

# IDENTIFICATION OF ASYMPTOTIC DECAY TO SELF-SIMILARITY FOR ONE-DIMENSIONAL FILTRATION EQUATIONS

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**Abstract.** The concern of this paper is the derivation and the analysis of a simple explicit numerical scheme for general one-dimensional filtration equations. It is based on an alternative formulation of the problem using the pseudo-inverse of the density's repartition function. In particular, the numerical approximations can be proven to satisfy a contraction property for a Wasserstein metric. Various numerical results illustrate the ability of this numerical process to capture the time-asymptotic decay towards self-similar solutions even for fast-diffusion equations.

**Key words.** Degenerate parabolic equation, porous medium equation, Wasserstein metric.

**AMS subject classifications.** 76S05, 65M06, 35K65.

**1. Introduction and examples.** The present paper focuses onto the numerical analysis of the following Cauchy problem,

$$(1.1) \quad \partial_t u = \partial_{xx} \Phi(u), \quad u(t=0, x) = u_0(x) \geq 0, \quad x \in \mathbb{R}, \quad t > 0,$$

where  $u_0 \in L^1(\mathbb{R})$  and  $\Phi \in C^2(\mathbb{R}^+)$ . It is also customary to assume  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  to be increasing. In the special case  $\Phi(u) = u^m$ ,  $m > 1$ , one speaks about *the porous media equation* which describes the flow of a gas through a porous interface according to some constitutive relation linking its velocity to the pressure like the Darcy's law. Another interesting situation corresponds to  $0 < m < 1$  and is referred to as *the fast-diffusion equation*. The general case of *the filtration equations* can be encountered within the theory of heat transfer assuming the thermal conductivity to be a function of the temperature. A comprehensive introduction to these topics is provided in [25].

The numerical analysis of (1.1) is delicate for at least two reasons: the appearance of singularities for solutions with compact support when  $\Phi'(0) = 0$ , and the so-called retention property, which means that its size keeps on growing as times increase; we shall briefly recall in §2 theoretical results which are useful on a computational level. Implicit discretizations are thus of common use after *e.g.* [6, 14, 20, 21, 17, 16, 23] (other references of interest are [1, 2, 11, 12, 13, 15, 18, 19, 22]); it leads to the resolution of a strictly elliptic problem for  $w = \Phi(u)$  at every time step  $\Delta t$ . Unfortunately, this very stable approach is of little help when investigating the **long-time behaviour** of (1.1). Indeed, because of its spreading dynamics, the equation will ask for repetitive regridding. We refer to [3, 8, 7, 9, 10, 24, 25, 26, 27] for theoretical background on the asymptotics of (1.1), mainly in the case  $\Phi(u) = u^m$ ,  $m > 1$ . We stress that one of the goals of the present work is to provide a tool which allows to achieve numerical studies for the cases still unknown nowadays.

This paper is therefore intended to introduce a new numerical approach able to solve both issues in a 1D context. Loosely speaking, it consists in considering the **repartition function**  $\varrho$  of the density  $u$ , which is a monotone function, discretizing its values and evolving in time its pseudo-inverse  $X(t, \varrho)$  which satisfies for  $t > 0$  the

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equation (3.4). This Lagrangian strategy is explained in detail in §3.1 whereas stability and convergence properties are stated in §3.2. Interestingly, a discrete contraction principle in a Wasserstein metric is shown in §3.3.

At last, §4 is concerned with numerical results: we check the decay onto a Gaussian distribution for the heat equation together with two cases of fast-diffusion equations. Then we come to present a decay onto the so-called Barenblatt-Pattle similarity profile for  $\Phi(u) = \frac{u^2}{2}$  and a doubly degenerate Buckley–Leverett equation. Finally, some concluding remarks are drawn together with possible extensions to *e.g.* viscous pressureless gas equations, [4] and radial solutions of 2D filtration equations

**2.  $L^1$  theory for general porous media equations.** We first notice that there is no restriction in assuming  $\Phi(0) = 0$  in (1.1). A **weak solution** is generally defined as a distribution  $u \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}))$  such that  $\Phi(u) \in L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}))$  thus satisfying (1.1) in a weak sense for test functions belonging to  $H^1(\mathbb{R})$ . Existence and uniqueness results in this framework are recalled for instance in [20, 21]; we shall not pursue in this direction here.

**2.1. Existence and uniqueness with nonnegative data.** We follow [25] and are concerned hereafter with the Cauchy problem for slow-diffusion equations:

$$(2.1) \quad \partial_t u = \partial_{xx}(u^m), \quad u(t=0, \cdot) = u_0 \in L^1(\mathbb{R}); \quad x \in \mathbb{R}, t > 0.$$

We assume  $m > 1$  and  $u_0 \geq 0$  in order to study nonnegative  $L^1$  solutions (*e.g.* densities) with finite mass. In this context, asymptotics for (2.1) are well-known by now.

**DEFINITION 2.1.** *A nonnegative function  $u \in C^0(\mathbb{R}^+; L^1(\mathbb{R}))$  is a strong solution of (2.1) if:*

- $u^m, \partial_t u, \partial_{xx}(u^m) \in L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{R}))$ ,
- $\partial_t u = \partial_{xx}(u^m)$  holds almost everywhere in  $\mathbb{R}^+ \times \mathbb{R}$ ,
- $u(t=0, \cdot) = u_0$ .

In contrast to strongly parabolic equations for which  $\Phi'(u) \neq 0$ , strong solutions aren't by non means classical; it is known that they are endowed with Hölder  $C^\alpha$  continuity in space only. In 1D, the exponent has been found to be  $\alpha = \min(1, 1/(m-1))$ , [26]. Relying on the regularity properties of strong solutions to (2.1), we deduce easily two important properties of strong solutions:

- $\int_{\mathbb{R}} u(t, x).dx = \int_{\mathbb{R}} u_0(x).dx$  for all  $t \in \mathbb{R}^+$  (conservation of mass),
- $\int_{\mathbb{R}} \max(0, u_1(t, x) - u_2(t, x)).dx \leq \int_{\mathbb{R}} \max(0, u_1(s, x) - u_2(s, x)).dx$  for all  $t \geq s \geq 0$  ( $L^1$ -contraction property).

This last property of course implies uniqueness of strong solutions in the sense of Definition 2.1. A quite general result reads as follows:

**THEOREM 2.2.** *For all  $0 \leq u_0 \in L^1(\mathbb{R})$ , there exists a unique strong solution of (2.1)  $u \in C^0(\mathbb{R}^+; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$  which satisfies:*

- $\forall 1 \leq p \leq +\infty, u_0 \in L^p(\mathbb{R}) \Rightarrow \|u(t, \cdot)\|_{L^p(\mathbb{R})} \leq \|u_0\|_{L^p(\mathbb{R})}, t > 0$ ,
- let  $v = \frac{mu^{m-1}}{m-1}$ ; it holds that  $\partial_{xx}v(t, \cdot) \geq \frac{1}{(1+m)t}$  (*semi-superharmonicity*).

Refined regularity properties are now given:

**PROPOSITION 2.3.** *Let  $u_0 \in L^1 \cap C^0(\mathbb{R})$  be strictly positive and  $u$  be its corresponding strong solution; then  $u \in C^\infty(\mathbb{R}^+ \times \mathbb{R}) \cap C^0(\mathbb{R}^+ \times \mathbb{R})$  is strictly positive and realizes a classical solution of (2.1).*

For instance,  $u_0(x) = \frac{1}{\pi(1+x^2)}$  generates a unique classical solution of (2.1); this class of initial data will be extensively studied numerically. More generally, one can consider  $u_0(x) = \frac{C_p}{(1+x^2)^p}, p \geq 1$ .

**2.2. Asymptotic decay towards source solutions.** We observe that equation (1.1) rewrites as:

$$\partial_t u = \partial_x(D(u)\partial_x u), \quad D(u) = \Phi'(u).$$

In the special case of (2.1),  $D(u) = mu^{m-1}$  is often called the **diffusivity**. It is a well-known fact that degeneracy levels for which  $\Phi'$  vanishes (e.g. at  $u = 0$ , for  $\Phi(u) = u^m$ ) induce a phenomenon called **finite speed of propagation**.

