KINETIC MODELS FOR THE TRADING OF GOODS

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Abstract. In this paper we introduce kinetic equations for the evolution of the probability distribution of two goods among a huge population of agents. The leading idea is to describe the trading of these goods by means of some fundamental rules in price theory, in particular by using Cobb-Douglas utility functions for the binary exchange, and the Edgeworth box for the description of the common exchange area in which utility is increasing for both agents. This leads to a Boltzmann-type equation in which the post-interaction variables depend in a nonlinear way from the pre-interaction ones. Other models will be derived, by suitably linearizing this Boltzmann equation. In presence of uncertainty in the exchanges, it is shown that the solution to some of the linearized kinetic equations develop Pareto tails, where the Pareto index depends on the ratio between the gain and the variance of the uncertainty. In particular, the result holds true for the solution of a drift-diffusion equation of Fokker-Planck type, obtained from the linear Boltzmann equation as the limit of quasi-invariant trades.

Keywords. Wealth and income distributions, kinetic models, Boltzmann equation, Fokker-Planck equation.

1. Introduction

In an effort to understand the emergent properties appearing in complex agent-based systems, various concepts and techniques of statistical mechanics have been fruitfully applied in the last twenty years to both social and economic fields. This is well documented both by recent books on these topics [4, 16, 20, 42, 47, 48], as well as by the introductory articles [17, 23, 30, 31, 50]. In particular, the point of view of collisional kinetic theory revealed to be a powerful instrument to describe the behavior of systems in which the mechanism of variation could be described mostly in terms of binary interactions between agents [18, 19, 32, 22, 46]. In kinetic theory of rarefied gas, where binary collisions between molecules are the dominant phenomenon, the evolution of the (spatially uniform) density of the molecules is described by the spatially homogeneous Boltzmann equation [13, 14],

$$\frac{\partial f(v,t)}{\partial t} = Q(f,f)(v,t). \tag{1.1}$$

In (1.1) the bilinear Q term accounts for all kinematically possible (those that conserve both momentum and energy) binary collisions. The post–collisional velocities (v^*, w^*) are consequently given by the linear (in terms of the pre–collisional ones

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$(v, w))$ relations

$$v^* = \frac{1}{2}(v + w + |v - w|\mathbf{n}), \qquad w^* = \frac{1}{2}(v + w - |v - w|\mathbf{n}), \tag{1.2}$$

in which $\mathbf{n}$ is a unit vector. The Boltzmann equation (1.1) describes the relaxation of the gas towards equilibrium, which in this case is given by the Maxwellian density [13, 14] of mass ρ, velocity U and temperature T, that is

$$M(v) := \frac{\rho}{(2\pi T)^{3/2}} \exp\left\{-\frac{|v - U|^2}{2T}\right\}. \tag{1.3}$$

One of the most used models for the collisional operator Q is the so-called Maxwellian molecules operator [8, 13, 14] given by

$$Q\big(f\big)(v) = \int_{\mathbb{R}^3 \times S^2} B\left((v - w) \cdot \mathbf{n}\right) \left[f(v^*) g(w^*)\right] dw d\mathbf{n} - \rho f(v). \tag{1.4}$$

In (1.4) $B(\cdot)$ is a measure of the collision frequency, $d\mathbf{n}$ denotes the *normalized* surface measure on the unit sphere S^2, and

$$\rho = \int_{\mathbb{R}^3} f(v)\, dv\ .$$

For Maxwellian molecules the operator simplifies, and the positive part of it takes the form of a generalized convolution. In reason of this, operator (1.4) has been used to model also binary interactions different from (1.2), like inelastic collisions [2, 5, 6, 7, 9].

Boltzmann-like equations for the evolution in time of the density of wealth in a market composed by many agents are easily recovered by identifying *molecules* with *agents*, *particles'energy* with *agents' wealth*, and *binary collisions* with *trade interactions*. In this way, the kinetic description of market models via a Boltzmann-type equation provides one possible explanation for the development of universal profiles in wealth distributions of real economies.

To obtain the kinetic description one usually resorts to few reasonable assumptions, that date back to the work of Angle [1]. First, agents are considered indistinguishable. Then, an agent's *state* at any instant of time $t \geq 0$ is completely characterized by his current wealth $w \geq 0$. When two agents encounter in a trade, their *pre-trade wealths* (v, w) change into the *post-trade wealths* (v^*, w^*) according to the (linear) rule

$$v^* = \pi_{1,1} v + \pi_{1,2} w, \quad w^* = \pi_{2,1} v + \pi_{2,2} w. \tag{1.5}$$

In most of the existing models, the *interaction coefficients* $\pi_{i,j}$, $i, j = 1, 2$, are non-negative random variables. While $\pi_{1,2} w$ denotes the fraction of the second agent's wealth transferred to the first agent, the difference $(\pi_{1,1} - \pi_{2,1})v$ aims to represent the relative gain (or loss) of wealth of the first agent due to market risks. One generally assumes that the $\pi_{i,j}$'s have fixed laws, which are independent of v and w, and of time.

In these one-dimensional models, the probability distribution $f(w, t)$ of wealth of the ensemble coincides with agent density and satisfies the associated spatially homogeneous Boltzmann equation of Maxwell type (1.4)

$$\frac{\partial f}{\partial t} = Q_+(f, f) - f, \tag{1.6}$$

on the real half line, $w \geq 0$. The collisional gain operator $Q_+(f,f)$, which quantifies the gain of wealth v at time t due to binary trades, acts on test functions $\varphi(w)$ as

$$
\begin{aligned}
Q_+(f,f)[\varphi] :=& \int_{\mathbb{R}_+} \varphi(w) Q_+\big(f,f\big)(w)\, dw \\
=& \frac{1}{2} \int_{\mathbb{R}_+^2} \langle \varphi(v^*) + \varphi(w^*) \rangle f(v) f(w)\, dv\, dw,
\end{aligned} \tag{1.7}
$$

with $\langle \cdot \rangle$ denoting the expectation with respect to the random coefficients $\pi_{i,j}$ in (1.5). The large-time behavior of the density is heavily dependent of the evolution of the average wealth [25, 40]

$$
M(t) := \int_{\mathbb{R}_+} w f(w,t)\, dw, \tag{1.8}
$$

Conservative models are such that the average wealth of the society is conserved with time, $M(t) = M$, where the value of M is finite. In terms of the interaction coefficients, this is equivalent to impose the conditions $\langle \pi_{1,i} + \pi_{2,i} \rangle = 1$ for $i = 1, 2$.

