# Fluid dynamic description of flocking via Povzner–Boltzmann equation

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**Abstract**

We introduce and discuss the possible dynamics of groups of indistinguishable agents, which are interacting according to their relative positions, with the aim of deriving hydrodynamic equations. These models are developed to mimic the collective motion of groups of species such as bird flocks, fish schools, herds of quadrupeds or bacteria colonies. Our starting model for these interactions is the Povzner equation [24], which describes a dilute gas in which binary collisions of elastic spheres depend of their relative positions. Following the Cucker and Smale model [10], we will consider binary interactions between agents that are dissipative collisions in which the coefficient of restitution depends on their relative distance. Under the assumption of weak dissipation, it is shown that the Povzner equation is modified through a correction in the form of a nonlinear friction type operator. Using this correction we formally obtain from the Povzner equation in a direct way a fluid dynamic description of a system of weakly interacting agents interacting in a dissipative way, with a coefficient of restitution that depends on their relative distance.

**Key words.** Swarming, Povzner equation, dissipative collisions.

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# 1 Introduction

The aim of this paper is to discuss some questions connected with the modeling of the evolution of groups of agents, whose interactions depend on their relative positions. In

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the pertinent literature, these models have been developed to mimic the collective motion of groups of species such as such as bird flocks, fish schools, herds of quadrupeds or bacteria colonies [5, 13, 16, 22, 23, 28], that are supposed to obey the same interaction rules, without the presence of a leader. Once initialized with a certain velocity distribution, these groups develop particular profiles with time, like flocking of birds, in which all birds fly with the same velocity close each other. Among other approaches, these groups can be described at a mesoscopic level using concepts borrowed from classical statistical mechanics, in particular, the methods of the kinetic theory of rarefied gases [17, 7]. In discrete models of flocking [10, 11] it clearly appears that the formation of flocking structures strongly depends of the loss of kinetic energy. This property remains true at a continuous level, where the loss of kinetic energy is used to describe the long time behavior of the kinetic equation [17, 7]. At a kinetic level, this establishes a link between the collective motion of groups of species and granular gases. In granular gases, the dissipation of kinetic energy causes in fact a number of non-trivial properties, such as the formation of clusters and other spatial structures [15, 14, 20], non-Maxwellian velocity distribution and anomalous diffusion.

While the use of the Boltzmann equation to obtain descriptions of the dynamics of granular gases has proven successful, one must remember that deterministic numerical simulations of the Boltzmann equations are prohibitively expensive. One may obtain approximate solutions by the Direct Simulation Monte Carlo method, but even here the computational cost is very high. In most applications, in fact, rapid granular flows are described at the macroscopic level by means of equations for fluid dynamics, modified to account for dissipation due to collisions among particles. In some applications use has been made of phenomenological equations of motion that account for particle dissipation in granular matter, without resorting to the description afforded by kinetic theory. Haff's mean field approach [18], which follows the pioneering Bagnold's work [2, 3], can serve as an example of such a description. In recent years, kinetic theoretical approaches, based on the inelastic Boltzmann or Enskog Boltzmann equations have been developed [25, 6], both providing hydrodynamic descriptions, the latter even to Burnett order.

The hydrodynamic description of a granular gas is well–understood in presence of weak dissipation [27]. Weakly dissipative granular gases with variable restitution coefficient, can be described by introducing a correction to the classical Boltzmann collision operator. This correction is represented by a nonlinear friction type operator, with a kernel which depends of the variable restitution coefficient. This representation allows one to obtain a formal derivation of hydrodynamic equations for weakly dissipative granular gases, by carrying out the classical Maxwellian closure of the conservation equations

We will adopt the same strategy here. The main difference between granular gas molecules and groups of agents is that in the former case, molecules typically interact within the distance of a diameter, while in the latter case, agents interact at any distance (long–range interactions). In all the other aspects, collisions are very similar. In a granular gas binary collision, mass and momentum

are conserved, while the energy is dissipated (the dissipation depending of the relative velocity). In a binary interaction of birds, following the well-known interaction model proposed by Cucker and Smale [10, 11], mass and momentum are conserved, while the energy is dissipated (the dissipation depending of the relative positions). More precise hypotheses will be made in the next section, where previous models will be checked in some detail.

In the physical space $\mathbb{R}^3$ the connection between solutions of the Euler equations for compressible fluids, and the solutions of an equation describing the dynamics of a system of particles undergoing elastic collisions at a stochastic distance has been elaborated some years ago by Lachowicz and Pulvirenti [19]. There, it was discovered that the underlying kinetic equation able to represent the one-particle dynamics as the number of particles tends to infinity is the Povzner equation [24], which was originally introduced by Povzner for purely mathematical reasons, considering an averaging process for the pair collisions. In Povzner equation, in fact, the unit vector which determines the postcollisional velocities depends of the relative positions of particles.

We will briefly present the result of [19] in Section 2, where also Povzner equation will be described in some detail. Corrections to the Povzner collision operator which take into account dissipation will be the main contribution concerning the flocking modeling. The passage to fluid dynamics will be described in Section 3, where the corrections to the classical Euler equations derived from the dissipative correction to the Povzner equation will be dealt with. A description of the possible steady states of these Euler equations, and their connection with the flocking phenomenon will conclude our analysis. Some numerical simulations in the one-dimensional case will be presented in Section 4.

# 2 The flocking dynamics of a bird population

## 2.1 The dynamics of a stochastic particle system

In [19] Lachowicz and Pulvirenti established an interesting connection between solutions of the Euler equations for compressible fluids, and the solutions of an equation describing the dynamics of a system of particles undergoing elastic collisions at random distances. More precisely, they consider density, velocity and temperature fields $\rho(x,t)$, $u(x,t)$ and $T(x,t)$ which constitute a (smooth) solution of the system of Euler equations (up to some time $t_0$ before the appearance of the first singularity), and construct a local Maxwellian function $\mathscr{M}$ whose mean density, velocity and temperature are given by $\rho$, $u$ and $T$, respectively, [9],

$$\mathscr{M}(x,v,t) = \frac{\rho(x,t)}{(2\pi T(x,t))^{3/2}} \exp\left(-\frac{(v-u(x,t))^2}{2T(x,t)}\right). \tag{1}$$

