Hydrodynamics from the dissipative Boltzmann equation

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1 Introduction

This paper deals with some questions related with the modelling of hydrodynamic equations for granular gases, at the light of recent mathematical results on the large-time behavior of the dissipative Boltzmann equation. This subject is relatively new, and the relevant mathematical theory is still restricted. In the pertinent literature [Duf01], rapid granular flows were frequently described at the macroscopic level by means of equations for fluid dynamics, modified to account for dissipation due to collisions among particles. This was the approach of Haff, which, in his pioneering paper [Haf83], gave a macroscopic description of the behavior of a granular material treating the individual grains as the molecules of a granular fluid, without resorting to the mesoscopic picture (the Boltzmann or Enskog kinetic equations).

In more recent years it became clear that, in agreement with the well established derivation of conservative fluid dynamics from the Boltzmann equation [BGL91, BGL93], kinetic theory was the basis for a deeper understanding of macroscopic equations even for dissipative flows. Kinetic theory is suitable to describe the evolution of materials composed of many small discrete grains, in which the mean free path of the grains is much larger than the typical particle size. In this regime, granular gases can be described within the concepts of classical statistical mechanics, by adapting methods of the kinetic theory of ideal gases [Kog69, CIP94].

Many authors (see Refs. [NY93, DLK95, BCP97, EP97, BCG00, BP00a] and the references therein) adopted this line of thought, by introducing and discussing the evolution of a system of partially inelastic rigid spheres through Boltzmann-like equations. A typical kinetic model for the study of the evolution of a granular material takes the following form: the unknown is a time dependent density in phase space $f(x, v, t)$ satisfying a Boltzmann–Enskog equation for inelastic hard–spheres, which for the force–free case reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\epsilon}\, C(f, f)(x, v, t). \tag{1}$$

Here $v \cdot \nabla_x f$ is the usual transport operator where $C(f, f)$ is the so–called granular collision operator, which describes the change in the density function due to creation and annihilation of particles in dissipative binary collisions. The ϵ-parameter (Knudsen number) represents a measure of the mean free path, and has to be assumed small in fluid dynamical regimes.

The loss of energy in the microscopic collision translates at a macroscopic level in the progressive cooling of the gas, a phenomenon which is responsible of most of the difficulties in extending methods of classical kinetic theory of ideal gases to granular ones. A clear understanding of the new problems one has to deal with in the derivation of macroscopic equations in dissipative kinetic theory can by obtained through the use of the splitting method, very popular in the numerical approach to the Boltzmann equation [GPT97, PR01]. If at each time step we consider sequentially the transport and relaxation operators in the Boltzmann equation (1), during this short time interval we recover the evolution of the density from the joint action of the relaxation

$$\frac{\partial f}{\partial t} = \frac{1}{\epsilon}\, C(f, f)(x, v, t), \tag{2}$$

and transport

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0. \tag{3}$$

In classical kinetic theory, the energy is conserved in collisions, and the relaxation (2) pushes the solution towards the Maxwellian equilibrium with the same mass, momentum and energy of the initial datum. Then, if ϵ is sufficiently small, one can easily argue that the solution to (2) is *sufficiently close* to the Maxwellian, and this Maxwellian can be used into the transport step (3). When dissipation is present, solutions of the inelastic Boltzmann equation lose energy until all particles travel at the same speed, and the relaxation (2) pushes the solution towards the asymptotic state represented by a δ function concentrated in the mean velocity of the initial value. It is evident that, if $\epsilon << 1$, so that the solution to (2) is close to this *poor* asymptotic state, substitution into the transport step (3) does not lead to any correct behavior.

To circumvent this difficulty, two different procedures have been proposed. The first one requires that microscopic collisions are weakly inelastic. In this case, one assumes that the collision operator $C(f, f)$ can be decomposed as

$$C(f, f) = B(f, f) + \beta I(f, f), \tag{4}$$

where $B(f, f)$ is the elastic collision operator, while $I(f, f)$ represents the inelastic correction. If β is of the same order of ϵ, so that $\beta/\epsilon \to \lambda$ as $\beta, \epsilon \to 0$, we can easily modify in this case the aforementioned splitting, putting the elastic collision operator in the relaxation step

$$\frac{\partial f}{\partial t} = \frac{1}{\epsilon}\, B(f, f)(x, v, t), \tag{5}$$

and including the granular correction into the transport

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\beta}{\epsilon}\ I(f, f). \tag{6}$$

As before, the relaxation (36) pushes the solution towards the Maxwellian equilibrium with the same mass, momentum and energy of the initial datum. Then, if ϵ is sufficiently small, $\beta/\epsilon \cong \lambda$, and the Maxwellian solution can be used in the transport step (6), to get explicitly computable equations for the macroscopic quantities.