The Boltzmann equation (1.6) belongs to the Maxwell type. As in operator (1.4) the collision frequency is independent of the relative velocity, and the loss term in the collision operator is linear. This introduces a great simplification, that allows to use most of the well established techniques developed for the three-dimensional spatially homogeneous Boltzmann equation for Maxwell molecules in the field of wealth redistribution [22, 24, 25, 26, 40, 41, 44].

In general, the richness of the steady states for kinetic market models is the main remarkable difference to the theory of Maxwell molecules [8]. While the Maxwell distribution (1.3) is the universal steady profile for the velocity distribution of molecular gases, the stationary profiles for wealth can be manifold, and are in general not explicitly known analytically. In fact, they depend heavily on the precise form of the microscopic modeling of trade interactions. Consequently, in investigations of the large-time behavior of the wealth distribution, one is typically limited to describe a few analytically accessible properties (e.g. moments and smoothness) of the latter.

This richness of steady states makes both the theoretical and numerical study of these models highly interesting. Indeed, it corresponds mathematically to identify the limit distribution in a problem which has many analogies with a generalized central limit theorem. Also, it allows to clarify the types of binary interactions which produce fat tails in the distribution at infinity (Pareto tails), which are shown to form in the profiles of the distribution of wealth in western societies.

On the other hand, the socio-economic behavior of a (real) population of agents is extremely complex. Apart from elements from mathematics and economics, a sound description — if one at all exists — would necessarily need contributions from various other fields, including psychology. Clearly, the majority of the available mathematical models are too simple to even pretend to reflect part of the real situation.

For a better understanding of some of the main outcomes of markets in western societies, in recent years the kinetic community started to improve the economic modeling of agent systems by including more realistic assumptions, without making too heavy the underlying Boltzmann equations. Two papers in particular fit into this line of thinking.

In [21] Cordier, Pareschi and Piatecki derive a kinetic description of the behavior of a simple financial market where each agent can create their own portfolio between two investment alternatives: a stock and a bond. In this case, the variation of density is not based on linear binary collisions like (1.5), but it is derived starting from the more realistic Levy-Levy-Solomon microscopic model for price formation [35, 36]. The model in [21] consists of a linear Boltzmann equation for the wealth distribution of the agents coupled with an equation for the price of the stock. From this model, under a suitable scaling, Cordier, Pareschi and Piatecki derive a Fokker-Planck equation and show that the equation admits a self-similar lognormal behavior. For the first time, the kinetic model in [21] attempts to join to simple financial rules a kinetic equation of Boltzmann type, able to describe a complex behavior that could then mimic the market and explain the price formation mechanism.

A second interesting example of this coupling has been recently proposed in [39] by Maldarella and Pareschi. They introduce a relatively simple model for a financial market characterized by a single stock or good and an interplay between two different trader populations, chartists and fundamentalists, which determine the price dynamics of the stock. The model has been inspired by the microscopic Lux–Marchesi model [37, 38]. The main novelty here is to couple the financial rules with the opinion of traders through a kinetic model of opinion formation recently introduced in [49]. Moreover, some psychological and behavioral components of the agents, like the way they interact with each other and perceive the risk, which may produce nonrational behaviors, are taken into account. This is done by means of a suitable *value function* in agreement with the Prospect Theory by Kahneman and Tversky [33, 34]. As they show, people systematically overreacting produces substantial instabilities in the stock market.

These two papers indicate that the methods of kinetic theory can be fruitfully used to understand the market behavior even in presence of more realistic ways of interactions between agents. The objective weakness of models based on linear binary trades of type (1.5) is linked to the fact that people trades according to rules which are in general difficult to justify from a microeconomic point of view. The common ingredients which appear in these trades are the saving propensity concept [15] (agents usually do not trade with all their wealth), and the risk concept [22] (there is no certainty of gain). While highly reasonable, this type of interaction does not clarify while agents are pushed to trade.

One of the fundamental assumption of prize theory in economics is that people trades to improve its utility. Among binary interactions which are motivated by an increasing of the individual utility, one of the most used pictures is furnished by the Edgeworth box [27], which is frequently used in general equilibrium theory, and can aid in representing the competitive equilibrium of a simple system or a range of such outcomes that satisfy economic efficiency. Edgeworth box can fruitfully be applied in presence of an agent-based system in which agents possess a finite number of goods of $n \geq 2$ different types. Inspired by this mechanism of increasing utility and competitive equilibrium, we will introduce in the following a kinetic equation of Boltzmann type for the evolution of the distribution density of the quantities of two goods, said (x, y), in a system of agents. In other words, we will be concerned with the time evolution of the probability density $f(x, y, t)$ of having x goods of the first type and y goods of the second type at time $t > 0$, given a known initial distribution of these goods at time $t = 0$. The variation in time of $f(x, y, t)$ will be entirely due to the microscopic binary interaction driven by the Edgeworth box structure. We will take into account the incomplete knowledge of the game in the binary interaction by allowing agents to be wrong with respect to the ideal outcome predicted by the increase of utility. This will be done by introducing a randomness in the Edgeworth box outcome.

It is remarkable that the basic problem of exchanging goods has been previously treated within the methods of statistical mechanics only in [45]. There, an exchange market consisting of many agents and two goods has been introduced, by allowing agents to have stochastic preferences between them. However, the model considered in [45] has steady state solutions which are described by Gamma distributions, and Pareto tails are not present. Gamma distributions have been recently obtained as exact steady solutions of a pure gambling model, in presence of stochastic redistribution in [3].

In more details, in the next Section we will resume the Edgeworth box interaction, and the consequent Boltzmann-type equation, together with its main properties. Surprisingly, on the contrary to most of the previous types of trade, this exchange rule leads to a highly nonlinear binary interaction, which is difficult to handle, if not numerically. For this reason, in Section 3 we will resort to a suitable linear Boltzmann equation, which is obtained by allowing the agent to interact (according to Edgeworth box), simultaneously with a sufficiently high number of agents in the market. By Fourier based methods, it will be shown that, under reasonable conditions on the initial data, this linear equation has a unique solution, and the steady states are concentrated along a well-defined line. Finally, in Section 4, we will resort to an asymptotic procedure (the so-called *quasi invariant trade limit*), to obtain a linear Fokker-Planck equation which describes the essentials of the trading of goods, and it is relatively easy to treat from a mathematical point of view. In particular, it will be shown that the solution converges towards a steady state with fat tails.