**THEOREM 2.4.** *Let  $0 \leq u_0 \in L^1 \cap L^\infty(\mathbb{R})$  and  $u$  be the corresponding strong solution of (2.1). Assume that  $u_0$  is supported in a bounded set of  $\mathbb{R}$ , then for any positive time  $t > 0$ , the support of  $u(t, \cdot)$  is also bounded.*

The support of  $u(t, \cdot)$  is generally strictly bigger than the one of  $u_0$ ; this is the **retention property**. Making use of modern analytical tools, one can be some more precise, [7]:

**THEOREM 2.5.** *Let  $(u_0, v_0) \in L^1 \cap L^\infty(\mathbb{R})$  be nonnegative with unit masses and  $u, v$  their corresponding strong solutions in the sense of Definition 2.1. We define*

$$\Omega_u(t) = \left\{ x \in \mathbb{R} \text{ such that } u(t, x) > 0 \right\},$$

and the analogue for  $v$ . Then it holds true that for all  $t > 0$ ,

- $|\inf(\Omega_u(t)) - \inf(\Omega_v(t))| \leq W_\infty^0,$
- $|\sup(\Omega_u(t)) - \sup(\Omega_v(t))| \leq W_\infty^0,$

where the constant  $W_\infty^0 \in \mathbb{R}^+$  depends only on  $m, u_0, v_0$ .

Its proof is based on a careful use of a Monge-Kantorowich related metric that we shall discuss in more detail later on, see §3.3 and [28]. Indeed, as a particular case of (2.1), one can make the following mild hypotheses on the data:

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} x \cdot u_0(x) \cdot dx = 0, \quad \Omega_u(0) \subset \text{compact of } \mathbb{R}.$$

Then, as  $t \rightarrow +\infty$ , the corresponding strong solution to (2.1) decays towards a **similarity** (or source-type) solution,

$$(2.2) \quad U(t, x, C) = \frac{1}{t^\mu} \max \left\{ 0, \left( C - k \frac{x^2}{t^{2\mu}} \right)^{\frac{1}{m-1}} \right\}, \quad \mu = \frac{1}{1+m}, \quad k = \mu \frac{m-1}{2m},$$

the normalization constant  $C > 0$  ensuring that  $U(t, \cdot, C)$  has unit mass. One can also define the so-called **similarity variable**  $\alpha(t)$  solution of

$$(2.3) \quad \alpha'(t) = \frac{1}{\alpha(t)^m}, \quad \alpha(0) = 0,$$

for which (2.2) reads:

$$(2.4) \quad U(t, x, C) = \frac{1}{\alpha(t)} \max \left\{ 0, \left( \tilde{C} - \frac{m-1}{2m} \frac{x^2}{\alpha(t)^2} \right)^{\frac{1}{m-1}} \right\}.$$

Of course, plugging  $U(t, x, \|u_0\|_{L^1})$ ,  $t \geq \tau > 0$  in place of  $v$  inside Theorem 2.5 yields an easy bound on the support of any strong solution of (2.1).

It has been recently shown that even for the general case (1.1) for which results are more sparse than (2.1), the third moment of  $u(t, x)$  can play the role of an auxiliary variable in order to investigate the long-time behavior:

**THEOREM 2.6.** (Toscani, [24]) Let  $0 \leq u_0 \in L^1 \cap L^\infty(\mathbb{R})$  be of compact support in  $\mathbb{R}$ ,  $u$  the corresponding strong solution of (2.1) and  $E(t)$  its third moment:

$$E(t) = \int_{\mathbb{R}} \frac{x^2}{2} u(t, x) \cdot dx.$$

Then the similarity variable  $\alpha(t)$  satisfies as  $t \rightarrow +\infty$ ,

$$\frac{E(t)}{\alpha(t)^2} \rightarrow E_B = \int_{\mathbb{R}} \frac{x^2}{2} U(t=1, x, C) \cdot dx,$$

where  $E_B$  is the third moment of the source solution (2.4) at time  $t=1$ .

Hence a feasible route to study numerically the long-time asymptotics of (1.1) is to consider its **scaled solutions**,

$$(2.5) \quad f(t, x) = \sqrt{E(t)} u\left(t, x\sqrt{E(t)}\right),$$

which can hopefully be expected to stabilize as  $t \rightarrow +\infty$  onto an asymptotic profile  $f_\infty(x)$  independent of  $t$ . Of course, in case one considers (2.1) with convenient initial data,  $f(t, \cdot)$  will converge onto the corresponding Barenblatt-Pattle similarity solution according to the decay results of *e.g.* [7, 26];  $t \mapsto E(t)$  is also expected to display a power-like behavior.

**3. An explicit numerical approximation.** We consider now a slightly more general problem than (2.1); namely (1.1) completed by  $0 \leq u_0 \in L^1 \cap L^\infty(\mathbb{R})$ , compactly supported with unit mass. We shall also assume for convenience that the second moment vanishes:

$$\int_{\mathbb{R}} x \cdot u_0(x) \cdot dx = 0.$$

This property propagates for  $t > 0$  as is easily seen from the formal computation:

$$\partial_t \int_{\mathbb{R}} x \cdot u(t, x) \cdot dx = - \int_{\mathbb{R}} \partial_x \Phi(u)(t, x) \cdot dx = 0.$$

**3.1. Derivation of the numerical process.** As we can already notice, the decay towards similarity solutions can be slow and because of the retention property, a direct simulation of (1.1) (or even (2.1)) will surely ask for quite a big computational domain with a possibly fine mesh. This clearly constitutes a numerical difficulty we propose to overcome in an original way as follows:

- Let us introduce the distribution function associated to the probability density  $u_0$ ,

$$\varrho_0(x) = \int_{-\infty}^x u_0(y) \cdot dy \in [0, 1], \quad \varrho_0 \in W_{loc}^{1,1}(\mathbb{R}),$$

which is obviously nondecreasing in the  $x$  variable. We can thus define its (nondecreasing) pseudo-inverse:

$$(3.1) \quad \begin{aligned} x_0 : [0, 1] &\rightarrow \mathbb{R} \\ \bar{\varrho} &\mapsto x_0(\bar{\varrho}) := \inf\{y \in \mathbb{R} \text{ such that } \varrho_0(y) = \bar{\varrho}\}. \end{aligned}$$

If (1.1) holds in the sense of distributions, then also

$$(3.2) \quad \partial_t \varrho = \partial_x (\Phi(\partial_x \varrho)), \quad \varrho(t=0, \cdot) = \varrho_0,$$

from which one gets  $u(t, x) = \partial_x \varrho(t, x)$ .

- For any  $\bar{\varrho} \in [0, 1]$ , we can define the **reciprocal mapping**,

$$\begin{aligned} X : \mathbb{R}^+ &\rightarrow \mathbb{R} \\ t &\mapsto X(t, \bar{\varrho}), \end{aligned}$$

by means of the implicit function theorem in case  $\partial_x \varrho \neq 0$ , such that

$$(3.3) \quad X(t=0, \bar{\varrho}) = x_0(\bar{\varrho}), \quad \varrho(t, X(t, \bar{\varrho})) = \bar{\varrho}.$$

From the second condition in (3.3), one deduces easily:

$$\frac{d}{dt} \varrho(t, X(t, \bar{\varrho})) = (\partial_t \varrho + \partial_t X \cdot \partial_x \varrho)(t, X(t, \bar{\varrho})) = 0.$$

This yields the time evolution of  $X(\cdot, \varrho)$ : (we drop the  $\bar{\cdot}$  for ease of reading)

$$(3.4) \quad \partial_t X = -\frac{\partial_t \varrho}{\partial_x \varrho} = -\frac{\partial_x(\Phi(\partial_x \varrho))}{\partial_x \varrho} \Rightarrow \partial_t X + \partial_\varrho \left( \Phi \left( \frac{1}{\partial_\varrho X} \right) \right) = 0,$$

since  $\partial_\varrho X = 1/\partial_x \varrho$  holds for smooth enough functions.