They also consider a system of $N$ particles located at the points $x_1, x_2, \ldots, x_N$ on a domain of $\mathbb{R}^3$, which move freely unless a pair of them undergo an elastic collision, expressed by the formula

$$v_i' = v_i - ((v_i - v_j) \cdot n_{ij}) n_{ij}\,, \quad v_j' = v_j + ((v_i - v_j) \cdot n_{ij}) n_{ij}. \tag{2}$$

where the unit vector $n_{ij}$ is given by

$$n_{ij} = \frac{x_i - x_j}{|x_i - x_j|}. \tag{3}$$

As usual, $v_i'$ and $v_j'$ denote the outgoing velocities, where the incoming velocities are given by $v_i$ and $v_j$, provided that $(v_i - v_j) \cdot n_{ij} < 0$. Each binary collision takes place according to a stochastic law. The collision times for each pair $i$ and $j$ of particles are independent Poisson processes with intensity given by $\varphi(x_i, x_j, v_i, v_j)|(v_i - v_j) \cdot n_{ij}|$, and $\varphi$ is given by

$$\varphi(x_i, x_j, v_i, v_j) = \frac{3}{N\delta^3}\frac{1}{\epsilon}\chi(|x_i - x_j| \le \delta)\chi(|v_i - v_j| \le \theta), \tag{4}$$

where $\epsilon$ is a measure of the mean free path and $\chi(I)$ is the characteristic function of the set $I$.

The evolution of the system of particles is described by the $N$-particle distribution function $f^N(x_1, v_1 \ldots, x_N, v_N, t)$ which gives the probability density for finding the $N$ particles in the points $x_1, \cdots, x_N$ with velocities $v_1, \cdots, v_N$ at time $t \ge 0$. Let the $s$-particle distribution functions be defined by the marginals

$$f^{N,s}(x_1, v_1 \ldots, x_s, v_s) = \int f^N(x_1, v_1 \ldots, x_N, v_N) dx_{s+1}\, dv_{s+1} \cdots dx_N\, dv_N.$$

Then, under some additional hypotheses on the regularity of the solutions to the Euler equations in the time interval $[0, t_0]$, it is proven in [19] that for all $\sigma > 0$ there exist $\epsilon_0(\sigma)$, $\delta_0(\sigma, \epsilon)$, $\theta_0(\sigma, \epsilon, \delta)$ and $N_0(\sigma, \epsilon, \delta, \theta)$ such that if $\epsilon \le \epsilon_0$, $\delta \le \delta_0$, $\theta \ge \theta_0$ and $N \ge N_0$,

$$\sup_{t \in [0,t_0]} \|\mathscr{M} - f^{N,1}\| < \sigma,$$

where $f^{N,1}$ is the 1-particle marginal corresponding to the $N$-particle distribution function $f^N(x_1, v_1 \ldots, x_N, v_N, t)$, with initial conditions

$$f^{N,s}(x_1, v_1 \ldots, x_s, v_s, t = 0) = \prod_{j=1}^{s} \mathscr{M}(0; x_j, v_j).$$

The analysis of [19] shows that, as the number of particles tends to infinity, the 1-particle marginal $f^{N,1} = f$ satisfies the (elastic) Povzner equation [24]. This kinetic equation was introduced by Povzner in 1962 through a modification of

the Boltzmann collision operator consisting in a averaging process for the pair collisions. The modified Povzner collision operator looks as follows

$$Q_P(f,f)(x,v) = \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw B(x-y, v-w) \left( f(x,v_*) f(y,w_*) - f(x,v) f(y,w) \right). \tag{5}$$

In (5) $B$ is a smooth collision kernel, while $(v_*, w_*)$ are the pre-collision velocities of the so–called inverse collision, which generate the pair $(v,w)$. The relationship between the pair $(v,w)$ and the post-collision velocities $(v^*, w^*)$ is expressed by

$$v^* = (I - A(x-y))v + A(x-y)w, \tag{6a}$$
$$w^* = A(x-y)v + (I - A(x-y))w, \tag{6b}$$

where $A$ is a $3 \times 3$ matrix and I the identity matrix. In Povzner equation the matrix $A$ is such that momentum and energy are preserved in the collision. The conservation of energy $(v^*)^2 + (w^*)^2 = v^2 + w^2$ yields straightforwardly

$$2(A-I)A(v-w) = 0, \quad \text{for all } v, w \in \mathbb{R}^3,$$

or

$$A^2 = A.$$

In what follows we choose the matrix $A$ to be rank of rank 1 and expressed with respect to the unit vector $n = n(x-y) = \frac{x-y}{|x-y|}$ as

$$A(x-y) = n \cdot n^\top.$$

Consequently,

$$v^* = v - (v-w) \cdot n(x-y) n(x-y), \qquad w^* = w + (v-w) \cdot n(x-y) n(x-y). \tag{7}$$

Note that the main difference between the classical Boltzmann equation and the Povzner one is that the positions of the particles enter into the interaction rule, so that the collision is non-local. From a mathematical point of view, Povzner equation was introduced to overcome one of the main difficulties in solving the classical Boltzmann equation, where binary collisions between particles happen at the same point $x = y$. The non-local character of the interaction, which is evident from the presence of products of type $f(v,x)f(w,y)$ into the collision operator (5), is such that, while the binary collision is elastic, and so preserves both momentum and energy, in Povzner equation only the mass is a collision invariant, while, as it happens in the Enskog equation [12], the momentum and the energy are not collision invariants. Maybe for this reason, Povzner equation has been usually ignored by the physicists, while only few mathematical papers deal with it [1, 8, 19].