2. The Edgeworth box

As discussed in the Introduction, most of the existing kinetic models for wealth distribution are based on rigid assumptions which, if on one hand can be shared, from the other hand are not deeply related to economic principles, like price theory [28]. The aim of this Section is to introduce a new framework for trades, which is derived directly from the basic principles of economy.

Individuals exchange goods. The benefits they receive depend on how much they exchange and on what terms. Price theory tries to give an answer to this fundamental question. In the case of a binary trade, there may be many different exchanges, each of which would be beneficial to both parties; some exchanges will be preferred by one person, some by the other. There are then two different questions to be settled. One is how to squeeze as much total gain as possible out of the opportunities for trade; the other is how that gain is to be divided. The two individuals who are trading have a common interest in getting as much total gain as possible but are likely to disagree about the division. Let us suppose in the following that agents in the system possess the same two types of goods, and there is no production, so that the total amount of goods remains unchanged. If (x_A, y_A) ((x_B, y_B) respectively) denote the quantity of goods of two agents A and B, the quantities

$$p_A = \frac{x_A}{x_A + x_B}, \quad q_A = \frac{y_A}{y_A + y_B} \tag{2.9}$$

are the percentages of goods of the first agent. Note that the point (p_A, q_A) belongs to the square $\mathcal{S} = [0,1] \times [0,1]$. In order to give a meaning to the reasons of trading, it is classical to assume that agent's behavior is driven by a utility function. One of the most popular of these functions is the Cobb-Douglas utility function

$$U(p,q) = p^\alpha q^\beta, \quad \alpha + \beta = 1. \tag{2.10}$$

Each agent will tend to maximize its utility by trading. The values α and β are linked to the preferences that the agent assigns to the two goods. If $\alpha > \beta$, the agent prefers to possess goods of the first type (numbered by x). The choice $\alpha = \beta = 1/2$ clearly means that the two goods are equally important for A. Given the percentage point (p_A, q_A) of the agent A, the curve

$$U(p,q) = U(p_A, q_A)$$

denotes the *indifference curve* for the agent A. Indeed, any point on the indifference curve has for A the same utility. Note that the indifference curve for A entirely belongs to $\mathcal{S}$, and splits the square into the two regions

$$U_A^- = \{(p,q) : U(p,q) < U(p_A, q_A)\}, \quad U_A^+ = \{(p,q) : U(p,q) > U(p_A, q_A)\}.$$

Clearly, any trade which will move the percentages of agent A into the region U_A^+ will increase its utility function, and will be acceptable for the agent.

In the case of a binary trade, also agent B has a Cobb-Douglas utility function, with parameters of preference which in general are different from the parameters of agent A. Likewise, it will be an indifference curve for B, and a good region in which the utility of B is increased after trading. Ultimately, a trade will be acceptable for both agents, if their percentages after the trade belong to the regions in which their utilities are increased.

An ingenious way of looking at such a situation is the Edgeworth Box, named after Francis Y. Edgeworth, the author of a nineteenth century work on economics called Mathematical Psychics [27].

The Edgeworth box is built up by rotating the square $\mathcal{S}$ in which is designed the indifference curve of agent B of 180° around the center of the square $(1/2, 1/2)$, and considering together the indifference curve of A and the rotated indifference curve of B on the same square.

Any point inside the box represents a possible division of the total quantity of A and B, with the share of A measured from the lower left-hand corner, and the share of B from the upper right-hand corner. Any possible trade is represented by a movement from one point in the box, to another.

By construction, since the Cobb-Douglas utility function is jointly convex, the utility of A increases if the point moves up and to the right; the utility of B increases if the point moves down and to the left. The region in which this happens is exactly the region common to U_A^+ and to U_B^+ rotated of 180°.

This operative way of trading furnished by the Edgeworth box, has interesting consequences. If the percentages of A and B are such that there is a possible region for trading, once the trade has been done, the point has moved inside the region, and the new region for trading is smaller. If the two agents were so smart to choose a point inside the region in which the two indifference curves are tangent each other, since they curve in opposite directions, this means that starting from this point, any point that is on a higher indifference curve for the first agent must be on a lower curve for the second; any trade that makes one better off makes the other worse off. The point in which the two curves are tangent each other is not unique. The set of all points from which no further mutually beneficial trading is possible (Pareto optimal points) is called the *contract curve*.

The contract curve is related to prices. At that tangency of the two indifference curves, in fact, the slope of the tangency line represents the relative prices for the two goods. Hence, there are relative prices that will be consistent with the Pareto optimum, and these prices will maximize the possible budget for both agents.

This short discussion on utility functions, trading and prices is the basis of our construction of the binary trade. In what follows, we suppose that all agents in the system have the same utility function (2.10). If agent A has percentages p and q in his Hedgeworth box, we consider as possible trades the movements of the point (p, q) into $(p^*, q^*) \in \mathcal{S}$, where

$$\begin{aligned} p^* &= p + \lambda\beta(q-p) + \mu(q-p) \\ q^* &= q + \lambda\alpha(p-q) + \tilde{\mu}(p-q). \end{aligned} \tag{2.11}$$

In (2.11) $0 < \lambda \leq 1$, and μ and $\tilde{\mu}$ are random variables with zero mean and finite variance. We will assume moreover that

$$0 < \lambda\beta + \mu < 1, \quad 0 < \lambda\alpha + \tilde{\mu} \leq 1. \tag{2.12}$$

Under this condition, the post-trade point (p^*, q^*) belongs to $\mathcal{S}$, and it is an admissible point for the Edgeworth box.

The dynamics consequent to (2.11) is such that

$$|p^* - q^*| = |1 - \lambda - (\mu + \tilde{\mu})|\, |p - q|. \tag{2.13}$$

Hence, if the random variables μ and $\tilde{\mu}$ satisfy

$$2 - \lambda < \mu + \tilde{\mu} < 1 - \lambda, \tag{2.14}$$

the difference between p and q decreases in time and the system moves asymptotically to the contract curve. In particular, this behavior holds true in absence of randomness.