Therefore our numerical approach to (1.1) with convenient (unit mass, centered) initial data stems from computing the pseudo-inverse of  $\varrho_0$ ,  $X(t=0, \cdot)$ , evolving it in time by means of an *explicit* marching scheme for (3.4) in order to deduce the values of  $\varrho(t, X(t, \cdot)) \in [0, 1]$  thanks to (3.3). Working on this pseudo-inverse  $X(t, \cdot)$  allows to pass through the expanding support issue for any arbitrary large time  $t > 0$  since the computational domain is now fixed,  $\varrho \in [0, 1]$ . The retention phenomenon manifests itself through the constant increase of  $|\sup_\varrho(X(t, \varrho))|$  and  $|\inf_\varrho(X(t, \varrho))|$  as  $t$  grows.

We now discretize the  $\varrho$  and  $t$  axes and define:

$$(3.5) \quad X_k^n \simeq X(t^n, \varrho_k), \quad t^n = n\Delta t \text{ for } k \in \mathcal{K} \subset \mathbb{N}, \quad n \in \mathbb{N}.$$

A numerical scheme for (3.4) reads:

$$(3.6) \quad X_k^{n+1} = X_k^n - \frac{\Delta t}{|C_k|} \left\{ \Phi \left( \frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n} \right) - \Phi \left( \frac{\varrho_k - \varrho_{k-1}}{X_k^n - X_{k-1}^n} \right) \right\},$$

where  $|C_k| = \varrho_{k+\frac{1}{2}} - \varrho_{k-\frac{1}{2}}$  stands for the width of the control cell centered on  $\varrho_k$  with  $\varrho_{k+\frac{1}{2}} = \varrho_0(x_{k+\frac{1}{2}})$ . As  $\varrho_0$  is at least absolutely and Lipschitz continuous, a convenient choice is given by linear interpolation,  $\varrho_{k+\frac{1}{2}} = \frac{1}{2}(\varrho_k + \varrho_{k+1})$ , which yields:

$$|C_k| = \frac{1}{2}(\varrho_{k+1} - \varrho_{k-1}).$$

(3.6) should be completed with boundary conditions at the edges of the computational domain  $\varrho \in [0, 1]$ . For convenience, we selected Neumann-type conditions:  $\Phi(u) = \Phi(\partial_x \varrho) = 0$  in  $\varrho = 0$  and  $\varrho = 1$ . This gives on the left side,

$$X_0^{n+1} = X_0^n - \frac{\Delta t}{|C_0|} \Phi \left( \frac{\varrho_1 - \varrho_0}{X_1^n - X_0^n} \right) \leq X_0^n,$$

together with a similar expression on the right side. This furthermore yields:

$$\forall n \in \mathbb{N}, \quad \sum_k |C_k| X_k^n = \sum_k |C_k| X_k^0 \simeq \int_0^1 X(t=0, \varrho) \cdot d\varrho = \int_{\mathbb{R}} x \cdot u_0(x) \cdot dx.$$

We stress that the  $\varrho_k$ 's stand for a cumulative mass variable and thus do **not** depend on time. In order to reconstruct  $\tilde{\varrho}(t^n, \cdot)$ , an approximation of  $\varrho(t, \cdot)$  at a given time  $t \simeq t^n$ , one has to interpolate the family of numerical values  $\varrho_k, X_k^n, t^n$  since

$$\tilde{\varrho}(t^n, X_k^n) = \varrho_k \simeq \varrho(t^n, X_k^n),$$

up to the numerical truncation errors on  $X_k^n$  coming from the discretization (3.6). Then one deduces  $u(t^n, \cdot)$  by centered divided differences; such a numerical differentiation process may weaken the convergence though. An obvious consequence of this discretization is that the total variation in space of  $\tilde{\varrho}(t, \cdot)$  is constant in time.

Other useful quantities for the study of the asymptotic behavior of (2.1) are the moments  $m_{2n+1}(t) = \int_{\mathbb{R}} x^{2n} u(t, x). dx$ ,  $n \in \mathbb{N}$ , which satisfy:

$$\frac{d}{dt} m_{2n+1}(t) = 2n(2n-1)m_{2n-1}(t) \text{ for } \Phi(u) = u^m.$$

In the general case of (1.1), one still has:

$$(3.7) \quad \frac{d}{dt} m_3(t) = 2 \int_{\mathbb{R}} \Phi(u)(t, x). dx, \quad m_3(t) = 2E(t) = \int_0^1 X(t, \varrho)^2 . d\varrho.$$

This last equality provides us with a very convenient way to compute the scaled solution  $f(t^n, \cdot)$  (2.5) relying on our marching scheme (3.6).

**3.2. Stability and consistency of the scheme.** To fix ideas, we introduce now a regular computational mesh determined by  $\Delta x > 0$ ,  $x_k := k\Delta x$ ,  $k \in \mathbb{N}$ . Then we compute the sequence  $u_k^0 = u_0(x_k)$  and thus  $X_k^0 = X(0, \varrho_k) = x_k$  with  $\varrho_0(x_k) = \varrho_k$ .

Of course, because of the retention property, the derivation of bounds for the  $X_k^n$ 's is doomed in advance because we expect  $\sup_{\varrho} X(t, \varrho)$  to diverge when  $t \rightarrow +\infty$ . However, we can prove that the scheme (3.6) is **monotonicity-preserving**.

LEMMA 3.1. *Let  $0 < u_0 \in L^1 \cap L^\infty(\mathbb{R})$  and  $\Phi \in C^1(\mathbb{R})$  be an increasing function; we denote  $0 < a := \inf_{k \in \mathcal{K}} (X_{k+1}^0 - X_k^0)$ . Then, under the CFL condition,*

$$(3.8) \quad \frac{\Delta t}{a^2} \sup_k \left\{ \frac{\varrho_{k+1} - \varrho_k}{|C_k|} \Phi' \left( \frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^0 - X_k^0} \right) \right\} \leq 1,$$

*the scheme (3.6) is monotonicity-preserving. Moreover, there hold for  $n \in \mathbb{N}$ ,*

$$(3.9) \quad \sum_k |\delta X_{k+\frac{1}{2}}^n| \left| \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right|^p \leq \|u_0\|_{L^p(\mathbb{R})}^p, \quad p \geq 1,$$

*and the uniform Lipschitz estimate,*

$$(3.10) \quad \sup_k \left| \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right| \leq \|u_0\|_{L^\infty(\mathbb{R})}.$$

The estimates (3.9) and (3.10) are of course the discrete analogues of the continuous ones recalled in Theorem 2.2.

*Proof.* We first want to prove that  $X_{k+1}^{n+1} - X_k^{n+1}$  is a positive combination of its neighbors at time  $t^n$ . To this end, we proceed by induction: let us assume that  $X_{k+1}^n - X_k^n \geq a > 0$ , from (3.6) we get:

$$\begin{aligned} X_{k+1}^{n+1} - X_k^{n+1} &= X_{k+1}^n - X_k^n - \left\{ \frac{\Delta t}{|C_{k+1}|} \left( \Phi \left( \frac{\varrho_{k+2} - \varrho_{k+1}}{X_{k+2}^n - X_{k+1}^n} \right) - \Phi \left( \frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n} \right) \right) \right. \\ &\quad \left. - \frac{\Delta t}{|C_k|} \left( \Phi \left( \frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n} \right) - \Phi \left( \frac{\varrho_k - \varrho_{k-1}}{X_k^n - X_{k-1}^n} \right) \right) \right\}. \end{aligned}$$