A second method is based on a more precise study of the asymptotic behavior of the solution in the relaxation step. This is obtained by looking for exact solutions to (2) (homogeneous cooling states), with the aim to use this exact solution in the transport (3).

The main advantage in working with small inelasticity is that one can easily include in the procedure any type of inelastic collisions, and in particular general coefficients of restitution. Various studies enlighten in fact the dependence of the cooling problem on the coefficient of restitution in the microscopic collision, and emphasize the effects of a non–constant restitution coefficient [BP00a, BP03a]. Special attention has been devoted in this respect to a system of viscoelastic spheres, a quite realistic model whose coefficient of restitution has been recently derived [RPBS99]. Moreover, the elastic collision dominated regime prevents the derivation to be sensible to the strength of spatial gradients. From a mathematical point of view, the granular Boltzmann equation has been object of some attention in a recent past [BC02b, BC03], and hydrodynamic closure in a weakly inelastic regime has been discussed already in [GS95, BCG00, BDKS98, Tos04], mostly at the level of Euler equations. A further step beyond the Euler level, in which closure is achieved by simply using the zero order solution (the equilibrium Maxwellian) for the distribution function, has been recently done in [RC02, BST04], by resorting to a Grad 13–moment expansion able to capture the relevant Grad equations. The idea of applying Grad's method to inelastic gases goes back to Jenkins and Richmann [JR85]. In this pioneering paper they outline the main ideas of Grad's derivation of hydrodynamics from a kinetic equation, using the Maxwellian distribution to close the hierarchy of transport equations. An important feature of Grad's equations is that they still contain collision terms, and are affected by the same small parameters as the kinetic equations. They lend themselves then to a classical asymptotic procedure of the Chapman–Enskog type, and provide as important byproduct hydrodynamic equations at the Navier–Stokes level. We refer to the recent paper [GK02] for a detailed treatment of the classical Chapman–Enskog derivation of hydrodynamics given in the framework of Grad's moment equations.

As far as the second method is concerned, the common assumption which has been at the basis of several recent papers on the matter is that there are only small spatial variations, so that the zero order approximation of the solution (and of any asymptotic expansion) is constituted by the so–called homogeneous cooling state (see for instance Ref. [BDKS98] and the references therein). A detailed theory of the homogeneous cooling state for viscoelastic particles in terms of expansion in Sonine polynomials has been recently developped [BP00b, BP00c]. Such spatially homogeneous solution turns out to depend not only on the similarity variable, as it would occur for constant restitution coefficient, but also on time explicitly. In addition, temperature has been shown to decay asymptotically according to a corrected Haff's law (see also Ref. [SP98]). Asymptotic expansions around the homogeneous cooling state have been used then as hydrodynamic closure for the macroscopic equations in order to achieve a Navier–Stokes level via a Chapman–Enskog procedure also for non–constant restitution coefficient [LS86]. In particular, the complete set of hydrodynamic equations and transport coefficients have been derived in this frame for a granular gas of viscoelastic particles [BP02, BP03b]. From a mathematical point of view, the possibility of using the homogeneous cooling state to close the transport equation (3) would require precise statements on the role of this exact solution, which has to be the intermediate solution of the homogeneous problem for a large *physical* class of initial densities. For a simplified model of the Boltzmann equation, the so–called Maxwellian model [BCG00], recent results showed that this holds true [BCT03, BCT05a]. These results were first motivated by a question posed by Ernst and Brito [EB02a, EB02b], concerning the fat tails of the self–similar solution for this model.