Moreover, unless $p = q$, and in absence of randomness, namely if

$$\begin{aligned} p^* &= p + \lambda\beta(q-p) \\ q^* &= q + \lambda\alpha(p-q), \end{aligned} \tag{2.15}$$

it is immediate to show that the trade increases the Cobb-Douglas utility function of agent A. Indeed

$$\frac{d}{d\lambda} U(p^*, q^*) = \alpha\beta(1-\lambda)(p-q)^2 p^{\alpha-1} q^{\beta-1} > 0,$$

which, coupled with the convexity of the utility function gives

$$U(p^*, q^*) = U(p, q) + \alpha\beta(p-q)^2 p^{\alpha-1} q^{\beta-1}\lambda + \frac{1}{2}\frac{d^2}{d\lambda^2} U(\tilde{p}^*, \tilde{q}^*)\lambda^2 \geq$$

$$U(p, q) + \alpha\beta(p-q)^2 p^{\alpha-1} q^{\beta-1}\lambda.$$

Last, note that the trade (2.15) is such that $p^* \neq q^*$, unless $p = q$.

Concerning agent B, since it starts from the point $(1-p, 1-q) \in \mathcal{S}$, according to (2.15) it moves to the point

$$\begin{aligned}\bar{p} &= 1-p+\lambda\beta(1-q-(1-p)) \\ \bar{q} &= 1-q+\lambda\alpha(1-p-(1-q)),\end{aligned} \tag{2.16}$$

which, proceeding as before, implies

$$U(\bar{p}, \bar{q}) > U(1-p, 1-q).$$

Hence, in absence of randomness, the trade (2.15) is such that the Cobb-Douglas utility functions of both agents are increased, locating the new point in $\mathcal{S}$ in the interior of the allowed area for trades predicted by the Edgeworth box. Clearly, this trade refers to the purely theoretical situation in which the traders A and B know all about the trade and its result. In general, this is not realistic, and what traders can hope, after a certain number of trades, is that the majority of their trades had the result of increasing their utility. This can be expressed by saying that a reasonable result for traders is to obtain, in a single trade, the increasing of their utility only in the mean. If one agrees with this, then trades of type (2.11) satisfy all constraints of Edgeworth box, but the single trade can move the point in a region which is convenient only for one of the two traders, as well as in the region which is not convenient for both.

Once the trade has been done, according to the rules of price theory, the quantities of goods of agents A and B have been changed accordingly. Making use of the constraints $x_A + x_B = x_A^* + x_B^*$ and $y_A + y_B = y_A^* + y_B^*$, (2.11) implies for the agents A and B the post-trade quantities of goods

$$\begin{aligned}x_A^* &= x_A + (\lambda\beta+\mu)\left(\frac{x_A+x_B}{y_A+y_B}y_A - x_A\right) \\ y_A^* &= y_A + (\lambda\alpha+\tilde{\mu})\left(\frac{y_A+y_B}{x_A+x_B}x_A - y_A\right) \\ x_B^* &= x_B + (\lambda\beta+\mu)\left(\frac{x_A+x_B}{y_A+y_B}y_B - x_B\right) \\ y_B^* &= y_B + (\lambda\alpha+\tilde{\mu})\left(\frac{y_A+y_B}{x_A+x_B}x_B - y_B\right),\end{aligned} \tag{2.17}$$

We remark that, on the contrary to what is usually assumed in the wealth trades of type (1.5), the post-trade quantities in (2.17) are related to the pre-trade quantities by nonlinear relations. It can be verified, however, that the constraints about the conservation of the total quantities of goods in the trade are verified, and

$$x_A^* + x_B^* = x_A + x_B, \quad y_A^* + y_B^* = y_A + y_B.$$

3. A linear Boltzmann equation for trading of goods

Let $f(x, y, t)$ denote the density of agents with quantities x and y of the two goods at time $t \geq 0$. Without loss of generality, we will assume in the following that (x, y) are nonnegative real numbers. A Boltzmann-like equation of Maxwell type for the time evolution of $f(x, y, t)$ can be written by observing that the *collision rules* are here given by (2.17). A useful way of writing the Boltzmann equation has been recalled in the introduction, where the Boltzmann equation (1.6) has been rewritten resorting to the weak form (1.7). It corresponds to write, for any given smooth function φ, the spatially homogeneous Boltzmann equation in the form

$$\frac{d}{dt}\int_{\mathbb{R}_+^2}\varphi(x,y)f(x,y,t)\,dx\,dy=\int_{\mathbb{R}_+^2}\varphi(x,y)Q(f,f)(x,y)\,dxdy. \tag{3.18}$$

The right-hand side of equation (3.18) describes the change of density due to collision of type (2.17). Given two agents with quantities of goods given by (x,y) and (x_1,y_1), one obtains from (2.17) the post-trade quantities

$$\begin{aligned} x^* &= x+(\lambda\beta+\mu)\left(\frac{x+x_1}{y+y_1}y-x\right)\\ y^* &= y+(\lambda\alpha+\tilde{\mu})\left(\frac{y+y_1}{x+x_1}x-y\right). \end{aligned} \tag{3.19}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}_+^2}\varphi(x,y)Q(f,f)(x,y)\,dxdy=\\ &\left\langle\int_{\mathbb{R}_+^4}(\varphi(x^*,y^*)-\varphi(x,y))f(x,y,t)f(x_1,y_1,t)\,dx\,dy\,dx_1\,dy_1\right\rangle. \end{aligned} \tag{3.20}$$

In alternative, one can use the identity

$$\begin{aligned} \int_{\mathbb{R}_+^2}\varphi(x,y)Q(f,f)\,dxdy=\frac{1}{2}\Big\langle\int_{\mathbb{R}_+^4}&(\varphi(x^*,y^*)+\varphi(x_1^*,y_1^*)\\ -\varphi(x,y)-\varphi(x_1,y_1))f(x,y,t)f(x_1,y_1,t)\,dx\,dy\,dx_1\,dy_1&\Big\rangle. \end{aligned} \tag{3.21}$$

By choosing $\varphi(x,y)=x$ (respectively $\varphi(x,y)=y$) one verifies that the mean values

$$\begin{aligned} m_x(t) &= \int_{\mathbb{R}_+^2} xf(x,y,t)\,dx\,dy\\ m_y(t) &= \int_{\mathbb{R}_+^2} yf(x,y,t)\,dx\,dy, \end{aligned} \tag{3.22}$$

remain constant in time $m_x(t)=m_x(0)$ (respectively $m_y(t)=m_y(0)$). These conservations are consequence of the constraints of conservation of the quantities of goods in the trades. Since the exchanges of goods of type (3.19) are nonlinear, the study of the properties of the solution to the Boltzmann equation (3.18) is difficult. In particular, it is cumbersome to find the evolution of the higher moments of the solution in a closed form. Hence, the Boltmann equation (3.18) is the starting point for a numerical study of the evolution of the density by means of MonteCarlo methods [43]. While leaving a more detailed analysis to a forthcoming paper [10], we will present some numerical results in the short Section 5. There, it is shown numerically both that the solution is concentrating on the contract line, and that the presence of tails in the solution to the nonlinear Boltzmann equation is linked to the degree of randomness in the interaction process. Indeed, fat tails are created by assuming a high degree of randomness.