Thanks to the hypothesis, we can apply the mean-value theorem to the function  $\Phi$  in the preceding expression. We introduce some notation:  $\delta X_{k+\frac{1}{2}}^n := X_{k+1}^n - X_k^n \geq 0$ ,  $\delta \varrho_{k+\frac{1}{2}} := \varrho_{k+1} - \varrho_k \geq 0$ , and so on. This boils down to:

$$\delta X_{k+\frac{1}{2}}^{n+1} = \delta X_{k+\frac{1}{2}}^n - \left\{ \frac{\Delta t}{|C_{k+1}|} \Phi'_{k+1} \left( \frac{\delta \varrho_{k+\frac{3}{2}} \delta X_{k+\frac{1}{2}}^n - \delta \varrho_{k+\frac{1}{2}} \delta X_{k+\frac{3}{2}}^n}{\delta X_{k+\frac{3}{2}}^n \delta X_{k+\frac{1}{2}}^n} \right) - \frac{\Delta t}{|C_k|} \Phi'_k \left( \frac{\delta \varrho_{k+\frac{1}{2}} \delta X_{k-\frac{1}{2}}^n - \delta \varrho_{k-\frac{1}{2}} \delta X_{k+\frac{1}{2}}^n}{\delta X_{k+\frac{1}{2}}^n \delta X_{k-\frac{1}{2}}^n} \right) \right\},$$

where  $\Phi'_{k+1}$ ,  $\Phi'_k$  stand for some mid-point values of the derivative of  $\Phi$  at time  $t^n$ . Now, taking into account for the signs of all the present quantities and rearranging terms, we obtain

$$\delta X_{k+\frac{1}{2}}^{n+1} = \delta X_{k+\frac{1}{2}}^n \left\{ 1 - \frac{\Delta t \Phi'_{k+1}}{\delta X_{k+\frac{3}{2}}^n \delta X_{k+\frac{1}{2}}^n} \frac{\delta \varrho_{k+\frac{3}{2}}}{|C_{k+1}|} - \frac{\Delta t \Phi'_k}{\delta X_{k+\frac{1}{2}}^n \delta X_{k-\frac{1}{2}}^n} \frac{\delta \varrho_{k-\frac{1}{2}}}{|C_k|} \right\} + \frac{\Delta t \Phi'_{k+1}}{\delta X_{k+\frac{3}{2}}^n \delta X_{k+\frac{1}{2}}^n} \frac{\delta \varrho_{k+\frac{1}{2}}}{|C_{k+1}|} \delta X_{k+\frac{3}{2}}^n + \frac{\Delta t \Phi'_k}{\delta X_{k+\frac{1}{2}}^n \delta X_{k-\frac{1}{2}}^n} \frac{\delta \varrho_{k+\frac{1}{2}}}{|C_k|} \delta X_{k-\frac{1}{2}}^n.$$

From this positive combination, we infer that,

$$\frac{\delta X_{k+\frac{1}{2}}^{n+1}}{\delta \varrho_{k+\frac{1}{2}}} = \frac{\delta X_{k+\frac{1}{2}}^n}{\delta \varrho_{k+\frac{1}{2}}} \left\{ 1 - \frac{\Delta t \Phi'_{k+1}}{\delta X_{k+\frac{3}{2}}^n \delta X_{k+\frac{1}{2}}^n} \frac{\delta \varrho_{k+\frac{3}{2}}}{|C_{k+1}|} - \frac{\Delta t \Phi'_k}{\delta X_{k+\frac{1}{2}}^n \delta X_{k-\frac{1}{2}}^n} \frac{\delta \varrho_{k-\frac{1}{2}}}{|C_k|} \right\} + \frac{\Delta t \Phi'_{k+1}}{\delta X_{k+\frac{3}{2}}^n \delta X_{k+\frac{1}{2}}^n} \frac{\delta \varrho_{k+\frac{3}{2}}}{|C_{k+1}|} \left( \frac{\delta X_{k+\frac{3}{2}}^n}{\delta \varrho_{k+\frac{3}{2}}} \right) + \frac{\Delta t \Phi'_k}{\delta X_{k+\frac{1}{2}}^n \delta X_{k-\frac{1}{2}}^n} \frac{\delta \varrho_{k-\frac{1}{2}}}{|C_k|} \left( \frac{\delta X_{k-\frac{1}{2}}^n}{\delta \varrho_{k-\frac{1}{2}}} \right),$$

which is the desired convex combination under the condition (3.8); this ensures  $X_{k+1}^{n+1} - X_k^{n+1} \geq a > 0$ . Since  $\mathbb{R}_*^+ \ni x \mapsto 1/x$  is a convex function, the estimates (3.9) and (3.10) follow by Jensen's inequality.  $\square$

We stress that the monotonicity property of the  $X_k^n$ 's is crucial in order to define an approximation  $\tilde{\varrho}(t^n, \cdot)$  as being the graph of a monovalued function, which is the least one may expect in this context. In particular, for  $p = 1$ , (3.9) boils down to  $\sup_k \varrho_k \leq 1$ .

Of course, in practice, one could obey to restriction (3.8) according to the *real* value of  $\inf_k (X_{k+1}^n - X_k^n)$  at time  $t^n$  in order to allow  $\Delta t$  to vary in an *adaptive* way as times increase. We took advantage of this in the numerical tests shown subsequently. In order to keep  $\Delta t > 0$ , one needs to assume that  $X(t, \cdot)$  is strictly increasing, which implies that  $u_0 > 0$ . This can be partially dropped in practice, see §4.

The next lemma is an important step towards the convergence result.

LEMMA 3.2. *The scheme (3.6) is consistent with the equation (3.2).*

*Proof.* Let us define the function  $\varrho^{\Delta x}$  defined by  $C^1$  interpolation of the values

$$(3.11) \quad \forall (k, n) \in \mathcal{K} \times \mathbb{N}, \quad \varrho^{\Delta x}(t^n, X_k^n) = \varrho_k.$$

From the very definition of  $\varrho^{\Delta x}$  (3.11) and the scheme (3.6), one derives:

$$\begin{aligned} \varrho^{\Delta x}(t^n, X_k^n) &= \varrho^{\Delta x}(t^{n+1}, X_k^{n+1}) \\ &= \varrho^{\Delta x} \left( t^{n+1}, X_k^n - \frac{\Delta t}{|C_k|} \left\{ \Phi \left( \frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n} \right) - \Phi \left( \frac{\varrho_k - \varrho_{k-1}}{X_k^n - X_{k-1}^n} \right) \right\} \right). \end{aligned}$$

Then the mean-value theorem gives for some  $\zeta_k^{n+1} = \lambda X_k^n + (1 - \lambda)X_k^{n+1}$ ,  $\lambda \in [0, 1]$ :

$$\begin{aligned} \varrho^{\Delta x}(t^{n+1}, X_k^{n+1}) &= \varrho^{\Delta x}(t^n, X_k^n) \\ &\quad + \frac{\Delta t}{|C_k|} \partial_x \varrho^{\Delta x}(t^{n+1}, \zeta_k^{n+1}) \left\{ \Phi\left(\frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n}\right) - \Phi\left(\frac{\varrho_k - \varrho_{k-1}}{X_k^n - X_{k-1}^n}\right) \right\}. \end{aligned}$$

We now observe that,

$$\frac{\partial_x \varrho^{\Delta x}(t^{n+1}, \zeta_k^{n+1})}{|C_k|} = \frac{\partial_x \varrho^{\Delta x}(t^{n+1}, \zeta_k^{n+1})}{\varrho^{\Delta x}(t^{n+1}, X_{k+\frac{1}{2}}^{n+1}) - \varrho^{\Delta x}(t^{n+1}, X_{k-\frac{1}{2}}^{n+1})} = \frac{1}{X_{k+\frac{1}{2}}^{n+1} - X_{k-\frac{1}{2}}^{n+1}},$$

up to high-order terms. Replacing the other values  $\varrho_k$  inside (3.6) by the corresponding  $\varrho^{\Delta x}(t^n, \cdot)$  leads to the following *finite volume* discretization of (3.2):

$$(3.12) \quad \left\{ \varrho^{\Delta x}(t^{n+1}, X_k^n) = \varrho^{\Delta x}(t^n, X_k^n) + \frac{\Delta t}{X_{k+\frac{1}{2}}^{n+1} - X_{k-\frac{1}{2}}^{n+1}} \times \right. \\ \left. \left\{ \Phi\left(\frac{\varrho^{\Delta x}(t^n, X_{k+1}^n) - \varrho^{\Delta x}(t^n, X_k^n)}{X_{k+1}^n - X_k^n}\right) - \Phi\left(\frac{\varrho^{\Delta x}(t^n, X_k^n) - \varrho^{\Delta x}(t^n, X_{k-1}^n)}{X_k^n - X_{k-1}^n}\right) \right\} \right\}. \quad \square$$

We now derive a time-modulus of equicontinuity for the aforementioned scheme.

