Variation on a theme by Bobylëv and Villani

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Abstract

It is shown that the collisional gain operator for a Maxwell gas does not increase the Fisher information. Our proof is a variant of the one given by Villani in [2], but it is shorter and based on Fourier techniques rather than direct estimates. The method we use also applies to general (non-symmetric) Wild convolutions. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

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1. Introduction

The collisional gain operator for Maxwellian molecules in $\mathbb{R}^N$ is defined by

$$Q_+[f,g](v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(n \cdot \hat{q}) f(v') g(v'_*) \, \mathrm{d}v_* \, \mathrm{d}\sigma(n). \tag{1}$$

In the homogeneous Boltzmann equation $\partial_t f = Q_+[f,f] - f$, the difference $Q_+[f,f] - f$ accounts for the changes in the velocity distribution f due to binary particle collisions. In (1), the cross section $B : [-1,1] \to \mathbb{R}_{\geq 0}$ determines the frequency at which collisions between particles of velocities v' and v'_* occur, and these pre-collisional velocities are related to the post-collisional ones, v and v_*, by

$$v' = \frac{1}{2}(v + v_* + |q|n), \quad v'_* = \frac{1}{2}(v + v_* - |q|n), \quad \text{with } q = v - v_* \text{ and } \hat{q} = q/|q|.$$

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In Maxwellian gases, B depends on $q \cdot \hat{q}$ but not on $|v - v_*|$, and

$$\int_{\mathbb{S}^{N-1}} B(\mathbf{e} \cdot n)\,\mathrm{d}\sigma(n) = 1 \quad \text{for every unit vector } \mathbf{e} \in \mathbb{R}^N. \tag{2}$$

Under the additional assumption that B is even, $B(s) = B(-s)$, it has been shown in [2] that Q_+ does not increase the Fisher information, defined on probability densities f by

$$\mathcal{F}[f] = \int_{\mathbb{R}^N} \frac{|\nabla f(v)|^2}{f(v)}\,\mathrm{d}v = 4 \int_{\mathbb{R}^N} |\nabla \sqrt{f}(v)|^2\,\mathrm{d}v.$$

More precisely, it has been shown that $\mathcal{F}[Q_+[f,g]] \leq (\mathcal{F}[f] + \mathcal{F}[g])/2$. Below, we prove that the hypothesis $B(s) = B(-s)$ can be removed at the price of replacing the original estimate by

$$\mathcal{F}\big[Q_+[f,g]\big] \leq \frac{1+\lambda_B}{2}\mathcal{F}[f] + \frac{1-\lambda_B}{2}\mathcal{F}[g] \text{ with } \lambda_B := \int_{\mathbb{S}^{N-1}} (\mathbf{e} \cdot n)\, B(\mathbf{e} \cdot n)\,\mathrm{d}\sigma(n) \in (-1,1). \tag{3}$$

For even B, we have $\lambda_B = 0$ and thus recover the estimate [2].

Our main contribution in this note, however, is a novel, very concise derivation of the following representation formula,

$$\nabla Q_+[f,g](v) = \frac{1}{2} \iint_{\mathbb{S}^{N-1} \times \mathbb{R}^N} B(\hat{q} \cdot n)\big\{Y'_+ + \mathrm{P}_{n,q}[Y'_-]\big\}\,\mathrm{d}\sigma(n)\,\mathrm{d}v_*, \quad Y'_\pm := \nabla f(v')\, g(v'_*) \pm f(v') \nabla g(v'_*), \tag{4}$$

which is at the heart of Villani's original proof. The "projection" operator P defined for $a, b, x \in \mathbb{R}^N$ by

$$\mathrm{P}_{a,b}[x] = (\hat{a} \cdot \hat{b})x - (\hat{a} \wedge \hat{b}) \cdot x, \quad \text{with } v \wedge w = vw^T - wv^T, \tag{5}$$

and $\hat{a} = a/|a|$ denotes the normalization of a vector $a \in \mathbb{R}^N$. Notice that $v \wedge w$ is an anti-symmetric $N \times N$-matrix, and in particular, in dimension $N = 3$, one has $(\hat{a} \wedge \hat{b}) \cdot x = (\hat{a} \times \hat{b}) \times x$.

Notation: Inside integrals, f, f', g_*, g'_* abbreviates $f(v)$, $f(v')$, $g(v_*)$, $g(v'_*)$, and Q_+ abbreviates $Q_+[f,g]$.