In order to derive a more treatable model, which maintains most of the features of the nonlinear one described by equation (3.18), let us study in more details the exchange rule driven by the Edgeworth box, assuming that, instead of single binary trades, any agent is allowed to exchange goods with part of the agent market, composed by a certain number of agents. We remark that the exchange rule (2.17) for the quantity of the first good can be fruitfully rewritten as

$$x_A^* = x_A + (\lambda\beta + \mu)\left(\frac{x_A + x_B}{y_A + y_B} y_A - x_A\right) = x_A + (\lambda\beta + \mu)\left(\frac{\frac{x_A + x_B}{2}}{\frac{y_A + y_B}{2}} y_A - x_A\right). \tag{3.23}$$

This writing clarifies that, in the post-trade quantity of goods in (3.23), the coefficient of y_A is the ratio between the mean values of the quantities x and y of the two agents exchanging goods.

If one agent exchanges goods, according to the Edgeworth box strategy, with a number of N agents, the quantity of the first good changes according to

$$x_A^* = x_A + (\lambda\beta + \mu)\left(\frac{x_A + x_1 + \cdots + x_N}{y_A + y_1 + \cdots + y_N} y_A - x_A\right),$$

so that, if N is sufficiently large,

$$\frac{x_A + x_1 + \cdots + x_N}{y_A + y_1 + \cdots + y_N} = \frac{\frac{x_A + x_1 + \cdots + x_N}{N+1}}{\frac{y_A + y_1 + \cdots + y_N}{N+1}} \cong \frac{m_x}{m_y}. \tag{3.24}$$

Since the ratio between the mean values is a conserved quantity, by using approximation (3.24) we obtain a consistent linear version of the trade of goods, clearly valid under the assumptions specified above. Therefore, by allowing agents to interact with part of the market, according to the rules of the Edgeworth box, the result of the trade can be well approximated by the linear exchange rules

$$\begin{aligned} x^* &= x + (\lambda\beta + \mu)\left(\frac{m_x}{m_y} y - x\right) \\ y^* &= y + (\lambda\alpha + \tilde{\mu})\left(\frac{m_y}{m_x} x - y\right), \end{aligned} \tag{3.25}$$

in which the action of the market appears only through the mean values m_x and m_y. Consequently, if $f(x, y, t)$ denotes the density of agents with quantities x and y of the two goods at time $t \geq 0$, $f(x, y, t)$ satisfies now the linear Boltzmann equation

$$\frac{d}{dt}\int_{\mathbb{R}_+^2} \varphi(x, y) f(t)\, dx dy = \left\langle \int_{\mathbb{R}_+^2} (\varphi(x^*, y^*) - \varphi(x, y)) f(t)\, dx dy \right\rangle. \tag{3.26}$$

Also for equation (3.26), it can be easily verified that the mean values (3.22) remain constant in time.

Due to its linearity, the Boltzmann equation (3.26) can be studied by resorting to Fourier transform, by standard methods available for the Boltzmann equation for Maxwell molecules, and their extensions to kinetic models of wealth distribution [25, 40]. Among other properties, the analysis of [40] in terms of Fourier-based metrics, allows to prove existence (and uniqueness) of a steady distribution $f_\infty(x, y)$, under few reasonable conditions on the initial distribution of goods, like existence of a certain number of principal moments.

A quite useful expression for the Fourier transformed equation follows by writing the linear interaction (3.25) as

$$\begin{aligned} m_y x^* &= m_y x + (\lambda\beta + \mu)(m_x y - m_y x) \\ m_x y^* &= m_x y + (\lambda\alpha + \tilde{\mu})(m_y x - m_x y). \end{aligned} \tag{3.27}$$

The trade (3.27) corresponds to a dissipative interaction of particles with a random addition proportional to the relative velocity, similar to the one studied in [11]. Then, the pair $(x' = m_y x, y' = m_x y)$ can be fruitfully used as new set of variables, and $f(x', y', t)$ still satisfies the Boltzmann equation (3.26). By choosing

$$\varphi(x, y) = \exp\left\{-i(x\xi + y\eta)\right\},$$

and using (3.27), one directly obtain from (3.26) the equation for the Fourier transform $\widehat{f}$ of f, that takes the form

$$\frac{\partial \widehat{f}(\xi, \eta)}{\partial t} = \langle \widehat{f}(\xi^*, \eta^*)\rangle - \widehat{f}(\xi, \eta). \tag{3.28}$$

In (3.28), the post-interaction variables are given by

$$\begin{aligned} \xi^* &= (1 - (\lambda\beta + \mu))\xi + (\lambda\alpha + \tilde{\mu})\eta \\ \eta^* &= (\lambda\beta + \mu)\xi + (1 - (\lambda\alpha + \tilde{\mu}))\eta. \end{aligned} \tag{3.29}$$

Note that, by conditions (2.12), the random coefficients in (3.29) are non negative. It is immediate to check that the post-trade Fourier variables satisfy the constraint

$$\xi^* + \eta^* = \xi + \eta. \tag{3.30}$$

This implies that the eventual stationary solutions are of the form

$$\widehat{f}_\infty(\xi, \eta) = \widehat{f}_\infty(\xi + \eta). \tag{3.31}$$

Hence, the solution to the Boltzmann equation (3.26) will concentrate for large times along the line $m_x y = m_y x$. Indeed, any generalized function on the plane x, y concentrated on the line $x = y$, can be written in the form

$$F(x, y) = \phi(x + y)\delta(x - y),$$

where as usual $\delta(x)$ represents a Dirac delta function concentrated in $x = 0$, and therefore has a Fourier transform equal to $\frac{1}{2}\widehat{\phi}\left(\frac{1}{2}(\xi + \eta)\right)$.