LEMMA 3.3. *Under the assumptions of Lemma 3.1 and the CFL restriction (3.8), the scheme (3.12) satisfies:*

$$(3.13) \quad \sup_k |\varrho^{\Delta x}(t^{n+1}, X_k^n) - \varrho^{\Delta x}(t^n, X_k^n)| = O(\sqrt{\Delta t}).$$

*Proof.* This readily follows from the expression (3.12), the CFL condition and the Lipschitz estimate (3.10).  $\square$

THEOREM 3.4. *Under the assumptions of Lemma 3.1 and the CFL restriction (3.8), the sequence of approximate solutions  $\varrho^{\Delta x}$  is relatively compact as  $\Delta x \rightarrow 0$  in  $L_{loc}^p(\mathbb{R}_*^+ \times \mathbb{R})$ ; it converges towards the unique solution in the sense of distributions of*

$$\partial_t \varrho = \partial_x (\Phi(\partial_x \varrho)), \quad \varrho(t = 0, \cdot) = \varrho_0 \in W^{1,p}(\mathbb{R}), \quad 1 \leq p \leq +\infty.$$

*Proof.* It is a bare consequence of the preceding lemmas together with a time-modulus of equicontinuity as we explain now. Let us start from (3.12); multiplying by a smooth function with compact support  $\varphi(t^{n+1}, X_k^n)$  and summing gives:

$$\begin{aligned} \frac{1}{\Delta t} \sum_{k,n} |X_{k+\frac{1}{2}}^{n+1} - X_{k-\frac{1}{2}}^{n+1}| \varphi(t^{n+1}, X_k^n) \left( \varrho^{\Delta x}(t^{n+1}, X_k^n) - \varrho^{\Delta x}(t^n, X_k^n) \right) = \\ \sum_{k,n} \varphi(t^{n+1}, X_k^n) \left\{ \Phi\left(\frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n}\right) - \Phi\left(\frac{\delta \varrho_{k-\frac{1}{2}}}{\delta X_{k-\frac{1}{2}}^n}\right) \right\}. \end{aligned}$$

Summing by parts yields

$$\begin{aligned} \frac{1}{\Delta t} \sum_{k,n} \varrho^{\Delta x}(t^n, X_k^n) \left( -\varphi(t^{n+1}, X_k^n) |X_{k+\frac{1}{2}}^{n+1} - X_{k-\frac{1}{2}}^{n+1}| + \varphi(t^n, X_k^n) |X_{k+\frac{1}{2}}^n - X_{k-\frac{1}{2}}^n| \right) \\ = \sum_{k,n} \Phi\left(\frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n}\right) \left( \varphi(t^{n+1}, X_k^n) - \varphi(t^{n+1}, X_{k+1}^n) \right) \end{aligned}$$

We deduce, with preceding notations,

$$\begin{aligned} & \frac{1}{\Delta t} \sum_{k,n} \varrho^{\Delta x}(t^n, X_k^n) \left( -\varphi(t^{n+1}, X_k^n) \delta X_k^{n+1} + \varphi(t^n, X_k^n) \delta X_k^n \right) \\ & - \sum_{k,n} \Phi \left( \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right) \left( \varphi(t^{n+1}, X_k^n) - \varphi(t^{n+1}, X_{k+1}^n) \right) = 0. \end{aligned}$$

We can rewrite that in integral form as follows:

$$\begin{aligned} & \sum_{k,n} \int_{t^n}^{t^{n+1}} \int_{X_{k-\frac{1}{2}}^n}^{X_{k+\frac{1}{2}}^n} \varrho^{\Delta x}(t^n, X_k^n) \left( -\frac{\varphi(t^{n+1}, X_k^n) - \varphi(t^n, X_k^n)}{\Delta t} \right) + \\ & \Phi \left( \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right) \left( \frac{\varphi(t^n, X_{k+1}^n) - \varphi(t^n, X_k^n)}{\delta X_k^n} \right) . dx . dt \\ & = \Delta t \sum_{k,n} \varrho(t^n, X_k^n) \varphi(t^n, X_k^n) (\delta X_k^{n+1} - \delta X_k^n) + \\ & \Phi \left( \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right) \left( \varphi(t^{n+1}, X_k^n) - \varphi(t^{n+1}, X_{k+1}^n) - \varphi(t^n, X_k^n) + \varphi(t^n, X_{k+1}^n) \right) \end{aligned}$$

At this point, we use the fact that  $\Phi(\delta \varrho / \delta X)$  and  $\varrho(t^n, X_k^n)$  are bounded, that  $\varphi$  is smooth in both variables; then we rewrite the first term of the right-hand side as

$$- \sum_{k,n} \Delta t \int_{X_{k-\frac{1}{2}}^n}^{X_{k+\frac{1}{2}}^n} (\varrho^{\Delta x} \varphi)(t^n, X_k^n) - (\varrho^{\Delta x} \varphi)(t^{n-1}, X_k^{n-1}) . dx,$$

and the second term like

$$- \sum_{k,n} \Delta t \int_{t^n}^{t^{n+1}} \int_{X_k^n}^{X_{k+1}^n} \Phi \left( \frac{\delta \varrho_{k+\frac{1}{2}}}{\delta X_{k+\frac{1}{2}}^n} \right) \partial_{tx} \varphi(\tau, \xi) . d\xi . d\tau.$$

Now, since  $0 < u_0 \in L^1 \cap L^\infty(\mathbb{R})$ ,  $\varrho_0$  is a strictly increasing Lipschitz function of  $x$  hence the family  $(X_k^n)_k$  covers the whole axis  $\mathbb{R}$ . Then, by regularity,  $|X_{k+1}^n - X_k^n| \rightarrow 0$  and  $|X_k^{n+1} - X_k^n| \rightarrow 0$  for  $n \in \mathbb{N}$  as  $\Delta x \rightarrow 0$  since by Lemma 3.1, (3.6) is a convex combination. This is enough to derive the weak form of the equation. Uniqueness in the limit follows from the classical argument of Oleinik for weak solutions, see [25].  $\square$

We close this section mentioning that the assumption  $u_0 > 0$  in Lemma 3.1 is essentially needed in order to ensure that  $(X_k^n)_{k \in \mathbb{N}}$  permits to cover the whole real line as  $\Delta x \rightarrow 0$ . We shall consider in §4.4 initial data of compact support which are strictly positive only inside their support; in this case, only the support of  $u(n\Delta t, \cdot)$  can be expected to be recovered.

**3.3. Study of the Wasserstein metric.** We mainly follow [7, 10, 28] to study *contraction properties* of the scheme (3.6) within the Wasserstein metric framework. Denoting  $\mathbb{P}_p(\mathbb{R})$  the set of all probability measures on  $\mathbb{R}$  with moments of order  $1 \leq p < +\infty$  and  $\Pi(\nu_1, \nu_2)$  any of the probability measures on  $\mathbb{R}^2$  admitting  $\nu_{1,2} \in \mathbb{P}_p(\mathbb{R})$  as marginal distributions, the **Wasserstein p-metric** reads:

$$(3.14) \quad W_p(\nu_1, \nu_2) := \left( \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int_{\mathbb{R}^2} |x - y|^p . d\pi(x, y) \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty.$$

Any probability measure admits a distribution function, which can be chosen right-continuous, nondecreasing and taking values inside  $[0, 1]$ . A nondecreasing pseudo-inverse can be defined as for (3.1). Hence if  $X_1, X_2$  stand for the pseudo-inverses of the repartition functions of  $\nu_1, \nu_2 \in \mathbb{P}_p(\mathbb{R})$ , the distance (3.14) rewrites:

$$(3.15) \quad W_p(\nu_1, \nu_2) := \left( \int_0^1 |X_1(\varrho) - X_2(\varrho)|^p \cdot d\varrho \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty.$$