2 Modelling dissipative Boltzmann equation

In a granular gas, the microscopic dynamics of grains is governed by the restitution coefficient e which relates the normal components of the particle velocities before and after a collision. If grains are identical perfect spheres of diameter $\sigma > 0$, (x, v) and $(x-\sigma n, w)$ are their states before a collision, where $n \in S^2$ is the unit vector along the center of both spheres, the post collisional velocities (v^*, w^*) are such that

$$(v^* - w^*) \cdot n = -e((v - w) \cdot n)\,. \tag{7}$$

Thanks to (7), and assuming the conservation of momentum, one finds the change of velocity for the colliding particles as

$$v^* = v - \frac{1}{2}(1+e)((v-w)\cdot n)n\,, \quad w^* = w + \frac{1}{2}(1+e)((v-w)\cdot n)n. \tag{8}$$

For elastic collisions one has $e = 1$, while for inelastic collisions e decreases with increasing degree of inelasticity.

In the literature, it is frequently assumed that the restitution coefficient is a physical constant. In real applications, however, the restitution coefficient may depend on the relative velocity in such a way that collisions with small relative velocity are close to be elastic. The simplest physically correct description of dissipative collisions is based on the assumption that the spheres are composed by viscoelastic material [BP00a, BP00d]. This analysis suggests that in general the coefficient of restitution is such that

$$1 - e = 2\beta\gamma\left(|(v-w)\cdot n|\right), \tag{9}$$

where $\gamma(\cdot)$ is a given function and β is a parameter which is small in presence of small inelasticity. For example, for small values of α, the velocity dependence of the restitution coefficient in a collision of viscoelastic spheres can be expressed at the leading order as in (9), choosing $\gamma(r) = r^{1/5}$ [BP00a]. In a rarefied regime, a general model of bilinear operator for dissipative collisions is obtained by choosing

$$C(f,f)(x,v,t) = G(\rho)\bar{Q}(f,f)(x,v,t), \tag{10}$$

where $\bar{Q}$

$$\bar{Q}(f,f)(v) = 4\sigma^2 \int_{I\!R^3}\int_{S_+} q\cdot n\left\{\chi f(v^{**})f(w^{**}) - f(v)f(w)\right\} dw\, dn. \tag{11}$$

In (10)

$$\rho(x,t) = \int_{I\!R^3} f(x,v,t)\, dv \tag{12}$$

is the density, and the function $G(\rho)$ is the statistical correlation function between particles, which accounts for the increasing collision frequency due to the excluded volume effects. We refer to [Cer95] for a detailed discussion of the meaning of the function G.

In (11), $q = (v-w)$, and S_+ is the hemisphere corresponding to $q\cdot n > 0$. The velocities (v^{**}, w^{**}) are the pre collisional velocities of the so–called inverse collision, which results with (v,w) as post collisional velocities. The factor χ in the gain term appears respectively from the Jacobian of the transformation $dv^{**}dw^{**}$ into $dvdw$ and from the lengths of the collisional cylinders $e|q^{**}\cdot n| = |q\cdot n|$. For a constant restitution coefficient, $\chi = e^{-2}$.

In what follow we write the operator (11) in weak form [Tos04]. More precisely, for all smooth functions $\varphi(v)$, it holds

$$\begin{aligned} &< \varphi\,,\, \bar{Q}(f,f) > = \int_{I\!R^3} \varphi(v)\bar{Q}(f,f)(v)\, dv = \\ &4\sigma^2 \int_{I\!R^3}\int_{I\!R^3}\int_{S_+} q\cdot n\left(\varphi(v^*) - \varphi(v)\right) f(v)f(w) dv\, dw\, dn = \\ &2\sigma^2 \int_{I\!R^3}\int_{I\!R^3}\int_{S^2} |q\cdot n|\left(\varphi(v^*) - \varphi(v)\right) f(v)f(w) dv\, dw\, dn.. \end{aligned} \tag{13}$$

The last equality follows since the integral over the hemisphere S_+ can be extended to the entire sphere S^2, provided the factor $1/2$ is inserted in front of the integral itself. Let (v', w') be the post collisional velocities in a elastic collision with (v,w) as incoming velocities,

$$v' = v - (q \cdot n)n\,, \quad w' = w + (q \cdot n)n. \tag{14}$$

Using (8) and (14) one obtains

$$v^* = v' + \frac{1}{2}(1-e)(q \cdot n)n\,, \quad w^* = w' - \frac{1}{2}(1-e)(q \cdot n)n. \tag{15}$$