To verify that this behavior holds true, one resorts to previous works on similar subjects. The standard Fourier-based norm to be used in connection with this type of equations [25] is

$$d_s(f, g) = \sup_{\xi,\eta \neq 0} \frac{|\widehat{f}(\xi, \eta) - \hat{g}(\xi, \eta)|}{(|\xi|^2 + |\eta|^2)^{s/2}}, \quad s > 0. \tag{3.32}$$

Relationships of this norm with other equivalent norms widely used in probability theory and mass transportation can be found in [12]. Uniqueness follows by the following argument. Let $\widehat{f}_1(\xi, \eta, t)$ and $\widehat{f}_1(\xi, \eta, t)$ two solutions to equation (3.28) corresponding to the initial data $\widehat{f}_{0,1}(\xi, \eta)$ and $\widehat{f}_{0,2}(\xi, \eta)$. Then it holds

$$\frac{\partial}{\partial t}\frac{\widehat{f}_1(\xi, \eta, t) - \widehat{f}_2(\xi, \eta, t)}{(|\xi|^2 + |\eta|^2)^{s/2}} + \frac{\widehat{f}_1(\xi, \eta, t) - \widehat{f}_2(\xi, \eta, t)}{(|\xi|^2 + |\eta|^2)^{s/2}} =$$

$$\left\langle \frac{\widehat{f}_1(\xi^*, \eta^*, t) - \widehat{f}_2(\xi^*, \eta^*, t)}{(|\xi^*|^2 + |\eta^*|^2)^{s/2}} \frac{(|\xi^*|^2 + |\eta^*|^2)^{s/2}}{(|\xi|^2 + |\eta|^2)^{s/2}} \right\rangle \leq$$

$$d_s(f_1, f_2) \sup_{\xi,\eta \neq 0} \left\langle \frac{(|\xi^*|^2 + |\eta^*|^2)^{s/2}}{(|\xi|^2 + |\eta|^2)^{s/2}} \right\rangle$$

Suppose that

$$C_s = \sup_{\xi,\eta\neq 0} \left\langle \frac{(|\xi^*|^2 + |\eta^*|^2)^{s/2}}{(|\xi|^2 + |\eta|^2)^{s/2}} \right\rangle < +\infty \tag{3.33}$$

Then, application of Gronwall's inequality gives (cfr. [29])

$$d_s(f_1(t), f_2(t)) \le d_s(f_{0,1}, f_{0,2}) \exp\{(C_s - 1))\,t\}\,. \tag{3.34}$$

From (3.34), uniqueness of solution in a fixed time interval follows. The case in which $C_s < 1$ leads to better decay. In fact, if this is the case, for all times $t \ge 0$ the d_s-distance of f_1 and f_2 decays exponentially in time, and, proceeding as in [40] it can be proven that, if $f(x, y, t)$ is a weak solution of the Boltzmann equation (3.26), which has initially finite moments up to order 2, then f converges exponentially fast in d_s to a steady state f_∞. In addition f_∞ has the same mean values, as given by (3.22), of the initial datum, and it is the only steady state with these averages.

To verify that the constant C_s can be strictly less than one, it is in general not direct. However, the following argument shows that we can easily prove that the d_s-norm is at least non increasing. Let us set

$$A = \lambda\beta + \mu, \quad B = \lambda\alpha + \tilde{\mu}.$$

In reason of condition (2.12), both A and B are strictly positive. It is immediate to recognize that

$$A\xi^* - B\eta^* = (1 - A - B)(A\xi - B\eta) \tag{3.35}$$

Therefore, if the random variables μ and $\tilde{\mu}$, in addition to (2.12) satisfy the additional constraint

$$\langle |1 - A - B|^s \rangle = \langle |1 - \lambda - \mu - \tilde{\mu}|^s \rangle < 1 \tag{3.36}$$

for some positive s, it holds

$$C_s = \left\langle \frac{(|\xi^* + \eta^*|^2 + |A\xi^* - B\eta^*|^2)^s}{(|\xi + \eta|^2 + |A\xi - B\eta|^2)^s} \right\rangle \le 1,$$

Hence, by passing to the new variables

$$\xi_1 = \xi + \eta, \quad \eta_1 = A\xi - B\eta$$

we can prove that, with respect to this set of variables the $d_s(f, g)$ is not increasing for all $s > 0$. This argument implies the global existence of a unique solution to the Boltzmann equation (3.26), but it is not conclusive with respect to the convergence of the global solution towards a unique (in terms of the initial density) steady state. Also, it is not clear if, along the line in which the measure steady solution is concentrated, there is a behavior at infinity with fat tails. This is related to the possibility that, under the constraint of conservation of the total number of goods in a binary exchange, there is a possibility to form a class of rich agents.

Instead of working directly on the Boltzmann equation (3.26), in the next Section we will deal with the derivation of a Fokker-Planck equation which, while keeping the main properties of the Boltzmann equation, will be easier to treat in connection with the aforementioned question about tails.

4. Fokker-Planck models

As often happens when dealing with models of multi-agent systems, one of the fundamental issues is to understand the emergent properties appearing in reason of the type of interactions. In the problem under study, this corresponds to the knowledge of the properties of the equilibrium distribution $f_\infty(x, y)$. To this aim, let us introduce a new set of variables, which are more adapted to our purposes. Starting from (3.27), let us define

$$v = m_y x + m_x y, \quad w = m_y x - m_x y. \tag{4.37}$$

Note that $v \in \mathbb{R}_+$, while $w \in \mathbb{R}$. In addition, by construction $|w| \le v$. With respect to the new variables (v, w), the trade induced by the Edgeworth box reads

$$\begin{aligned} v^* &= v + [\lambda(\alpha - \beta) + \tilde{\mu} - \mu] w \\ w^* &= (1 - \lambda + \tilde{\mu} + \mu)\, w. \end{aligned} \tag{4.38}$$

These new variables better enlighten the outcome of the exchange, and the consequences we described by Fourier transform methods in Section 3. In absence of randomness, the interaction is dissipative in the w variable, since

$$|w^*| = (1 - \lambda)|w|.$$

The same property remains true if the random variables μ and $\tilde{\mu}$ are such that, for a given λ

$$\left\langle (1 - \lambda + \tilde{\mu} + \mu)^2 \right\rangle = (1 - \lambda)^2 + \left\langle (\tilde{\mu} + \mu)^2 \right\rangle < 1. \tag{4.39}$$

Note that condition (4.39) is the analogous of condition (3.36), which relates the smallness of the support of the random variables to the size of λ. If (4.39) holds true, in view of the dissipation, the solution to the linear Boltzmann equation (3.26) will concentrate in time on the line $m_y x = m_x y$.