According to (1.1), a formal computation leads easily to a contraction property for the metric  $W_2(\cdot, \cdot)$ . Let  $X(t, \varrho), Y(t, \varrho)$  stand for two reciprocal mappings associated to nonnegative and centered initial data of (1.1)  $u_0, v_0 \in L^1 \cap L^\infty(\mathbb{R})$  with unit mass,

$$\begin{aligned} \frac{d}{dt} \int_0^1 |X(t, \varrho) - Y(t, \varrho)|^2 \cdot d\varrho &= -2 \int_0^1 (X - Y) \partial_\varrho \left\{ \Phi\left(\frac{1}{\partial_\varrho X}\right) - \Phi\left(\frac{1}{\partial_\varrho Y}\right) \right\} (t, \varrho) \cdot d\varrho \\ &= 2 \int_0^1 \partial_\varrho (X - Y) \left\{ \Phi\left(\frac{1}{\partial_\varrho X}\right) - \Phi\left(\frac{1}{\partial_\varrho Y}\right) \right\} (t, \varrho) \cdot d\varrho \\ &\leq 0, \end{aligned}$$

because  $\Phi$  is increasing. Then a similar property can be shown to hold for the outcome of the explicit scheme (3.6):

**THEOREM 3.5.** *Let  $u_0, v_0$  be two nonnegative initial data in  $L^1 \cap L^\infty(\mathbb{R})$  for (1.1) and  $X, Y$  their reciprocal mappings. Under the CFL restriction (3.8), the scheme (3.6) is contractive in any Wasserstein metric  $W_p$ ; more precisely, there holds:*

$$(3.16) \quad \forall n \in \mathbb{N}, \quad \sum_k |C_k| |X_k^{n+1} - Y_k^{n+1}|^p \leq \sum_k |C_k| |X_k^n - Y_k^n|^p, \quad p \geq 1.$$

*Proof.* Mimicking the preceding formal computation, we aim at establishing:

$$\delta W_p := \frac{1}{\Delta t} \sum_k |C_k| \left\{ |X_k^{n+1} - Y_k^{n+1}|^p - |X_k^n - Y_k^n|^p \right\} \leq 0.$$

We get from (3.6) that:

$$\begin{aligned} X_k^{n+1} - Y_k^{n+1} &= X_k^n - Y_k^n - \left\{ \frac{\Delta t}{|C_k|} \left( \Phi\left(\frac{\varrho_{k+1} - \varrho_k}{X_{k+1}^n - X_k^n}\right) - \Phi\left(\frac{\varrho_k - \varrho_{k-1}}{X_k^n - X_{k-1}^n}\right) \right) \right. \\ &\quad \left. - \frac{\Delta t}{|C_k|} \left( \Phi\left(\frac{\varrho_{k+1} - \varrho_k}{Y_{k+1}^n - Y_k^n}\right) - \Phi\left(\frac{\varrho_k - \varrho_{k-1}}{Y_k^n - Y_{k-1}^n}\right) \right) \right\} \\ &= X_k^n - Y_k^n - \frac{\Delta t}{|C_k|} \left\{ \left( \Phi_{k+\frac{1}{2}}(X_{k+1}^n - X_k^n) - \Phi_{k+\frac{1}{2}}(Y_{k+1}^n - Y_k^n) \right) \right. \\ &\quad \left. - \left( \Phi_{k-\frac{1}{2}}(X_k^n - X_{k-1}^n) - \Phi_{k-\frac{1}{2}}(Y_k^n - Y_{k-1}^n) \right) \right\}, \end{aligned}$$

where we used the notation,

$$\Phi_{k+\frac{1}{2}}(\delta X) := \Phi\left(\frac{\varrho_{k+1} - \varrho_k}{\delta X}\right).$$

Thanks to the bound given by Lemma 3.1, we know that  $\delta X \geq a > 0$ , so the function  $\Phi_{k+\frac{1}{2}}$  is smooth and the mean-value theorem can be applied. The outcome is:

$$\begin{aligned} X_k^{n+1} - Y_k^{n+1} &= (X_k^n - Y_k^n) \left( 1 + \frac{\Delta t}{|C_k|} (\Phi'_{k+\frac{1}{2}} + \Phi'_{k-\frac{1}{2}}) \right) \\ &\quad - \frac{\Delta t}{|C_k|} \Phi'_{k+\frac{1}{2}} (X_{k+1}^n - Y_{k+1}^n) - \frac{\Delta t}{|C_k|} \Phi'_{k-\frac{1}{2}} (X_{k-1}^n - Y_{k-1}^n), \end{aligned}$$

with  $\Phi'_{k+\frac{1}{2}}$  standing for some mid-point value of the derivative of  $\Phi_{k+\frac{1}{2}}$  with respect to  $\delta X$ . Hence since

$$\Phi'_{k+\frac{1}{2}}(\delta X) = -\frac{\varrho_{k+1} - \varrho_k}{\delta X^2} \Phi' \left( \frac{\varrho_{k+1} - \varrho_k}{\delta X} \right),$$

the CFL condition (3.8) ensures that the last expression is a convex combination. By means of Jensen's inequality, and thanks to the fact that the fluxes are null on the borders of the domain, this yields  $\delta W_p \leq 0$  and we are done.  $\square$

A consequence of this is that in case one would want to use the discretization (3.6) for a problem (1.1) with a partly atomic probability measure, one can initialize the scheme with a somewhat smoother initial data relying on this contraction property. Moreover, this also ensures that the propagation speed of the free boundaries is correct relying on Theorem 2.5.

We stress that the estimate (3.16) doesn't imply a decay of the support  $(X_k^n)_{k \in \mathcal{K}}$  which would somewhat contradict the retention phenomenon. Indeed, the CFL condition (3.8) cannot allow to choose  $Y_k^n \equiv 0$ .

**4. Numerical results.** All the forthcoming tests have been carried out relying on the explicit scheme (3.6); the initial data  $u_0$  is sampled on a set of 257 points, which gives a space-step  $\Delta x$  equal to the length of the domain divided by 256. The  $\varrho_k$ 's are then deduced by numerical quadrature. The time-step is chosen in an adaptive way as explained after the proof of Lemma 3.1.

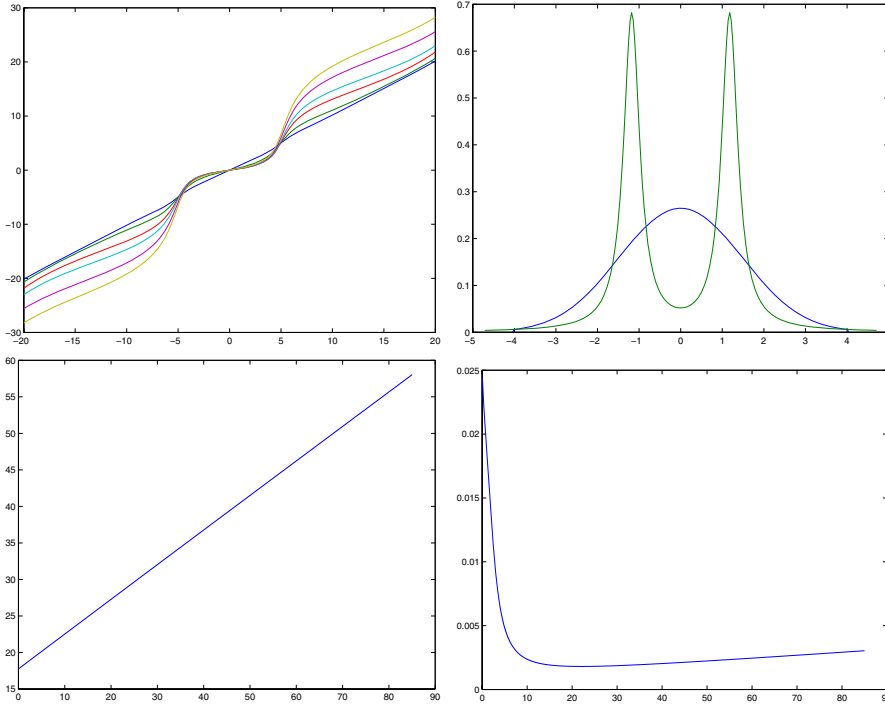


FIG. 4.1. Numerical values for  $X(t, x)$  for  $t = 1, 5, 15, 25, 45, 65$ ; scaled initial data  $f(t = 0, \cdot)$  (green) and stationary solution  $f_\infty$  (blue); evolution of  $E(t)$  and  $\Delta t$  with time  $t$  (left to right, top to bottom).