If we assume that the coefficient of restitution satisfies (9),

$$v^* - v' = \beta\gamma\left(|q \cdot n|\right)(q \cdot n)n. \tag{16}$$

Let us consider a Taylor expansion of $\varphi(v^*)$ around $\varphi(v')$. Thanks to (16) we get

$$\varphi(v^*) = \varphi(v') + \beta\nabla\varphi(v') \cdot \gamma\left(|q \cdot n|\right)(q \cdot n)n + O(\beta). \tag{17}$$

If the collisions are nearly elastic, $\beta << 1$, and we can cut the expansion (17) after the first–order term. Inserting (17) into (13) gives

$$\begin{aligned} &< \varphi\,,\, \bar{Q}(f,f) > = 2\sigma^2 \int_{I\!R^3}\int_{I\!R^3}\int_{S^2} |q \cdot n| \times \\ &(\varphi(v') - \varphi(v) + \beta\nabla\varphi(v') \cdot \gamma\left(|q \cdot n|\right)(q \cdot n)n)\, f(v)f(w)dv\, dw\, dn = \\ &< \varphi\,,\, Q(f,f) > \ +\beta < \varphi\,,\, I(f,f) > \ . \end{aligned} \tag{18}$$

It is a simple matter to recognize that in (18) $Q(f,f)$ is the classical Boltzmann collision operator for elastic hard–spheres molecules [CIP94],

$$Q(f,f)(v) = 2\sigma^2 \int_{I\!R^3}\int_{S^2} |q \cdot n| \left\{f(v')f(w') - f(v)f(w)\right\} dw\, dn. \tag{19}$$

In fact, the velocity v' into (18) is obtained from (v,w) through the elastic collision (14).

The second contribution to the inner product (18) can be easily computed [Tos04]. For weak inelasticity the granular correction is the nonlinear friction operator $\beta I(f,f)(v)$, where

$$I(f,f)(v) = 2\sigma^2 \mathrm{div}_v \int_{I\!R^3}\int_{S^2} n(q \cdot n)|q \cdot n|\gamma\left(|q \cdot n|\right) f(v')f(w')dw\, dn. \tag{20}$$

The nonlinear friction operator is such that mass and momentum are collisional invariant, while the energy is not. If the restitution coefficient satisfies (9), the Enskog–Boltzmann equation can be modelled at the leading order as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = G(\rho)Q(f,f)(x,v,t) + G(\rho)\beta I(f,f)(x,v,t), \tag{21}$$

where Q is the classical elastic Boltzmann collision operator, and I is a dissipative nonlinear friction operator which is based on elastic collisions between particles. An interesting property of the collisional integral I, connected with the passage to fluid dynamics, is that it leads to exact computations in correspondence to a locally Maxwellian function

$$M(x,v,t) = \frac{\rho(x,t)}{(2\pi T(x,t))^{3/2}} \exp\left(-\frac{(v-u(x,t))^2}{2T(x,t)}\right). \tag{22}$$

In (22) $\rho(x,t)$, $u(x,t)$ and $T(x,t)$ represent its mass, mean velocity and temperature, respectively. In particular one can explicitly evaluate moments. Let us consider the case in which $\gamma(r) = r^p$, with $p \geq 0$. Then [Tos04]

$$< \frac{1}{2}v^2 \, , \, I(M,M) > = -\frac{\Gamma(2+p/2)2^{7+p}\sqrt{\pi}}{(4+p)}\rho^2 T^{(3+p)/2}. \tag{23}$$

3 Hydrodynamic limit and the Euler equations.

Let us fix the coefficient of restitutions to satisfy $\gamma(r) = r^p$. This choice will include both the constant coefficient of restitution and the physically relevant case of the viscoelastic spheres. The procedure discussed in the introduction allows to formally derive the fluid dynamical equations in the regime of small inelasticity. Since Q is the classical elastic Boltzmann collision operator, from (2) we obtain that the solution is close to the Maxwellian equilibrium with the same mass, momentum and energy of the initial datum provided ϵ is sufficiently small. Inserting this Maxwellian into the transport step (3), gives

$$\int_{I\!R^3} \psi(v)\left(\frac{\partial M}{\partial t} + v\cdot\nabla_x M - g(\rho)\frac{\beta}{\epsilon} I(M,M)(x,v,t)\right) dv = 0, \tag{24}$$