2. Fourier representation

Our starting point is the following famous identity by Bobylëv [1]:

Lemma 2.1 *Given two probability densities f and g, then*

$$\widehat{Q}_+[f,g](\xi) = \int_{\mathbb{S}^{N-1}} B(n \cdot \hat{\xi}) \hat{f}(\xi_+) \hat{g}(\xi_-)\,\mathrm{d}\sigma(n) \quad \text{with } \xi_\pm = \frac{1}{2}(\xi \pm |\xi| n). \tag{6}$$

The key observation is that representation (6) in combination with the elementary relation (7) below — which admits a one-line proof — yields the Fourier analogue of (4).

Lemma 2.2 *For arbitrary $\xi \in \mathbb{R}^N$, and with $\xi_\pm$ defined in* (6),

$$(\mathbf{1} + \mathrm{P}_{n,\xi})[\xi_+] + (\mathbf{1} - \mathrm{P}_{n,\xi})[\xi_-] = 2\xi. \tag{7}$$

Proof. On one hand, $\xi_+ + \xi_- = \xi$ follows directly from (6). And on the other hand, also

$$\mathrm{P}_{n,\xi}[\xi_+ - \xi_-] = \mathrm{P}_{n,\xi}[|\xi| n] = |\xi|\big((n \cdot \hat{\xi})n - n(n \cdot \hat{\xi}) + \hat{\xi}(n \cdot n)\big) = |\xi|\hat{\xi} = \xi,$$

since $n \cdot n = 1$, and $\hat{\xi} := \xi/|\xi|$ by definition. □Inserting the relation (7) under the integral in (6) gives

$$\widehat{\nabla Q}_+(\xi) = i\xi \widehat{Q}_+(\xi) = \frac{1}{2} \int_{\mathbb{S}^{N-1}} B(\hat{\xi} \cdot n)\big((\mathbf{1} + \mathrm{P}_{n,\xi})[i\xi_+] + (\mathbf{1} - \mathrm{P}_{n,\xi})[i\xi_-]\big) \hat{f}(\xi_+) \hat{g}(\xi_-)\,\mathrm{d}\sigma(n). \tag{8}$$

We shall now show that the Fourier transform of (8) is (4). Substituting

$$\hat{f}(\xi_+) = \int_{\mathbb{R}^N} e^{-iv\cdot\xi_+} f(v)\,\mathrm{d}v, \quad i\xi_+\hat{f}(\xi_+) = \int_{\mathbb{R}^N} e^{-iv\cdot\xi_+} \nabla f(v)\,\mathrm{d}v,$$

and respective expressions for $\hat{g}(\xi_-)$, $i\xi_-\hat{g}(\xi_-)$ under the integral in (8), gives, with $Y_\pm := \nabla f g_* \pm f \nabla g_*$,

$$\begin{aligned}\widehat{\nabla Q}_+(\xi) = i\xi\widehat{Q}_+(\xi) &= \frac{1}{2}\iiint_{\mathbb{S}^{N-1}\times\mathbb{R}^N\times\mathbb{R}^N} B(\hat{\xi}\cdot n)\big\{Y_+ + \mathrm{P}_{n,\xi}[Y_-]\big\} e^{-i(v\cdot\xi_+ + v_*\cdot\xi_-)}\,\mathrm{d}\sigma(n)\,\mathrm{d}v\,\mathrm{d}v_* \\ &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left(\int_{\mathbb{S}^{N-1}} e^{i|\xi|(q\cdot n)} B(\hat{\xi}\cdot n)\big\{Y_+ + \mathrm{P}_{n,\xi}[Y_-]\big\}\,\mathrm{d}\sigma(n)\right) e^{-i\xi\cdot(v+v_*)/2}\,\mathrm{d}v\,\mathrm{d}v_*.\end{aligned}$$

Next, we apply a particular change of variables — which has been designed by Bobylëv [1] — inside the n-integral to exchange the roles of ξ and q. For the corresponding treatment of the projection operator, we need

Lemma 2.3 *For arbitrary vectors $q, \xi \in \mathbb{R}^N\setminus\{0\}$, and for any measurable function $A : [-1,1]\times[-1,1] \to \mathbb{R}$, one has*

$$\int_{\mathbb{S}^{N-1}} A(\hat{q}\cdot n, \hat{\xi}\cdot n)\hat{q}\wedge n\,\mathrm{d}\sigma(n) = -\int_{\mathbb{S}^{N-1}} A(\hat{\xi}\cdot n, \hat{q}\cdot n)\hat{\xi}\wedge n\,\mathrm{d}\sigma(n). \tag{9}$$

In fact, both (matrix-valued) integrals are multiples of $\xi\wedge q$, and vanish if ξ and q are linearly dependent.