As outlined at the end of Section 3, the interesting question to answer is to describe the equilibrium profile in the v-variable, to understand if the action of the interaction (4.38) could produce or not Pareto tails. Since the new variables (v, w) seem more suitable to describe the problem, let us set $g(v, w, t) = f(x, y, t)$. The condition $|w| \le v$ implies that the new density vanishes outside the allowed set of values, so that

$$g(v, w, t) = 0, \quad \text{if } |w| > v. \tag{4.40}$$

On the set $|w| \le v$, g satisfies the linear Boltzmann equation

$$\frac{d}{dt} \int \varphi(v, w) g(t)\, dv dw = \left\langle \int (\varphi(v^*, w^*) - \varphi(v, w)) g(t)\, dv dw \right\rangle. \tag{4.41}$$

In the rest of the Section, let us assume that the random variables μ and $\tilde{\mu}$ are identically distributed, with zero mean and variance σ^2. Thanks to conditions (2.12), the random variables possess in addition moments bounded of any order. Let us expand the smooth function $\varphi(v^*, w^*)$ in Taylor series up to order two. We obtain

$$\langle \varphi(v^*, w^*) - \varphi(v, w)) \rangle =$$
$$\lambda \left[(\alpha-\beta) w \frac{\partial \varphi}{\partial v} - w \frac{\partial \varphi}{\partial w} \right] + \frac{1}{2}\sigma_1^2 w^2 \frac{\partial^2 \varphi}{\partial v^2} + \frac{1}{2}\sigma_2^2 w^2 \frac{\partial^2 \varphi}{\partial w^2} +$$
$$\frac{1}{2}\lambda^2 \left[(\alpha-\beta)^2 w^2 \frac{\partial^2 \varphi}{\partial v^2} + w^2 \frac{\partial^2 \varphi}{\partial w^2} - 2(\alpha-\beta) w^2 \frac{\partial^2 \varphi}{\partial v \partial w} \right] + R(v, w).$$

Clearly, $R(v, w)$ denotes the remainder of the Taylor expansion.

Let us set $\tau = \epsilon t$, and let us consider the situation in which $\lambda \to \epsilon\lambda$, while $\mu \to \sqrt{\epsilon}\mu$ and $\tilde{\mu} \to \sqrt{\epsilon}\tilde{\mu}$. Then $h_\epsilon(v, w, \tau) = g(v, w, t)$ satisfies

$$\frac{d}{d\tau} \int \varphi(v, w) h_\epsilon(v, w, \tau) dv dw =$$
$$\int \left[\lambda(\alpha-\beta) w \frac{\partial \varphi}{\partial v} - \lambda w \frac{\partial \varphi}{\partial w} + \frac{1}{2}\sigma_1^2 w^2 \frac{\partial^2 \varphi}{\partial v^2} + \frac{1}{2}\sigma_2^2 w^2 \frac{\partial^2 \varphi}{\partial w^2} \right] h_\epsilon(\tau) dv dw +$$
$$\epsilon \frac{1}{2}\lambda^2 \int \left[(\alpha-\beta)^2 w^2 \frac{\partial^2 \varphi}{\partial v^2} + w^2 \frac{\partial^2 \varphi}{\partial w^2} - 2(\alpha-\beta) w^2 \frac{\partial^2 \varphi}{\partial v \partial w} \right] h_\epsilon(\tau) + R_\epsilon(v, w).$$

We remark that, by construction, the remainder $R_\epsilon(v, w)$ depends in a multiplicative way on higher moments of the random variables $\sqrt{\epsilon}\mu$ and $\sqrt{\epsilon}\tilde{\mu}$, so that $R_\epsilon(v, w)/\epsilon \ll 1$ for $\epsilon \ll 1$ (cfr. the discussion in [22, 49], where similar computations have been done explicitly).

As $\epsilon \to 0$ $h_\epsilon(v, w, \tau) \to h(v, w, \tau)$ satisfying

$$\frac{d}{d\tau} \int \varphi(v, w) h(v, w, \tau) dv dw =$$
$$\int \left[\lambda(\alpha-\beta) w \frac{\partial \varphi}{\partial v} - \lambda w \frac{\partial \varphi}{\partial w} + \frac{1}{2}\sigma_1^2 w^2 \frac{\partial^2 \varphi}{\partial v^2} + \frac{1}{2}\sigma_2^2 w^2 \frac{\partial^2 \varphi}{\partial w^2} \right] h(\tau) dv dw. \tag{4.42}$$

Equation (4.42) is the weak form of the Fokker-Planck equation

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\sigma_1^2 w^2 \frac{\partial^2 h}{\partial v^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 (w^2 h)}{\partial w^2} - \lambda(\alpha-\beta) w \frac{\partial h}{\partial v} + \lambda \frac{\partial (wh)}{\partial w}. \tag{4.43}$$

The coefficients in equation (4.43) are given by $\sigma_1^2 = \langle (\mu - \tilde{\mu})^2 \rangle$, and $\sigma_2^2 = \langle (\mu + \tilde{\mu})^2 \rangle$. Clearly $\sigma_1^2 = \sigma_2^2 = 2\sigma^2$ if the errors are uncorrelated.

The formal derivation of the Fokker-Planck equation (4.43) can be made rigorous repeating the analogous computations of [22, 49], which refer to one-dimensional models. The meaning of this derivation is that we allow for small changes in a single trade of goods. Then, in order that a macroscopic change would be visible, the system needs to wait a sufficiently long time. We call this procedure *quasi-invariant trade limit*. It is interesting to remark that the balance $\sigma^2/\lambda = C$ is the right one which maintains in the limit equation both the effects of the exchange of goods in terms of the intensity λ, as well as the effects of the randomness (through the variance σ). As explained in [49] for the case of opinion formation, different balances give in the limit purely diffusive equations, of purely drift equations.

The Fokker-Planck equation (4.43) is reminiscent of all parameters which contributed to the exchange mechanism. In particular, it contains the values α and β, with $\alpha + \beta = 1$, which are the exponent in the Cobb-Douglas utility function. It is remarkable that these parameters appear only in terms of their difference, so that the case $\alpha = \beta$ leads to a simplification of the equation. Also, the values of λ and

σ^2 are linked to the differential terms of the equation, the former to the drift, and the latter to the diffusion.