for any test function ψ. It is well-known that system (24) for the moments of f, which is in general not closed, is closed by assuming f to be a locally Maxwellian function like (22) [CIP94]. Since the dissipative operator I is such that $\psi = 1, v$ are collisional invariants, choosing $\psi = 1, v, \frac{1}{2}v^2$ we obtain from (24) the dissipative Euler equations for density $\rho(x,t)$, bulk velocity $u(x,t)$ and temperature $T(x,t)$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) &= 0 \\
\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p &= 0 \\
\frac{\partial T}{\partial t} + (u\cdot\nabla)T + \frac{2}{3}T\mathrm{div}u &= -\frac{\beta}{\epsilon}C_p g(\rho)\rho T^{(3+p)/2}
\end{aligned} \tag{25}$$

where $p = \rho T$, and

$$C_p = \frac{\Gamma(2+p/2)2^{7+p}\sqrt{\pi}}{3(4+p)}, \tag{26}$$

This approximation is valid when both $\epsilon << 1$, $\beta << 1$ in such a way that $\beta/\epsilon = \lambda$. This is clearly a nearly elastic regime. If we assume this relationship between β and ϵ, and $p = 1/5$, we obtain the Euler system for a weakly dissipative system of viscoelastic spheres

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) &= 0 \\ \frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p &= 0 \\ \frac{\partial T}{\partial t} + (u \cdot \nabla) T + \frac{2}{3} T \mathrm{div} u &= -\lambda C_{1/5} g(\rho) \rho T^{8/5} \end{aligned} \tag{27}$$

Further applications of this idea lead to higher order hydrodynamic equations [BST04]. Now, the sought approximate closure for the collision term is achieved by replacing, in the evaluation of higher order moments, both in $Q(f,f)$ and $I(f,f)$ the actual distribution function f with the Grad distribution function [Gra49], which, in the spatially one–dimensional case, reads as

$$f_G(\mathbf{v}) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \,\mathrm{e}^{-\frac{c^2}{2T}} \left[1 + \frac{p}{2\rho T^2} \left(-\frac{1}{2} c^2 + \frac{3}{2} c_z^2 \right) + \frac{4}{5} \frac{q}{2\rho T^2} c_z \left(\frac{c^2}{2T} - \frac{5}{2} \right) \right], \tag{28}$$

and constitutes the weighted polynomial approximation to f sharing the same moments up to heat flux. Consistently with our hypothesis of small ϵ and small β, (28) represents a perturbation to a Maxwellian distribution, solution to the elastic problem in the hydrodynamic limit.

We remark that at present there are no rigorous results which justify the derivation of macroscopic equations in the case of dissipative collisions. As a matter of fact, in classical elastic kinetic theory, one of the main ingredients in the proof is the well–known Boltzmann H-theorem, which guarantees that, at any fixed positive time, the solution is close to the Maxwellian equilibrium provided the Knudsen number is small enough. In recent years, many efforts have been done to obtain explicit computable formulas which allow to quantify the space homogeneous time decay of the solution towards the Maxwellian in terms of the time decay of the relative entropy

$$H(f|M) = \int_{\mathbb{R}^3} f(v) \log \frac{f(v)}{M(v)}\, dv, \tag{29}$$

where M is the Maxwellian function with the same constant mass ρ, drift velocity u and temperature T of f. In particular, lower bounds on the entropy production

$$-D(f) = \int_{\mathbb{R}^3} \log f(v) Q(f,f)(v) dv$$

in terms of the relative entropy have been obtained in [TV99a]

The main problem here is the lack of H-theorem for the dissipative Boltzmann equation. A semi–formal discussion on the behavior of the Boltzmann entropy in this case can be found in [BST04]. We only recall that, since the granular gas is cooling, the correct relative entropy in this case should be

$$H(f|M)(t) = \int_{I\!R^3} f(v,t)\log f(v,t)\,d\mathbf{v} - \rho\,\log\rho + \frac{3}{2}\,\rho\log\Big[2\pi \mathrm{e}\,T(t)\Big], \qquad (30)$$

with $H(f|M) \geq 0$.

4 Hydrodynamics from homogeneous cooling states

As described in the previous Section, the macroscopic description of a rapid granular flow in a weakly inelastic regime by means of the closure with a local Maxwellian is easy to handle, and reasonable in many physical situations. The alternative to this approach relies in closing the equations with respect to the homogeneous cooling state, namely an exact solution to the dissipative Boltzmann equation. In principle, this alternative way of closure requires a knowledge of its stability which in many cases is far from being understood.