Going back to the result of [19], it was proven that, in presence of a stochastic law for binary interactions, if the particles are initially identically and independently distributed according to a distribution density $F = F(x, v)$, then at later times they are identically and independently distributed according to a solution of a kinetic equation with collision integral given by the Povzner operator (5), with initial datum $f_o = F$, and rate function

$$B(x - y, v - w) = \frac{1}{2\delta^3}\frac{1}{\epsilon}\chi(|x - y| < \delta)\chi(|v - w| < \theta)\,|(v - w) \cdot n|\,,$$

where $\delta$ and $\theta$ are positive constants, and $\chi(E)$ denotes the characteristic function of the set $E \subseteq \mathbb{R}^3$. Note that, at least formally, the Boltzmann equation for the hard sphere model is obtained from the Povzner equation for $\theta = +\infty$ by making $\delta$ tend to zero

$$Q_P(f, f)(x, v) \to Q_B(f, f)(x, v),$$

where

$$Q_B(f, f)(x, v) = \frac{1}{2\epsilon}\int_{\mathbb{R}^3} dw \int_{S^2} dn|(v - w) \cdot n|\,(f(x, v_*)f(x, w_*) - f(x, v)f(x, w))\,. \tag{8}$$

## 2.2 A dissipative correction

In [10, 11], Cucker and Smale proposed a model to describe a population of $N$ agents, which, while interacting according to their relative positions, are developing particular profiles with time, like flocking of birds, in which all birds fly with the same velocity close each other. In [10], the main hypothesis which justifies the long–time behavior of the population is that every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds. That is, given a population of $k$ birds, at time $t \in I\!N$, and for $i$-th bird,

$$v_i(t + 1) - v_i(t) = \sum_{j=1}^{k} a_{ij}\,(v_j(t) - v_i(t))\,. \tag{9}$$

where the weights $a_{ij}$ quantify the way the birds influence each other. In [10] it is assumed that this influence is a function of the distance between birds, namely

$$a_{ij} = \frac{K}{(\lambda + |x_i - x_j|^2)^\beta} \tag{10}$$

for some fixed $K$, $\lambda > 0$ and $\beta \geq 0$.

Condition (9) can be conveniently rephrased in a different way [7]. Suppose that we have a population comprising two birds, say $i$ and $j$. Then, if their

velocities are modified according to (9),

$$v_i(t+1) = (1-a_{ij})v_i(t) + a_{ij}v_j(t), \tag{11a}$$
$$v_j(t+1) = a_{ij}v_i(t) + (1-a_{ij})v_j(t), \tag{11b}$$

the momentum is preserved after the interaction, so that

$$v_i(t+1) + v_j(t+1) = v_i(t) + v_j(t),$$

but the energy increases or decreases according to the value of $a_{ij}$

$$v_i^2(t+1) + v_j^2(t+1) = v_i^2(t) + v_j^2(t) - 2a_{ij}\,(1-a_{ij})\,(v_1 - v_j)^2. \tag{12}$$

If $a_{ij} < 1$, the energy is dissipated. Note that in this case the relative velocity decreases, since

$$|v_i(t+1) - v_j(t+1)| = |1-2a_{ij}||v_i(t) - v_j(t)| < |v_i(t) - v_j(t)|, \tag{13}$$

and the velocities of the two birds tend towards the mean velocity $(v_i + v_j)/2$.

In the general case of a population of $k$ birds, the binary law (11) is taken into account together with the assumption that the $i$-th bird modifies its velocity giving the same weight to all the other velocities. In consequence of this,

$$v_i(t+1) = \frac{1}{k}\sum_{j=1}^{k}\left\{(1-a_{ij})v_i(t) + a_{ij}v_j(t)\right\}, \tag{14}$$

that is a different way to write formula (9).

The stochastic dynamics of particles introduced in [19] is perfectly adaptable to the context of a population of species. Substituting birds for particles, and changing in (4) the interaction intensity $\varphi$, accordingly, allows to obtain a reasonable model to describe the time-space evolution of a population of birds, and, at the same time it establishes an interesting connection with the fluid dynamic picture (Euler equations), in the presence of a large population.

To adapt the particle system by Lachowicz and Pulvirenti to the present context, we need however to introduce various modifications. The first one is related to the fact that, since the flocking phenomena are heavily dependent on dissipation, the elastic interaction considered in [19] has to be suitably modified to account for dissipation.

Second, the interaction rules (11) considered according to the Cucker–Smale model, while dissipative and able to reproduce the flocking phenomenon, do not represent a dissipative correction of an elastic collision of type (2). In fact, the limit when $a_{ij}$ are zero in (11) does not produce an elastic collision, but a collision in which birds simply do not exchange their velocities. This is related to the fact that the interaction rule given by (11) is such that the relative position influences

the interactions only through the modulus of the distance, while the angles formed by the relative position and the respective velocities do not play any role.

To overcome this unpleasant effect, a different more sophisticated interaction can be considered, such that agents tend to dissipate their relative velocity along the relative direction. This mathematical constraint agrees with the reasonable assumption that birds which are approaching tend to diminish their relative velocity along their relative positions, and the same happens in the opposite situation where they are going away.

We assume that the microscopic dynamics of two agents $(x, v)$ and $(y, w)$ is governed by the interaction coefficient $0 < e(|x-y|) < 1$ which relates the components of the agents velocities along before and after an interaction. The post interaction velocities $(v^*, w^*)$ are such that

$$(v^* - w^*) \cdot \frac{x-y}{|x-y|} = -e(|x-y|)\,(v-w) \cdot \frac{x-y}{|x-y|}\,. \tag{15}$$

Thanks to (15), and assuming the conservation of momentum, one finds the change of velocity for the interacting agents as

$$v^* = v - \frac{1}{2}(1+e)((v-w)\cdot n)n\,, \quad w^* = w + \frac{1}{2}(1+e)((v-w)\cdot n)n, \tag{16}$$

where $n = n(x-y) = (x-y)/|x-y|$. A Povzner–type (conservative) interaction, that coincides with (2) is obtained for $e = 1$. For dissipative interactions $e$ decreases with increasing degree of dissipation.

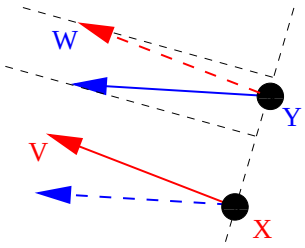


Figure 1: Illustration of the Cucker-Smale-Povzner-type interaction rule, where a particle located in $x$ with velocity $v$ averages its velocity with a particle located in $y$ with velocity $w$, according to (16).

The choice

$$e(|x-y|) = 1 - \gamma a(|x-y|), \tag{17}$$

where $a(|x-y|)$ is given like in (10) by

$$a(|x-y|) = \frac{K}{(\lambda + |x-y|^2)^\beta}, \tag{18}$$

is consistent with the Cucker–Smale approach. Note that the constant $\gamma$ has to be chosen so that $e(|x-y|) < 1$.