**4.1. Validation: the heat equation.** In order to test the scheme on a simple and well-known case, we set up equation (1.1) with  $\Phi(u) = \frac{u}{2}$ . The initial data is chosen rather far away from the expected equilibrium state:

$$(4.1) \quad u_0(x) = \frac{1}{2} \left( \frac{1}{\pi(1+(x-5)^2)} + \frac{1}{\pi(1+(x+5)^2)} \right), \quad x \in [-20, 20].$$

We observe that even if  $\int_{\mathbb{R}} x^2 \cdot u_0(x) \cdot dx$  isn't bounded, one can set up the scheme (3.6) for  $x$  inside a compact interval of  $\mathbb{R}$ . The results at time  $t = 85$  are displayed on Fig.4.1. Along with the evolution of  $X_k^n$  as  $t^n = 1, 5, 15, 25, 45, 65$ , we observe a linear increase of  $t \mapsto E(t)$  as shown theoretically and a correct decay onto a Gaussian distribution for the scaled solution  $f(t, \cdot)$ . We show the corresponding (numerically) stationary profile. The time-step decreases a lot when the two bumps merge but it increases afterwards as the solution  $u(t, \cdot)$  doesn't change its shape no more.

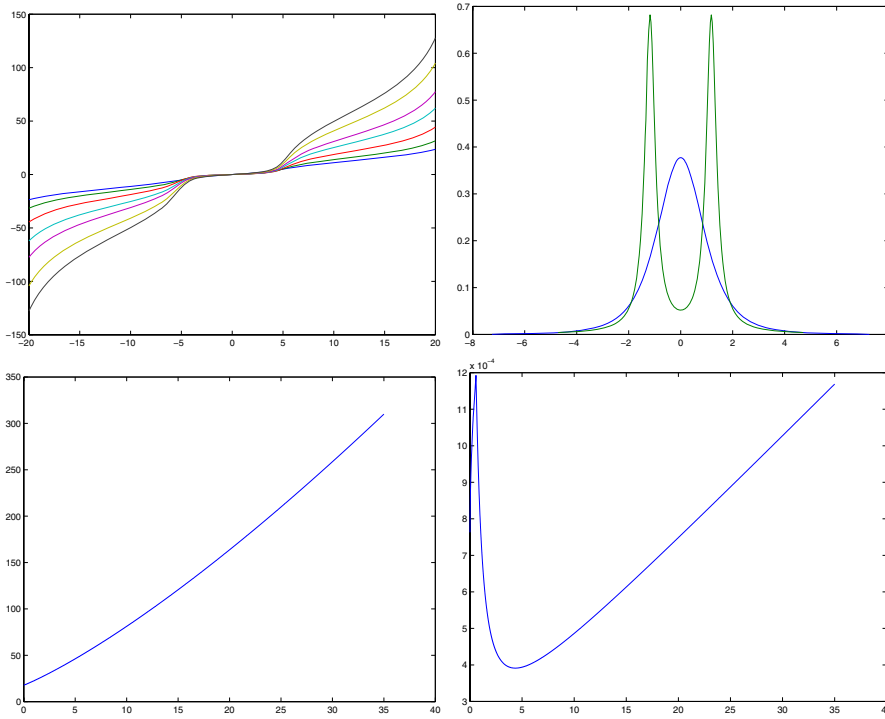


FIG. 4.2. Numerical values for  $X(t, x)$  for  $t = 0.5, 2, 5, 10, 15, 25, 35$ ,  $m = 0.5$ ; scaled initial data  $f(t = 0, \cdot)$  (green) and stationary solution  $f_\infty$  (blue); evolution of  $E(t)$  and  $\Delta t$  with time  $t$  (left to right, top to bottom).

**4.2. Two cases of fast-diffusion equations.** We now display on Figs.4.2 and 4.3 a similar experiment with two fast-diffusion equations, respectively  $\Phi(u) = \sqrt{u}$  and  $\Phi(u) = u^{\frac{1}{4}}$ . The initial data and the computational domain are still given by (4.1). Several major differences show up in this case compared to the heat equation:

- the support of the solution extends much more quickly as the exponent  $m$  is decreased as can be seen on the graphs of the  $X_k^n$ 's,
- the scaled solutions  $f(t, \cdot)$  stabilize at much earlier times ( $t \simeq 25-30$ ),
- the asymptotic profile is more peaked with a lower value of  $m$ ,

- the mapping  $t \mapsto E(t)$  looks now convex.

However, as for the heat equation, the asymptotic solution has infinite support and is thus  $C^\infty$  as a consequence of Proposition 2.2, see also [26]. We stress that the tails of the initial data can be seen to be close to the ones of the asymptotic profile, see [24] for remarks in this direction.

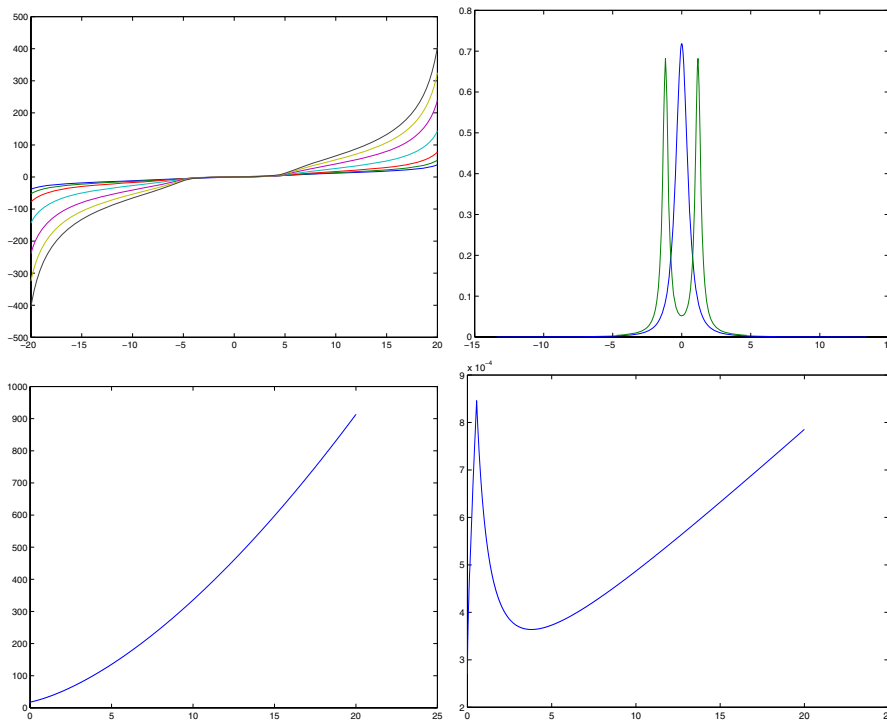


FIG. 4.3. Numerical values for  $X(t, x)$  for  $t = 0.5, 1, 2, 5, 10, 15, 20$ ,  $m = 0.25$ ; scaled initial data  $f(t = 0, \cdot)$  (green) and stationary solution  $f_\infty$  (blue); evolution of  $E(t)$  and  $\Delta t$  with time  $t$  (left to right, top to bottom).

**4.3. The porous media equation and Barenblatt’s solution.** We investigated the case of the classical porous medium equation, namely  $\Phi(u) = \frac{u^2}{2}$  with the data (4.1). Since it isn’t compactly supported, we didn’t observe the well-known decay onto the corresponding Barenblatt-Pattle solution, but instead,  $f_\infty$  exhibits a similar profile with two tails on each side as shown in Fig.4.4. Also the mapping  $t \mapsto E(t)$  looks like being concave and the stabilization time is much greater than in the two preceding examples ( $t \simeq 200$ ). The variations of the time-step are moderate in comparison with the fast-diffusion equations.