In many fields of evolution equations, exact solutions of self–similar type play an important role as attractors of wide classes of other solutions. Among others, this is the case of nonlinear diffusions, where Barenblatt type self–similar solutions have been recognized as intermediate solutions for the large–time behavior of porous medium equations [Vaz83, Vaz03]. Unlikely, this is not the case in kinetic theory of rarefied gases, where various examples show that self–similar solutions are not always stable with respect to a reasonably large class of physical data. The first example is furnished by the well–known Bobylev–Krook–Wu mode of the Maxwell–Boltzmann equation [Bob88, KW76], a special self–similar solution of the elastic Boltzmann equation for Maxwellian molecules. In fact, even if has been conjectured that the BKW–mode attracts solutions of the Boltzmann equation for a large class of initial data, before these solutions reach the Maxwellian equilibrium, this conjecture has never been verified. More recently, Caglioti and Villani [CV02] gave a precise mathematical proof of the very weak stability properties of the self–similar solution of the nonlinear friction equation introduced by McNamara and Young [NY93, BCP97] as one–dimensional model for the cooling of a granular gas. After their analysis, it is obvious to questioning about the role of homogeneous cooling states, at least in the case of too simplified dissipative models.

As far as the hydrodynamics closure is concerned, the first step in the validation of the closure around the homogeneous cooling state requires a proof of the intermediate asymptotic property of the cooling state itself. In other words, one has to be sure that, (at least in homogeneous situatons) if $f_\infty(v,t)$ denotes the homogeneous cooling state of equation (2), the solution to this equation is such that

$$f(v,t/\epsilon) = f_\infty(v,t/\epsilon) + O(\epsilon^2). \qquad (31)$$

Relation (31) insures that the solution to the dissipative Boltzmann equation reaches the self–similar profile before reaching the asymptotic state which has the form of a Dirac delta. A proof of this result is difficult due to the fact that there is no H–theorem, while this theorem is at the basis of similar results in the nonlinear diffusion case [CT00, CJMTU01].

The program has been realized in the case of solutions of the homogeneous Boltzmann equation for the inelastic Maxwell molecules introduced in Ref. [BCG00], where in (1)

$$C(f,f) = B\sqrt{\theta(t)}\tilde{Q}(f,f)\,. \tag{32}$$

Here, $\tilde{Q}(f,f)$ is the inelastic Boltzmann collision operator with Maxwellian molecules,

$$(\varphi, \tilde{Q}(f,f)) = \frac{1}{4\pi}\int_{I\!R^3}\int_{I\!R^3}\int_{S^2} f(v)f(w)\Big[\varphi(v') - \varphi(v)\Big]dv\,dw\,dn. \tag{33}$$

In expression (32), the factors B and the temperature of f in front of Q,

$$\theta(t) = \frac{1}{3}\int_{I\!R^3} |v|^2 f(v,t)\,dv,$$

allow the Maxwell model to have the same loss of temperature law of the inelastic hard-spheres model (11).

In (33) the outgoing velocities assumed by a particle in the collision defined by the ingoing velocities v, w and the angular parameter $n \in S^2$:

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|n, \tag{34}$$

$$w' = \frac{1}{2}(v+w) - \frac{1-e}{4}(v-w) - \frac{1+e}{4}|v-w|n\,. \tag{35}$$

The restitution coefficient e is here assumed to be constant.

Inelastic Maxwell models are of interest for granular fluids in spatially homogeneous states because of the mathematical simplifications resulting from their energy–independent collision rate. For this reason, after its introduction in [BCG00], equation (32) has been widely studied with or without energy supplies.