## 2.3 Boltzmann–like model for dissipative interactions

According to the rule (16), the one-particle distribution function $f(x, v, t)$ considered by Lachowicz and Pulvirenti [19], in presence of a large population ($N \to \infty$) satisfies a Povzner–type equation for dissipative interactions, which reads [4]

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \bar{Q}_P(f, f)(x, v, t), \tag{19}$$

where $\bar{Q}_P$ is now a dissipative collision operator, which describes the change in the density function due to creation and annihilation of agents velocities in binary interactions

$$\bar{Q}_P(f, f)(x, v) =$$

$$\frac{1}{\epsilon} \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw B \left( \Gamma(|x - y|) f(x, v_*) f(y, w_*) - f(x, v) f(y, w) \right). \tag{20}$$

Like in Povzner equation, in (20) the velocities $(v_*, w_*)$ are the pre-interaction velocities. The factor $\Gamma$ in the gain term appears from the Jacobian of the transformation $dv^* dw^*$ into $dvdw$. For a restitution coefficient which depends only on positions, like in (17), $\Gamma(|x - y|) = e(|x - y|)^{-1}$. Finally, $B(\tau)$ represents the collision rate function, which gives the probability that a collision between agents happen at a distance $\tau$. We recall that the model considered by Lachowicz and Pulvirenti [19] leads to $B = B(x - y, v - w) = \chi(|x - y| < \delta)/(\delta^3)|(v - w) \cdot n|$. We also stress that, differently from (5), we do not assume that the collision rate function depends on the term $|(v - w) \cdot n|$, which is not considered here.

To avoid the presence of the function $\Gamma$, and to study approximations to the Povzner operator (20) it is extremely convenient to write the operator (20) in weak form. More precisely, let us define with $\langle \cdot\,, \cdot \rangle$ the inner product in $L_1(\mathbb{R}^3)$. For all smooth functions $\psi(v)$, it holds

$$\begin{aligned} \langle \psi\,, \bar{Q}_P(f, f)(x, v) \rangle \; &= \int_{\mathbb{R}^3} \psi(v) \bar{Q}_P(f, f)(x, v)\, dv \\ = \frac{1}{\epsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & B(|x - y|) \left( \psi(v^*) - \psi(v) \right) f(x, v) f(y, w) dv\, dw\, dy \end{aligned}$$

Let $(v', w')$ be the post-interaction velocities in a Povzner elastic interaction with $(v, w)$ as incoming velocities. Denoting by $q$ the relative velocity, $q = v - w$,

$$v' = v - (q \cdot n)n\,, \quad w' = w + (q \cdot n)n. \tag{21}$$

As before, the unit vector is $n = n(x - y) = \frac{x-y}{|x-y|}$. Using (16) and (21) one obtains

$$v^* = v' + \frac{1}{2}(1 - e)(q \cdot n)n\,, \quad w^* = w' - \frac{1}{2}(1 - e)(q \cdot n)n. \tag{22}$$

If we assume that the coefficient of restitution has the form (17), then

$$v^* - v' = \gamma a(|x-y|)(q\cdot n)n. \tag{23}$$

Let us consider a Taylor expansion of $\psi(v^*)$ around $v'$. Thanks to (23) we get

$$\begin{aligned}&\psi(v^*) = \psi(v') + \gamma a(|x-y|)\nabla\psi(v')\cdot(q\cdot n)n + \\ &\frac{1}{2}\gamma^2 a(|x-y|)^2\sum_{i,j}\frac{\partial^2\psi(v')}{\partial v_i'\partial v_j'}(q\cdot n)^2 n_i n_j + \dots\end{aligned} \tag{24}$$

If the interactions are nearly elastic, so that $\gamma << 1$, we can truncate the expansion (24) after the first–order term. Inserting (24) into (21) gives

$$\begin{aligned}\langle\psi\,,\,\bar{Q}_P(f,f)\rangle \quad &\approx \quad \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|x-y|)\,(\psi(v')-\psi(v)\\ &\quad +\gamma\nabla\psi(v')\cdot a(|x-y|)(q\cdot n)n)\,f(x,v)f(y,w)dv\,dw\,dy\\ &\quad = \langle\psi\,,\,\mathscr{Q}_P(f,f)\rangle\;+\gamma\langle\psi\,,\,\mathscr{I}(f,f)\rangle\;.\end{aligned} \tag{25}$$

It is a simple matter to recognize that in (25) $\mathscr{Q}_P(f,f)$ is a Povzner collision operator of the type (5), since the post-interaction velocity $v'$ into (25) is obtained from the pre-interaction velocities $(v,w)$ through the elastic interaction (21).

Let us now study in more detail the second contribution to the inner product (25). First of all, let us set

$$b(|x-y|) = B(|x-y)|)a(|x-y|). \tag{26}$$

Using the properties of the transformation (21), we obtain

$$\begin{aligned}&\langle\psi\,,\,\mathscr{I}(f,f)\rangle\\ &= \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla\psi(v')\cdot n\,b(|x-y|)(q\cdot n)f(x,v,t)f(y,w,t)dv\,dw\,dy\\ &= -\frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla\psi(v)\cdot n\,b(|x-y|)(q\cdot n)f(x,v',t)f(y,w',t)dv\,dw\,dy\\ &= \frac{1}{\epsilon}\int_{\mathbb{R}^3} dv\,\psi(v)\mathrm{div}_v\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n\,b(|x-y|)(q\cdot n)f(x,v',t)f(y,w',t)dw\,dy\right)\end{aligned} \tag{27}$$

In fact, the transformation $dv\,dw$ into $dv'\,dw'$ given by (21) is such that $q'\cdot n = -q\cdot n$, while its Jacobian is equal to unity.
The last equality follows from the divergence theorem. The second contribution to the inner product (25) defines the dissipative interaction operator

$$\mathscr{I}(f,f)(x,v,t) = \frac{1}{\epsilon}\mathrm{div}_v\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n(q\cdot n)b\left(|x-y|\right)f(x,v',t)f(y,w',t)dw\,dy\right). \tag{28}$$

Finally, for nearly elastic interactions, with a restitution coefficient satisfying (17), the dissipative Povzner equation can be modeled at the leading order as

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(x, v, t) = \mathscr{Q}_P(f, f)(x, v, t) + \gamma \mathscr{I}(f, f)(x, v, t), \qquad (29)$$

where $\mathscr{Q}_P$ is inspired by the classical elastic Povzner collision operator, and $\mathscr{I}$ is a dissipative nonlinear friction operator which is based on elastic interactions between agents.

