective, *Recent Developments in Statistical Mechanics* (Trivandrum Press) 2002.

[11] Gabetta E., Pareschi L. and Toscani G., *SIAM J. Numer. Anal.*, **34** (1997) 2168.

[12] Hörmander L., *Acta Math.*, **119** (1967) 147.

[13] Kolobov V.I., *Comput. Mater. Sci.*, **28** (2003) 302.

[14] Okubo A., *Diffusion and ecological problems: Mathematical models*, edited by Springer-Verlag, (Berlin, Germany) 1980.

[15] Pareschi L. and Russo G., *SIAM J. Sci. Comput.*, **23** (2001) 1253.

[16] Slanina F., *Phys. Rev. E*, **69** (2004) 046102.

[17] Toscani G., *Appl. Math. Lett.*, **12** (1999) 19.

[18] Toscani G., *Monatsh. Math.*, **142** (2004) 179.

[19] Villani C., *Proceedings of the International Congress of Mathematicians*, edited by European Mathematical Society Ed., Vol. **3** (Madrid, Spain) 2006, p. 473.