Among others, one of the interesting features of granular flows, which can be observed in the framework of Maxwellian molecules, is the knowledge of many of the properties of self-similar solutions in the homogeneous cooling problem, and the non-Maxwellian behavior of these solutions, which display power-like decay for large velocities. Inelastic Maxwell models allow one to take advantage of the powerful Fourier transform methods. Using these techniques for the self–similar scaling problem, existence of solutions with power–like tails has been proven by several authors [BMP02, BK00, BK02, EB02a]. A systematic approach to the existence of self-similar profiles for both elastic and inelastic interactions was subsequently proposed by Bobylev and Cercignani in [BC03], who obtained also results of convergence towards the self-similar solution. Later on, these results have been improved in [BCT03], by showing that convergence towards the self–similar profile occurs for all solutions corresponding to initial data which have more that two moments bounded. In both papers, however, no rate of convergence was found. We remark here that, in view of the passage to hydrodynamics, the knowledge of the rate of convergence is of paramount importance to prove conditions of type (31)

Concerning the problem of the Boltzmann equation with an energy source, Bobylev and Cercignani [BC02a] found steady solutions to the inelastic Maxwell model with a heat bath, that behave like $\exp(-r|v|)$. The problem of convergence towards the steady solution has been subsequently dealt with in [BCT05a]. By means of the contraction property of a suitable metric in the set of probability measures, existence, uniqueness, boundedness of moments and regularity of the steady state have been derived. Furthermore, explicit decay rates of general solutions towards the stationary state were obtained.

Using ideas from [BCT05a] in [BCT05b] various problems related to the convergence towards the self–similar profile were solved, reckoning precise rates of convergence in terms of suitable metrics defined in terms of the Fourier transform. Contraction properties of these metrics allow to show existence and uniqueness of the similarity solution, as well as various properties of the solution itself. In particular, it has been possible to discuss in detail the conjecture on the self–similar solution formulated by Ernst and Brito in [EB02a, EB02b].

A crucial role in the analysis of [BCT05b] is played by the weak norm convergence, which is obtained by further pushing the development of a method first used in [GTW95] to control the exponential convergence for Maxwellian molecules in certain weak norms.

Easy computations show that $(\varphi(v), Q(f, f)) = 0$ whenever $\varphi(v) = 1, v$, while $(\varphi(v), Q(f, f)) < 0$ if $\varphi(v) = v^2$. This corresponds to conservation of mass and momentum, and, respectively, to loss of energy for the solution to equation (32). For this reason, if we fix the initial data to be a centered probability density function, the solution will remain centered at any subsequent time $t > 0$.

Let $\mathcal{P}_s(I\!R^3)$ be the set of probability measures with bounded s-moment. The Fourier-based metrics d_s, for any $s > 0$, are defined as

$$d_s(\hat{f}, \hat{g}) = \sup_{k \in I\!R^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s}$$

for any pair of probability measures in $\mathcal{P}_s(I\!R^3)$. As usual, $\hat{f}$ is the Fourier transform of the density $f(v)$,

$$\hat{f}(k, t) = \int_{I\!R^3} f(v, t) \mathrm{e}^{-iv\cdot k}\, dv.$$

By simple Taylor expansion, one shows that the distance is well-defined and finite for any pair of probability measures with equal moments up to order $[s]$, where $[s]$ denotes the integer part of s. Moreover, in case $s \geq 1$ be an integer, it suffices equality of moments up to order $s-1$ for being d_s finite. In fact, d_s with $s \geq 2$ topology is equivalent to the weak-star topology for measures plus convergence of moments up to order $[s]$ [TV99b], and can be related to the Wasserstein distance between probability measures. This distance can be considered as a Lyapunov functional in this case.

To fix ideas, let us set $\epsilon = 1$ in the Boltzmann equation (2), with the collision operator satisfying (32). The spatially homogeneous dissipative Boltzmann equation then reads

$$\frac{\partial f}{\partial t} = B\sqrt{\theta(t)}\tilde{Q}(f,f)\,. \tag{36}$$

Self–similar solutions of equation (32) are obtained through a suitable scaling of both time and velocity in such a way that energy of the solution is conserved. If

$$f(v,t) = \theta^{-\frac{3}{2}}(\tau)\, g(v\,\theta^{-\frac{1}{2}}(\tau),\tau),$$

where

$$\tau = \frac{B}{E}\int_0^t \theta^{\frac{1}{2}}(w)\,dw, \tag{37}$$

and $E = 8/(1-e^2)$, g satisfies the equation

$$\frac{\partial g}{\partial \tau} = -\nabla_v \cdot \Big(v\,g(v)\Big) + EQ(g,g). \tag{38}$$

This equation now may admit a nontrivial steady state, because the energy dissipative effects of the Boltzmann collision operator are balanced by the energy input coming from the term $-\nabla_v \cdot \Big(v\,g(v)\Big)$. Self–similar solutions of the original Boltzmann equation correspond to stationary solutions g_∞ of equation (38).