**Remark 2.1** The model kinetic equation obtained here as a first-order correction to an elastic Povzner–type equation is valid, at least formally, for weakly dissipative interactions, i.e., when the value of $\gamma$ is sufficiently small. Other moderately dissipative regimes would require the inclusion of higher order terms in the expansion (24). In particular, the second–order term in this expansion gives a diffusive correction to the Povzner equation. The analysis done in this section is close to the analogous one done for the Boltzmann equation for granular dissipative gases [27]. This last problem has been studied systematically both from a numerical and theoretical point of view (cfr. [27] and the references therein). Even if numerical computations for the full three–dimensional Boltzmann equation are still missing, in a closely related problem it has been shown numerically in [21] that for a one-dimensional (in the velocity space) dissipative Boltzmann equation the results relative to the full equation are in good agreement with those relative to the equation with a first–order correction, even in regimes of moderate inelasticity. The same analysis showed that higher–order corrections (of order bigger than two) introduce problems in the numerical spectral approximation, without essential improvements in the accuracy of the solution. This suggests that also in presence of agents interactions the first–order correction gives a reasonably good approximation to the dissipative Povzner equation.

**Remark 2.2** The dissipative correction (28) to the elastic Povzner equation appears very similar to the corresponding one obtained in [17] as the mean field limit of the Cucker–Smale model. Also, the Povzner–like model introduced in [7] is apparently close to the one considered in this paper. These models, which are based on the Cucker–Smale interaction (11), however are not consistent with the elastic description of the system introduced in [19]. In fact, the continuous-time version of the Cucker–Smale interaction (11),

$$v^* = (1 - \gamma a)v + \gamma a w, \qquad w^* = av + (1 - \gamma a)w,$$

leads to $v^* = v$ and $w^* = w$ as $\gamma \to 0$. Consequently, the limit $\gamma \to 0$ gives a dynamics in which all particles in the system move freely without interactions.

# 3 Passage to fluid dynamics

## 3.1 Exact computations

The main purpose of this short section is to study in some detail the main properties of the nonlinear friction operator $\mathscr{I}$. Choosing $\psi = 1$ in (27) shows that

$$\langle 1\,,\, \mathscr{I}(f,f)(x,t)\rangle$$

$$= \frac{1}{\epsilon}\int_{\mathbb{R}^d} \mathrm{div}_v \left( \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n(q\cdot n) b\left(|x-y|\right) f(x,v',t) f(y,w',t) dw\, dy \right) dv \;\; = 0, \tag{30}$$

so that the mass is a collision invariant. Choosing now $\psi = v_i$ in (27), and using the first equality, owing to the fact that $\nabla\psi(v')\cdot n = n_i$ we obtain

$$\langle v\,,\, \mathscr{I}(f,f)(x,t)\rangle \;\; = \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n\, b(|x-y|)(q\cdot n) f(x,v,t) f(y,w,t) dv\, dw\, dy$$

$$= \frac{1}{\epsilon}\int_{\mathbb{R}^3} n\, b(|x-y|)\left(u(x,t)-u(y,t)\right)\cdot n\, \rho(x,t)\rho(y,t) dy. \tag{31}$$

In (31), $\rho(x,t)$ and $u(x,t)$ denote as usual the local mass density and bulk velocity, that is

$$\rho(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\, dv, \qquad u(x,t) = \frac{1}{\rho(x,t)}\int_{\mathbb{R}^3} v f(x,v,t)\, dv. \tag{32}$$

Note that, in agreement with the analogous property of Povzner equation, the mean velocity is not a local collision invariant, while by symmetry

$$\int_{\mathbb{R}^3} \langle v\,,\, \mathscr{I}(f,f)(x,t)\rangle\, dx = 0. \tag{33}$$

Finally, taking $\psi = v^2/2$ into (27), and using the identity $v'\cdot n = w\cdot n$ we obtain

$$\begin{aligned}
& \langle v^2/2\,,\, \mathscr{I}(f,f)(x.t)\rangle \\
= \;& \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} w\cdot n\, b(|x-y|)((v-w)\cdot n) f(x,v,t) f(y,w,t) dv\, dw\, dy \\
= \;& \frac{1}{\epsilon}\int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t)\, n\cdot u(x,t)\, n\cdot u(y,t) dy \\
- \;& \frac{1}{\epsilon}\rho(x,t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b(|x-y|)|w\cdot n|^2 f(y,w,t)\, dw\, dy.
\end{aligned} \tag{34}$$

We remark that, while equality (31) is expressed in terms of macroscopic quantities, the same is not true for equality (34), where one can not in general express the last integral in terms of macroscopic quantities. This is possible at least when

the density $f(x,v,t)$ is isotropic in the velocity variable, $f(x,v,t) = f(x,|v|,t)$. In this case, in fact

$$\int_{\mathbb{R}^3} |v\cdot n|^2 f(x,v,t)\,dv = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f(x,v,t)\,dv,$$

and

$$\int_{\mathbb{R}^3} |v|^2 f(x,v,t)\,dv = \rho(x,t)|u(x,t)|^2 + 2\rho(x,t)e(x,t). \tag{35}$$

In (35), $e(x,t)$ is the internal energy per unit mass, given by

$$e(x,t) = \frac{3}{2}T(x,t) = \frac{1}{2\rho(x,t)}\int_{\mathbb{R}^3} |v-u(x,t)|^2 f(x,v,t)\,dv. \tag{36}$$

A second useful representation can be obtained, owing to the second line equality in (27),

$$\begin{aligned} &\langle v^2/2\,,\,\mathscr{I}(f,f)(x,t)\rangle \\ =\;& -\frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v\cdot n\, b(|x-y|)((v-w)\cdot n) f(x,v',t)f(y,w',t)dv\,dw\,dy \end{aligned} \tag{37}$$

In this case, since the change of variables $v \leftrightarrow w$, and, at the same time $x \leftrightarrow y$ gives $v' \leftrightarrow w'$ and $n \leftrightarrow -n$

$$\begin{aligned} &\int_{\mathbb{R}^3}\langle v^2/2\,,\,\mathscr{I}(f,f)(x,t)\rangle\,dx \\ =\;& -\frac{1}{2\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b(|x-y|)((v-w)\cdot n)^2 f(x,v',t)f(y,w',t)dv\,dw\,dx\,dy < 0. \end{aligned} \tag{38}$$

Hence, the nonlinear friction operator is globally dissipative.