Before proving (9), we show that it indeed concludes the calculation started above. First, observe that (notice the change of order in the subscripts)

$$\mathrm{P}_{n,\xi} = (n\cdot\hat{\xi})\mathbf{1} - n\wedge\hat{\xi} \quad \text{and} \quad \mathrm{P}_{q,n} = (n\cdot\hat{q})\mathbf{1} + n\wedge\hat{q}.$$

We substitute (9) under the n-integral above and observe that its value does not change upon replacing n by its mirror image in the hyperplane orthogonal to $\hat{\xi} - \hat{q}$,

$$\begin{aligned}\widehat{\nabla Q}_+(\xi) &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left(\int_{\mathbb{S}^{N-1}} e^{i|q|(\xi\cdot n)} B(\hat{q}\cdot n)\big\{Y_+ + \mathrm{P}_{q,n}[Y_-]\big\}\,\mathrm{d}\sigma(n)\right) e^{-i\xi\cdot(v+v_*)/2}\,\mathrm{d}v\,\mathrm{d}v_* \\ &= \frac{1}{2}\iint_{\mathbb{S}^{N-1}\times\mathbb{R}^N\times\mathbb{R}^N} B(\hat{q}\cdot n)\big\{Y_+ + \mathrm{P}_{q,n}[Y_-]\big\}\,\mathrm{d}\sigma(n) e^{-i\xi\cdot v'}\,\mathrm{d}v\,\mathrm{d}v_*.\end{aligned}$$

Formula (4) is now obtained by performing a change of variables $(v, v_*) \leftrightarrow (v', v'_*)$ under the integral. This substitution is of determinant one, it changes $Y_\pm$ into $Y'_\pm$, and it exchanges $\hat{q}$ with n — as desired.

Proof of Lemma 2.3. Let $X \subset \mathbb{R}^N$ be the subspace spanned by ξ and q. Denote its orthogonal complement by $X^\perp$. We start by proving the second claim, namely that

$$I := \int_{\mathbb{S}^{N-1}} A(\hat{q}\cdot n, \hat{\xi}\cdot n)\hat{q}\wedge n\,\mathrm{d}\sigma(n) \quad \text{and} \quad J := \int_{\mathbb{S}^{N-1}} A(\hat{\xi}\cdot n, \hat{q}\cdot n)\hat{\xi}\wedge n\,\mathrm{d}\sigma(n)$$

are both scalar multiples of $\xi\wedge q$. Obviously, I and J inherit the anti-symmetry of their integrands, so

$$v^T I w = -w^T I v \quad \text{and} \quad v^T J w = -w^T J v \tag{10}$$

holds for arbitrary $v, w \in \mathbb{R}^N$. We shall now show that these products are actually zero whenever $w \in X^\perp$. Indeed, for $w \in X^\perp$,

$$Iw = \int_{\mathbb{S}^{N-1}} A(\hat{q}\cdot\tilde{n}, \hat{\xi}\cdot\tilde{n})\hat{q}n^T w\,\mathrm{d}\sigma(n). \tag{11}$$

Perform a change of variables $n = R^T\tilde{n}$ with an orthogonal matrix R under the integral such that $Rx = x$ for $x \in X$ and $Ry = -y$ for $y \in X^\perp$. This change leaves the spherical measure invariant, and the integrand in (11) changes to

$$A(\hat{q}\cdot R^T\tilde{n}, \hat{\xi}\cdot R^T\tilde{n})\hat{q}(R^T\tilde{n})^T w = A(R\hat{q}\cdot\tilde{n}, R\hat{\xi}\cdot\tilde{n})\hat{q}\tilde{n}^T(Rw) = -A(\hat{q}\cdot\tilde{n}, \hat{\xi}\cdot\tilde{n})\hat{q}\tilde{n}^T w,$$

which shows $Iw = -Iw = 0$. Thus I is an anti-symmetric matrix that is trivial on $X^\perp$. But the space of anti-symmetric matrices on X is (at most) one-dimensional, and is spanned by $q \wedge \xi$. So I and — by a similar argument — J are scalar multiples of $q \wedge \xi$. To prove (9), consider another orthogonal change of variables $n = R\tilde{n}$, in which $R\hat{\xi} = \hat{q}$ and $R\hat{q} = \hat{\xi}$. We find