Let us consider the pressure tensor for the solutions $f(v,\tau)$ of equation (36), namely, for $i \neq j$ the quantity

$$p_{ij}(\tau) = \int_{I\!R^3} v_i v_j f(v,\tau)\,dv.$$

If $\hat{\Phi}(k,\tau)$ is defined as

$$\hat{\Phi}(k,\tau) = \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{i\neq j} p_{ij}(\tau)k_i k_j & if \quad |k| \leq 1 \\ 0 & if \quad |k| > 1 \end{cases} \tag{39}$$

the following theorem holds:

Theorem 1. [BCT05b] *Let $g(v,t)$ denote a solution to (38) corresponding to the initial value $g(0)$ with unit mass and zero mean velocity, then the following exponential decay towards the corresponding steady state g_∞ holds:*

$$\begin{aligned} d_2(\hat{g}(\tau), \hat{g}_\infty) \leq \frac{C_{2\,,2+\alpha}}{\theta_0} & \left[2\, d_{2+\alpha}(\hat{g}(0) - \hat{\Phi}(0), \hat{g}_\infty) + C_1\right]^{2/(2+\alpha)} \\ \times \exp\left\{ - \frac{2}{2+\alpha}\, C(\alpha, e)\, \tau\right\} + & \ \frac{C_2}{\theta_0} \exp\left\{ - \frac{1+e}{1-e}\, \tau\right\}. \end{aligned} \tag{40}$$

The previous theorem gives explicit rates of exponential convergence to the steady state of the d_s-distance. The rate of convergence towards the self–similar solution (homogeneous cooling state) is easily derived from the previous theorem by coming back to the original time variable t. The evolution equation for the temperature yields

$$\theta(t) = \left\{ \theta_0^{-\frac{1}{2}} + \frac{1-e^2}{8} Bt \right\}^{-2},$$

hence time scaling (37) is nothing but

$$\tau = \log\left[1 + \frac{B}{E\theta_0^{-\frac{1}{2}}}\ t\right].$$

Therefore, to any exponential decay in the variable τ there corresponds an algebraic decay in t. From Theorem 1 we get the following estimate for the convergence of each solution $f(v,t)$ towards the homogeneous cooling state $f_\infty(t)$:

$$\begin{aligned} d_2(\hat{f}(t), \hat{f}_\infty(t)) \leq C_{2\,,2+\alpha} & \left[2\, d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_\infty(0)) + C_1\right]^{2/(2+\alpha)} \\ \times \left[1 + \frac{B}{E\theta_0^{-\frac{1}{2}}}\ t\right]^{-(2(1-A(\alpha,e))E)/(2+\alpha)} & + \ C_2 \left[1 + \frac{B}{E\theta_0^{-\frac{1}{2}}}\ t\right]^{-\,(3-e)/(1-e)}. \end{aligned}$$

The previous estimate guarantees that the homogeneous cooling state for the dissipative Boltzmann equation for Maxwell molecules attracts all solutions with finite moments of order $2+\delta$, so that it is highly reasonable to use this cooling state to close fluid dynamics equations.

Analogous results for the dissipative Boltzmann equation with rigid–spheres kernel are not available. Recent mathematical studies [MM05a, MM05b] however, proved existence (without uniqueness) and various properties of the cooling state in this case.

5 Conclusions

In this note, we briefly discussed the passage to hydrodynamics for rapid granular flows, described at a kinetic level in terms of the dissipative Boltzmann equation. This problem requires new hints on the mathematical tools necessary to justify rigorously the target macroscopic equations. Two possibilities are discussed. The first is linked to the presence of low inelasticity in the system, which allows to use locally Maxwellian functions to close macroscopic equations. We are confident in this case the passage can be justified by using methods close to thats applicable to the elastic Boltzmann equation. The second requires small spatial variations, and makes use of the homogeneous cooling state to close macroscopic equations. The first step in the justification of this procedure requires a proof of stability of the cooling state with respect to a large class of physical data. Recent mathematical results for the simplified model of the Boltzmann equation for Maxwell molecules then show that this stability property holds with respect to a distance equivalent to the weak* convergence of measures [BCT03]. The general situation is still unclear, and a result analogous to that for Maxwell molecules looks completely hopeless for the moment.

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