To close this section, we remark that, since the Maxwellian function $\mathscr{M} = \mathscr{M}(x,v,t)$ defined in (1) is isotropic in the velocity variable, hence

$$\begin{aligned} &\int_{\mathbb{R}^3}\langle v^2/2\,,\,\mathscr{I}(\mathscr{M},\mathscr{M})(x,t)\rangle\,dx \\ =\;& \frac{1}{\epsilon}\int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t)\left[n\cdot u(x,t)\,n\cdot u(y,t) - \left(\frac{1}{3}|u(y,t)|^2 + T(y,t)\right)\right]dy. \end{aligned} \tag{39}$$

## 3.2 Hydrodynamic limit and the Euler equations.

The goal of this section is to derive, in some particular regimes, a fluid dynamic description of the population of agents. In analogy with the basic concepts of

kinetic theory of gases [9], a fluid dynamic description of the population requires that agents could be treated as point particles. If a population of small birds is considered, their typical dimension is of the order of few centimeters. Consequently, a reasonable unit length for studying the flocking phenomenon has to be at least of the order of $10^5$ centimeters. Also, it is evident that mean free path of the phenomenon (the average distance birds travel between interactions with other birds of the population) can be assumed of the same order of the typical dimension of birds, namely of order $10^{-5}$ with respect to the typical unit length. Within these assumptions, which result in choosing the parameters $\delta \cong \epsilon \ll 1$, the analysis of [19] follows. The consequent fluid dynamic description that can be obtained from the Povzner–type equation is able to describe the flocking phenomenon at least in presence of well behaved interaction functions. First, the collision rate $B(\tau)$ in the collision integral (5), which is here a long-range interaction rate function, has to take into account (for small values of the parameter $\delta$) mainly interactions which happen at a small distance. Second, the influence function $a(\tau)$ has to maintain all properties of the function (18), including the polynomial decay at infinity, independently of $\delta$.

Among others, a choice of interactions functions which satisfies all these requirements is the following

$$B(|x-y|) = B_\delta(|x-y|) = \frac{1}{\delta^3}\left(1+\frac{|x-y|^2}{\delta^2}\right)^{-(1+\nu)}, \tag{40}$$

where $\nu > 1/2$ , and

$$a(|x-y|) = a_\delta(|x-y|) = \frac{1}{\delta^{2\nu-1}}\frac{K\ (\delta^2+|x-y|^2)^{1+\nu}}{(\lambda+|x-y|^2)^{1+\beta}}. \tag{41}$$

with $\beta > \nu$. The previous choices are heavily justified from the modeling point of view. The interaction kernel $B_\delta$ in (40), that is integrable on $\mathbb{R}^3$, takes into account also long-range interactions between agents, even if the interactions are more frequent if the distance between agents is small, and this property improves as soon as $\delta \to 0$. This is clearly related to the fact that in a dense group of agents a single agent modifies its velocity mainly according to the nearest agents (in reality, the dimension of the agents allow to see only the nearest ones).
As far as the Cucker and Smale type influence function $a_\delta(|x-y|)$ is concerned, it decays at a rate $|x-y|^{2(\beta-\nu)}$ independently of $\delta$, while, for a fixed distance between agents, it increases as soon as $\delta$ decreases. This last property is related to the fact that, the energy dissipation in a single interaction has to increase in correspondence to a dense population to avoid possible impacts.
Note that, under these assumptions

$$b(|x-y|) = B_\delta(|x-y|)a_\delta(|x-y|) = \frac{K}{(\lambda+|x-y|^2)^{\beta+1}}. \tag{42}$$

is independent of $\delta$, while

$$B_\delta(|x-y|)a_\delta^2(|x-y|) \leq \frac{C}{\delta^{2\nu-1}}. \tag{43}$$

Since the constant $\gamma$ in (17) has to be chosen so that $e(|x-y|) < 1$, this condition implies that $\gamma \leq D\delta^{2\nu-1}$, where $D = \lambda^{1+\beta}/(K\delta^{2+2\nu})$. Consequently, the higher order term in the expansion (24), which behaves like $\gamma^2 B_\delta a_\delta^2$ remains uniformly small. In this case we can formally derive the fluid dynamics equations in the regime of small dissipative interactions.

Let us make use of the splitting method, very popular in the numerical approach to the Boltzmann equation. To solve the complete dissipative Boltzmann–Povzner equation (19), at each time step we consider sequentially the elastic Boltzmann–Povzner equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathscr{Q}_P(f,f)(x,v,t), \tag{44}$$

and the dissipative correction

$$\frac{\partial f}{\partial t} = \frac{\gamma}{\epsilon}\mathscr{I}(f,f)(x,v,t). \tag{45}$$

Thanks to the result of Lachowicz and Pulvirenti [19] we know that the solution to (44) is well approximated by a Maxwellian function $\mathscr{M}$ whose moments satisfy the Euler equations. Substituting this Maxwellian function in equation (45), the right–hand side can be evaluated exactly using (31) and (38). Hence, when $\gamma$ and $\epsilon$ are of the same order, so that $\gamma/\epsilon \to \lambda$, with $\lambda > 0$ constant, we obtain that, for large values of $N$ and small values of $\epsilon$ the stochastic system is well approximated by the following system of Euler equations for density $\rho(x,t)$, bulk velocity $u(x,t)$ and temperature $T(x,t)$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0, \\
\frac{\partial}{\partial t}(\rho u_i) + \operatorname{div}(\rho u u_i + \rho T e_i) &= \lambda \mathscr{A}_i(\rho,u)(x,t), \\
\frac{\partial}{\partial t}\left(\rho(\frac{2}{3}T + \frac{1}{2}u^2)\right) + \operatorname{div}\left(\rho u(\frac{1}{2}u^2 + \frac{5}{2}T)\right) &= \lambda \mathscr{B}(\rho,u,T)(x,t).
\end{aligned} \tag{46}$$

In (46) $e_i$ is the component of the unit vector $e$ in the $i$-th direction,

$$\mathscr{A}(\rho,u)(x,t) = \int_{\mathbb{R}^3} n\, b(|x-y|)\,(u(x,t)-u(y,t))\cdot n\,\rho(x,t)\rho(y,t)dy, \tag{47}$$

and

$$\mathscr{B}(\rho,u,T)(x,t) =$$

$$= \int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t)\left[n\cdot u(x,t)\, n\cdot u(y,t) - \left(\frac{1}{3}|u(y,t)|^2 + T(y,t)\right)\right] dy. \tag{48}$$

**Remark 3.1** In [17], starting from the kinetic model obtained as the mean field limit of the Cucker and Smale model, Ha and Tadmor tried to introduce a set of hydrodynamical equations, which are given in the form of a system of Euler-type equations, with correction terms which are very similar to the $\mathscr{A}$ and $\mathscr{B}$ terms considered here. Their equations, however, constitute a system for the thirteen moments, one obtains evaluating the evolution of mass, momentum and energy, so that the system is not closed.