**4.4. Buckley-Leverett’s doubly degenerate equation.** Finally, we studied a more singular problem given by equation (1.1) with

$$\Phi(u) = \frac{u^2}{u^2 + 0.5(1 - u)^2}.$$

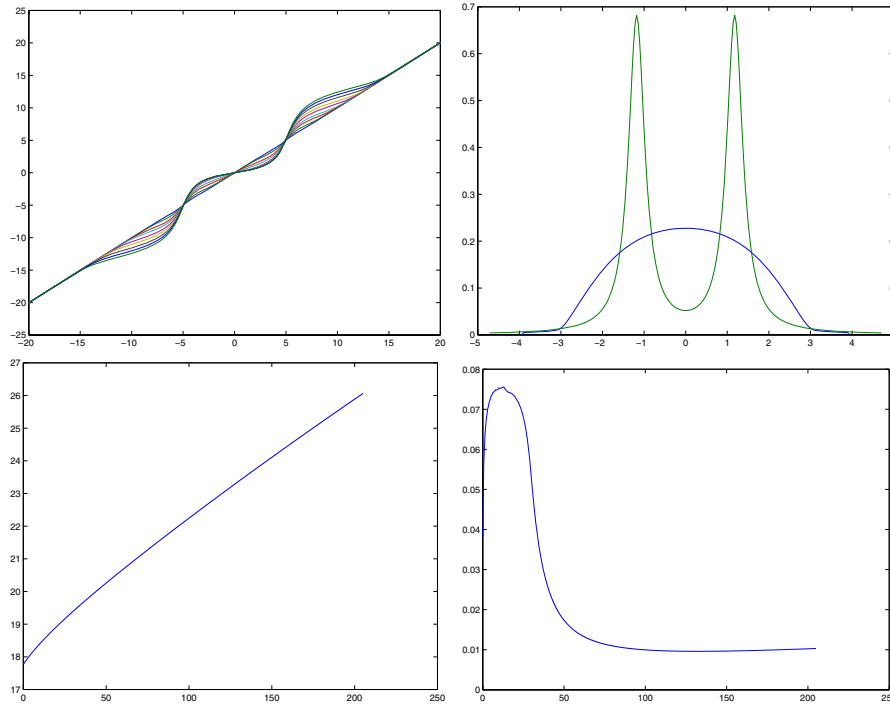


FIG. 4.4. Numerical values for  $X(t, x)$  for  $t = 1, 5, 15, 25, 45, 65, 95, 125, 155$ ; scaled initial data  $f(t = 0, \cdot)$  (green) and stationary solution  $f_\infty$  (blue); evolution of  $E(t)$  and  $\Delta t$  with time  $t$  (left to right, top to bottom).

The derivative  $\Phi'$  vanishes at two points  $u = 0$  and  $u = 1$ . We set up the following smooth initial data extended by zero outside of  $[-1, 1]$ :

$$u_0 = \cos(\pi x/2)^2, \quad x \in [-1, 1].$$

In this case, even if this hasn't been rigorously proven yet, we may expect to observe a decay of this compactly supported function towards a Barenblatt-Pattle profile asymptotically. This can be observed on Fig.4.5. We can also check that this problem shares other features with the slow-diffusion equation since the mapping  $t \mapsto E(t)$  seems like being concave. However, the support of the solution grows more quickly as times increase.

**5. Conclusion and outlook.** We introduced and studied analytically in this paper a new numerical scheme for one-dimensional filtration equations of the type (1.1). As a main feature, it allows to observe the asymptotic decay of solutions towards self-similar ones without requesting important changes of the computational domain (as it would be the case for a conventional discretization, see *e.g.* [14, 17, 16, 18, 20, 21, 23]). Moreover, a contraction property in Wasserstein metrics can be easily established. As a final remark, let us stress that a similar derivation can be applied to one-dimensional nonlinear Fokker-Planck equations,

$$\partial_t u = \partial_x(\partial_x V(t, x)u) + \partial_{xx}(\Phi(u)),$$

for which the evolution of the reciprocal mapping  $X(t, \varrho)$  would be given by (see [10]),

$$\partial_t X + (\partial_x V)(t, X) + \partial_\varrho(\Phi(1/\partial_\varrho X)) = 0,$$

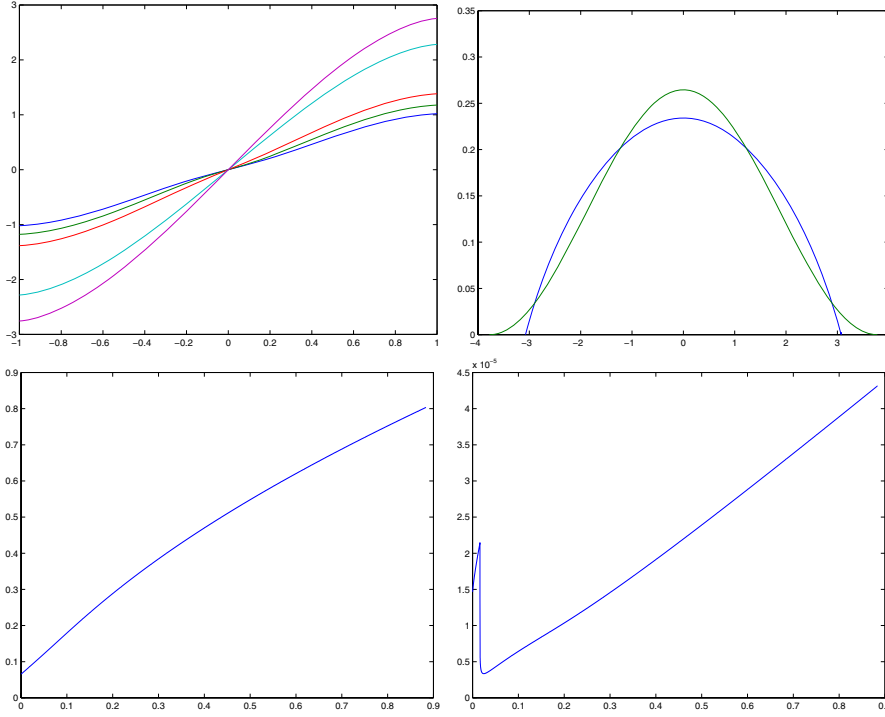


FIG. 4.5. Numerical values for  $X(t, x)$  for  $t = 0.05, 0.1, 0.2, 0.5, 0.88$ ; scaled initial data  $f(t = 0, \cdot)$  (green) and stationary solution  $f_\infty$  (blue); evolution of  $E(t)$  and  $\Delta t$  with time  $t$  (left to right, top to bottom).

thus only asking for mild changes with respect to (3.4)–(3.6). Very fast diffusion equations could also be handled within the present framework, *i.e.*  $\partial_t u + \partial_{xx}(u^{-m}) = 0$ ,  $0 < m < 1$ .

From this last computation, one can moreover extract information concerning radial solutions of 2D equations; let  $u(t, x, y)$  solve  $\partial_t u = \Delta \Phi(u)$ ,  $\Delta$  standing for the Laplace operator in  $\mathbb{R}^2$ , while meeting the requirement that  $u(t, x, y) = \tilde{u}(t, r)$ ,  $r = \sqrt{x^2 + y^2}$ . Then one deduces easily an equation on  $\tilde{u}$ , namely  $\partial_t \tilde{u} = \partial_{rr} \Phi(\tilde{u}) + \frac{1}{r} \partial_r \Phi(\tilde{u})$ , for which the last term creates a difficulty with respect to the aforementioned Fokker-Planck computation. This can be circumvented as follows: one observes that,

$$\varrho(t, r) = \int_0^r s \tilde{u}(t, s) ds, \quad \partial_t \int_0^\infty s \tilde{u}(t, s) ds \equiv 0.$$

Then it is possible to define  $R$  as the reciprocal mapping such that,

$$\forall \bar{\varrho} \in [0, 1], \quad \varrho(t, R(t, \bar{\varrho})) = \bar{\varrho},$$

and satisfying the equation:

$$\partial_t R + R \partial_\varrho \Phi \left( \frac{1}{R \partial_\varrho R} \right) = 0.$$

A third possible extension is suggested in [4, 5] for "viscous pressureless gas equations"; in this case, the pseudo-inverse evolves according to

$$\partial_t X = v_0(\varrho) - \partial_\varrho(\Phi(1/\partial_\varrho X)),$$

where  $\Phi$  can be computed explicitly from the Eulerian viscosity term. The function  $\Phi(u) = u \ln(u)$  is of special interest as it corresponds to a linear perturbation in Eulerian coordinates. In this context,  $v_0(\varrho)$  is implicitly defined from the initial velocity in Eulerian coordinates via  $v_0(\varrho) = u_0 \circ X(t = 0, \varrho)$ .

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