$$I = \int_{\mathbb{S}^{N-1}} A(\hat{q} \cdot R^T \tilde{n}, \hat{\xi} \cdot R^T \tilde{n}) \hat{q} \wedge (R^T \tilde{n}) \, \mathrm{d}\sigma(\tilde{n}) = \int_{\mathbb{S}^{N-1}} A(R\hat{q} \cdot \tilde{n}, R\hat{\xi} \cdot \tilde{n}) R^T \big((R\hat{q}) \wedge \tilde{n} \big) R \, \mathrm{d}\sigma(\tilde{n}) = R^T J R.$$

Since $R^T(q \wedge \xi)R = \xi \wedge q = -q \wedge \xi$, it follows that $I = -J$. □

3. Estimate on the Fisher information

In order to arrive at (3), we employ (4) in the same way as done in [2]. We adopt the abbreviations $f' = f(v')$, $g'_* = g(v'_*)$ etc. By definition of Q_+, and since B is a Maxwellian kernel, the quotient $B(n \cdot \hat{q}) f' g'_* / Q_+[f,g]$ defines — for every v — a probability density for integration w.r.t. $\mathrm{d}v_* \, \mathrm{d}\sigma(n)$. Now rewrite the quotient $\nabla Q_+[f,g]/Q_+[f,g]$ using (4) and apply Jenses inequality to find

$$\left| \frac{\nabla Q_+(v)}{Q_+(v)} \right|^2 \leq \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left| \frac{Y'_+ + \mathrm{P}_{n,q}[Y'_-]}{f' g'_*} \right|^2 \frac{B(n \cdot \hat{q}) f' g'_*}{Q_+(v)} \, \mathrm{d}v_* \, \mathrm{d}\sigma(n).$$

Multiply this expression by $Q_+[f,g]$, integrate w.r.t. v, and change variables $(v, v_*) \leftrightarrow (v', v'_*)$ again to obtain an estimate on the Fisher information:

$$\mathcal{F}[Q_+[f,g]] \leq \frac{1}{2} \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} B(\hat{q} \cdot n) \frac{\big| Y_+ + \mathrm{P}_{q,n}[Y_-] \big|^2}{2 f g_*} \, \mathrm{d}\sigma(n) \, \mathrm{d}v \, \mathrm{d}v_*. \tag{12}$$

To finish the proof, two properties of the operators P are needed: the first is simply

$$\mathrm{P}_{q,n} + \mathrm{P}_{q,n}^T = 2(\hat{q} \cdot n)\mathbf{1}, \tag{13}$$

which follows from the anti-symmetry $(\hat{q} \wedge n)^T = -\hat{q} \wedge n$. The second is taken from [2, Lemmata 3&4]:
Lemma 3.1 *For arbitrary vectors $a, b \in \mathbb{R}^N \setminus \{0\}$ and $x \in \mathbb{R}^N$,*

$$|\,\mathrm{P}_{ab}[x]| \leq |x|. \tag{14}$$

Expand the square under the integral in (12), using (14) and (13):

$$\begin{aligned} \frac{\big| Y_+ + \mathrm{P}_{q,n}[Y_-] \big|^2}{2 f g_*} &= \frac{|Y_+|^2 + |\,\mathrm{P}_{q,n}[Y_-]|^2 + Y_+ \cdot (\mathrm{P}_{q,n} + \mathrm{P}_{q,n}^T)[Y_-]}{2 f g_*} \\ &\leq \frac{|Y_+|^2 + |Y_-|^2 + 2(\hat{q} \cdot n) Y_+ \cdot Y_-}{2 f g_*} = |\nabla \sqrt{f}|^2 g_* + f |\nabla \sqrt{g}_*|^2 + (\hat{q} \cdot n) \big\{ |\nabla \sqrt{f}|^2 g_* - f |\nabla \sqrt{g}_*|^2 \big\}. \end{aligned}$$

To arrive at (3), insert this expansion into (12), use the Maxwell property (2), the definition of λ_B in (3), and the fact that f and g are probability densities.

References

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[2] Villani, C. *Fisher information estimates for Boltzmann's collision operator.* J. Math. Pures Appl. **77** (1998), 821–837.