**Remark 3.2** The procedure we used to recover the system of Euler equations (46) is based on several truncations and asymptotics of the true Povzner-type equation (19). While the final result is largely formal, nevertheless the main steps of this procedure can be justified. In particular, the truncation of Taylor's formula (24) after the first–order term does not affect Euler equations in all cases in which the coefficient $\gamma^2 B_\delta a_\delta^2 = o(\gamma)$ uniformly with respect to the position variable. In this case, in fact, $\gamma^2 B_\delta a_\delta^2/\epsilon = o(\gamma)/\epsilon \to 0$ if $\gamma/\epsilon \to \lambda$, and we lose the contribution of this term in the Euler system. Note that the choice of the interaction term $a(|x-y|)$ of the type considered by Cucker and Smale (decay of order $\beta$), implies a weaker interaction term at the level of Euler equations. In these equations the interaction term $b(|x-y|)$ decays at the order $\beta+1$.

**Remark 3.3** A different choice of the interaction parameters leads to different correction terms in the Euler equations. In particular, leaving the interaction function $a(|x-y|)$ unchanged, i.e., without any dependence on $\delta$ to compensate the vanishing of $B_\delta$ for $\delta \to 0$, implies that the correction terms $\mathscr{A}$ and $\mathscr{B}$ collapse into the analogous ones obtained for a granular gas [27]. In this case

$$\mathscr{A}_\delta(\rho,u)(x,t) = \int_{\mathbb{R}^3} n\, B_\delta(|x-y|)a(|x-y|)\,(u(x,t)-u(y,t))\cdot n\,\rho(x,t)\rho(y,t)dy \to 0 \tag{49}$$

as $\delta$ tends to 0, while

$$\mathscr{B}_\delta(\rho,u,T)(x,t) \to -\frac{4}{3}\rho^2(x,t)T(x,t). \tag{50}$$

## 3.3 Long-time solutions

The presence of the corrections to the classical Euler equations, allows to classify some possible steady solutions of system (46). Let $\Omega$ be a bounded subset of $\mathbb{R}^3$. Then, it is immediate to conclude that

$$\rho(x) = 0, \qquad \text{if } x \in \mathbb{R}^3 \setminus \Omega,$$

and

$$u(x) = \bar{u}, \qquad \rho(x) = \bar{\rho}(x) \qquad \text{if } x \in \Omega$$

imply $\mathscr{A}(\rho, u)(x) = 0$. Moreover, under the same hypotheses on $\rho$ and $u$,

$$\mathscr{B}(\rho, u, T) = 0 \qquad \text{if } x \in \mathbb{R}^3 \setminus \Omega,$$

and

$$\mathscr{B}(\rho, u, T) = -\bar{\rho}(x) \int_{\mathbb{R}^3} b(|x-y|)\bar{\rho}(y)T(y)\, dy \qquad \text{if } x \in \Omega.$$

Therefore, $\rho(x) = \bar{\rho}(x),\ u(x) = \bar{u},\ T(x) = 0$ in $\Omega$ implies $\mathscr{B}(\rho, u, T) = 0$. Then, the condition

$$\operatorname{div}(\bar{\rho}(x)\bar{u}) = 0 \qquad \text{if } x \in \Omega \tag{51}$$

is enough to guarantee that $\rho(x) = \bar{\rho}(x),\ u(x) = \bar{u},\ T(x) = 0$ in $\Omega$ and $\rho(x) = 0 \in \mathbb{R}^3 \setminus \Omega$ is a steady solution of the Euler equations. Note that condition (51) only implies that $\bar{\rho}(x)$ is constant along the direction of the constant mean velocity $\bar{u}$. Last, let us remark that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathscr{B}(\rho, u, T)(x,t)\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t) \left[n\cdot u(x,t)\, n\cdot u(y,t) - \left(\frac{1}{3}|u(y,t)|^2 + T(y,t)\right)\right] dy\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t) \left[n\cdot u(x,t)\, n\cdot u(y,t) - \left(\frac{1}{3}|u(x,t)|^2 + T(x,t)\right)\right] dy\, dx \\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t) \left[T(x,t) + T(y,t) + (n\cdot u(x,t) - n\cdot u(y,t))^2\right] dy\, dx.
\end{aligned} \tag{52}$$

Therefore, integrating with respect to the $x$-variable the third equation in (46) we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} \left(\rho(\frac{2}{3}T + \frac{1}{2}u^2)\right)(x,t)\, dx = \int_{\mathbb{R}^3} \mathscr{B}(\rho, u, T)(x,t)\, dx < 0. \tag{53}$$

Hence the functional

$$\mathcal{F}(t) = \int_{\mathbb{R}^3} \left(\rho(\frac{2}{3}T + \frac{1}{2}u^2)\right)(x,t)\, dx \tag{54}$$

is a Lyapunov (energy) functional for system (46). Moreover, the entropy production

$$\mathcal{E}(t) = -\int_{\mathbb{R}^3} \mathscr{B}(\rho, u, T)(x,t)\, dx > 0, \tag{55}$$

Considering now that $\mathcal{E}(t) = 0$ if and only if either both $T(x) = 0$ and $u(x) = \bar{u}$, or $\rho(x) = 0$, starting with a configuration which is compactly supported, the solution to system (46) would be forced both to remain compactly supported in space and to converge towards one of the steady states characterized above.

# 4 Numerical experiments

In this section we would like to illustrate the properties of the system (46) in a simple one dimensional situation. Actually, we were expecting that the contributions of the right-hand side operators $\mathscr{A}$ and $\mathscr{B}$ could produce a regularization and stabilization effect, especially for the expected dissipative properties of $\mathscr{B}$, but while developing the numerical experiments we realized that the presence of these additional terms in the Euler equations imposed us a much more conservative approach, with the need of very small time steps, resulting in a rather high computational cost. This tendency to instable behavior currently prevents us to consider 2D simulations which need *ad hoc* new methods for addressing the system (46), and are a matter of current investigation.

We considered the problem on a bounded interval $\Omega = [0, 1]$, by imposing additionally periodic boundary conditions. We used a ENO scheme for the approximation of fluxes in the space discretization and a third-order Runge-Kutta method for the time discretization, see [26] for an introduction. Interestingly, the results seem to be less oscillatory when we use lower order approximation in the ENO scheme, so eventually we used just the first order approximation. We tried an approach both with and without transformation to the characteristic variables, and the option without transformation gives more stable results. The integral operators $\mathscr{A}$ and $\mathscr{B}$ on the right-hand side are approximated by a simple first order quadrature formula, in order to limit the computational costs. We considered a 100 grid point discretization in space and a rather conservative time step $\Delta t = 10^{-6}$ in order to ensure the stability of the scheme for longer time before blow-up; note that for the Euler equations with homogenous right-hand side (i.e., setting $\mathscr{A} = 0$ and $\mathscr{B} = 0$), we can use $\Delta t = 10^{-3}$ with the same scheme and therefore the source of possible instabilities are necessarily related to the particular form of the right-hand side. For our experiment reported in Figure 2 and Figure 3 we considered initial data $\rho_0(x) = \rho(x, 0)$ being a Gaussian centered at 0.5, the initial velocity $u_0(x) = u(x, 0) = -\sin(2\pi x)$, and the initial temperature $T_0(x) = T(x, 0) = 1$. Note that in this case $\bar{u} = 0$, hence, if a steady state is reached, the velocity should be zero as well as the temperature, as conjectured in Section 3.3. The parameter $\beta$, measuring the strength of the interaction $a(|x - y|)$, is chosen $\beta = 0.1$.

The evolution starts with the initial density concentrated around $x = 0.5$ which first splits apart, then it merges at $x = 0$ and $x = 1$ (periodically), then again in $x = 0.5$, and so on. The height of the peak at $x = 0.5$ is growing at (almost) every new merging. Periodically the peak becomes so concentrated that the gradient blows up, and eventually the numerics breaks down. Therefore we believe that the last part of the numerical simulation is not completely stable and reliable. Nevertheless, this numerical test confirms the dissipative nature of the

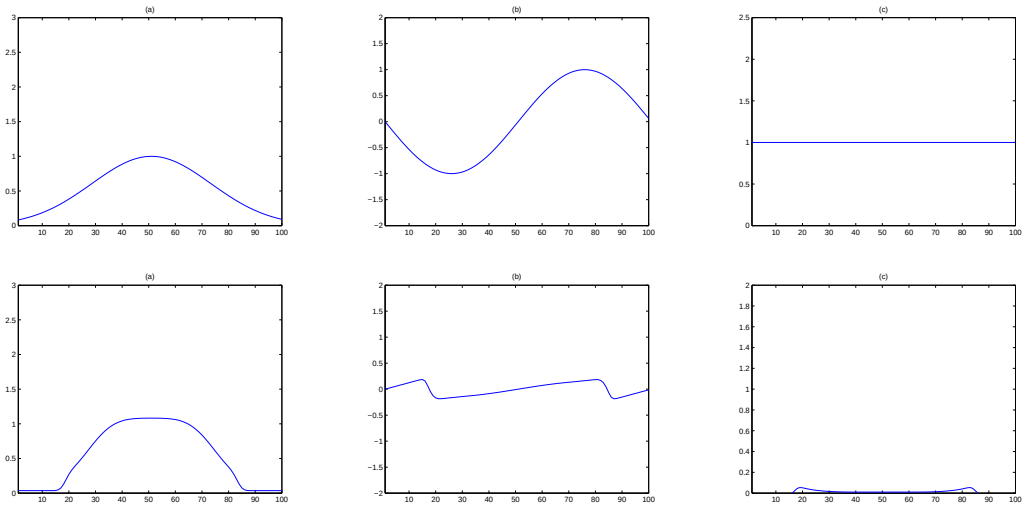


Figure 2: We illustrate respectively in (a) the density $\rho(x,t)$, in (b) the velocity $u(x,t)$, and in (c) the temperature $T(x,t)$ at time $t = 0$ (top) and after $t = 3100$ time-units of the numerical simulation (bottom).

system (46) since the total energy of the system decreases in time, in particular, both velocity and temperature are tending to 0, see Figure 2 and Figure 3, as expected in Section 3.3.

# 5 Conclusions

In this paper we derived Euler–type equations for a population of agents mutually interacting at a kinetic level in such a way that their mutual velocities are dissipated along the joining line position. The underlying kinetic model is the Povzner equation [24], which describes long–range elastic collisions between molecules lying in different positions in space, suitably corrected to take into account the dissipation. Owing to the pioneering result of Lachowicz and Pulvirenti [19], one can resort to the standard fractional step method to obtain the modified system of Euler equations, in which new correction terms are present into the equations for velocity and energy. These corrections are highly nonlocal, and their strength depends of the interaction function introduced by Cucker and Smale [10]. From a formal point of view, it is easily seen that the evolution of macroscopic quantities is driven by the dissipation of energy, which forces the evolution of the system towards a configuration in which the mean velocity is constant, and the internal energy vanishes. It would be certainly interesting to render this behavior rigorous. Numerical simulations show indeed that this is the

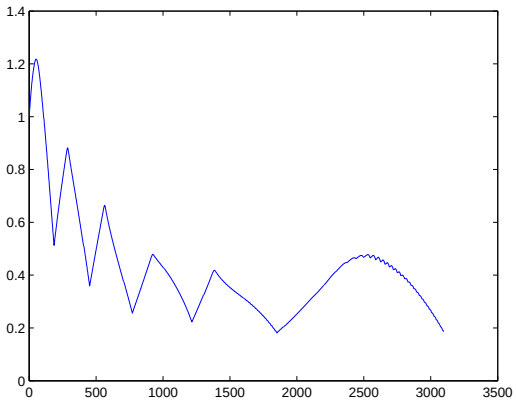


Figure 3: We illustrate the time decay of the maximal velocity, i.e., $U(t) := \max_{x\in\Omega} |u(x,t)|$.

case.

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