Maybe the most important characteristic of the solution to the Fokker-Planck equation, is that it is immediate to recognize that it develops fat tails. This property follows by evaluating moments relative to the w variable. Let us set $\varphi(v,w) = |w|^{1+r}$, with $r > 0$. Then, assuming that the solution is rapidly vanishing at infinity, one obtains

$$\frac{d}{d\tau}\int |w|^{1+r} h(v,w,\tau)\,dv\,dw =$$

$$\left(\frac{1}{2} r\sigma_2^2 - \lambda\right)(1+r)\int |w|^{1+r} h(v,w,\tau)\,dv\,dw.$$

The sign of the right-hand side determines the large-time behavior of the solution. If $r \geq 2\lambda/\sigma_2^2$ the moment $m_{1+r} = \int |w|^{1+r} h(v,w,t)\,dv\,dw$ blows up as time goes to infinity. On the contrary, in agreement with the behavior of the Fourier transformed Boltzmann equation studied in Section 3, if $r < 2\lambda/\sigma_2^2$ the solution concentrates along the line $w = 0$, that is $m_y x = m_x y$. Moreover, since $h(v,w,t) = 0$ whenever $w > v$,

$$\int v^{1+r} h(v,w,\tau)\,dv\,dw \geq \int |w|^{1+r} h(v,w,\tau)\,dv\,dw. \tag{4.44}$$

Hence, if $r \geq 2\lambda/\sigma_2^2$, also the principal moment of order r with respect to the v variable blows up when time goes to infinity, and the stationary solution, which is concentrated along the line $w = 0$ has Pareto tails.

5. Numerical examples

To illustrate the relaxation behavior and to study the influence of the different model parameters, we have performed some kinetic Monte Carlo simulations for the nonlinear Boltzmann model (3.18) presented in Section 3. Generally, in this kind of simulations, known as direct simulation Monte Carlo (DSMC) or Bird's scheme, pairs of agents are randomly and non-exclusively selected for binary trades, and exchange goods according to the rule under consideration. In all our experiments, every agent possesses initially the same quantities of goods. The relaxation occurs exponentially fast. Hence, to compute a good approximation of the steady state it suffices to carry out the simulation for about 10^4 time steps, and then average the wealth distribution over another 1000 time steps. In every experiment, we average over $M = 100$ such simulation runs. We show below some numerical example, relative to the nonlinear Boltzmann equation (3.18), corresponding to $\lambda = 1$ and to different values of $\mu, \tilde{\mu}$.

As it happens in the linear case, predicted by the Fokker-Planck like equation introduced in Section 4, the formation of Pareto tails is consequent to the presence of randomness in the exchanges, which in addition has to be sufficiently strong. Figure 5.1 refers to the case in which there is absence of randomness. The solution here is concentrating on the contract line, while along this line the density function exhibits a profile rapidly decreasing at infinity. On the contrary, Figure 5.2 refers to exchanges in which there is presence of randomness sufficient to develop Pareto tails. Note that the solution still concentrates along the contract line, as in the case described by Figure 5.1.

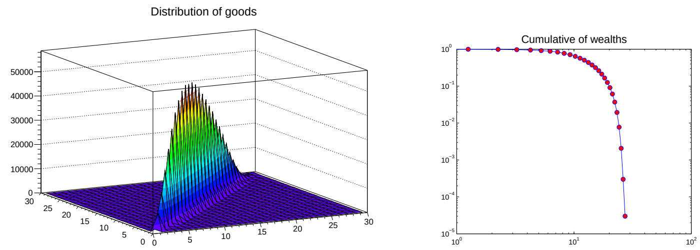

Figure 5.1. Left figure (color on–line): distribution of goods in the nonlinear case ($\lambda = 1, \mu = \tilde{\mu} = 0$; $\alpha = \beta = 0.5$). Right figure: cumulative distribution of goods in $log - log$-scale. No tails in this case.

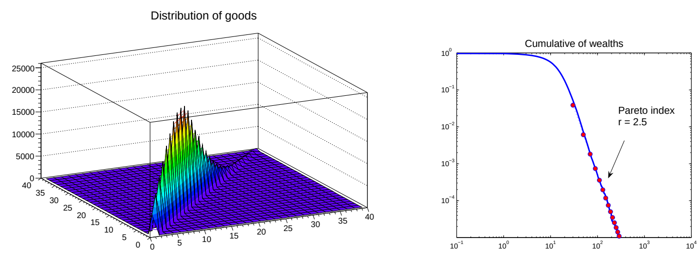

Figure 5.2. Left figure (color on–line): distribution of goods in the nonlinear case ($\lambda = 1$, $\mu = \tilde{\mu} = 0.6$; $\alpha = \beta = 0.5$). Right figure: cumulative distribution of goods in $log - log$-scale. Evidence of Pareto tails.

6. Final remarks

In this paper we introduced kinetic equations for the evolution of the probability distribution of two goods among a huge population of agents. The leading idea was to describe the trading of these goods by means of some fundamental rules in prize theory, in particular by using Cobb-Douglas utility functions for the binary exchange, and the Edgeworth box for the description of the common exchange area in which utility is increasing for both agents. Also, to take into account the intrinsic risks of the market, we introduced randomness in the trade, without affecting the microscopic conservations of the trade, that is the conservation of the total number of each good in the binary exchange.

The linear Boltzmann equation we introduced in Section 3 and its asymptotic Fokker-Planck limit studied in Section 4, showed that, despite the microscopic constraints, sufficiently high moments of the solution blow up with time. This is in contrast with the behavior of the simplified one-dimensional models for wealth distribution expressed by equation (1.6), consequent to trades of type (1.5), where the microscopic conservation of wealth in the binary exchange has been shown to imply an exponential decay of the stationary profile at infinity [25, 40], thus excluding the possibility of Pareto tails.

The model we introduced in this paper can be generalized in many ways. First, the binary trade (2.11) suggested by the Edgeworth box and considered in Section 2 can be modified in many ways, which continue to satisfy all constraints imposed by price theory. One of the possible modifications is given by

$$\begin{aligned} p^* &= p(1+\mu) + \lambda\beta(q-p) \\ q^* &= q(1+\tilde{\mu}) + \lambda\alpha(p-q). \end{aligned} \tag{6.45}$$

Analogously to (2.11), in (6.45) $0 < \lambda \le 1$, and μ and $\tilde{\mu}$ are random variables with zero mean and finite variance, which satisfy conditions (2.12). At difference with the previous interactions, here the randomness modifies the point in the Edgeworth box proportionally to the percentages of agents (not to their difference). The main consequence of this choice relies in the fact that now $p = q$ does not imply $p^* = q^*$, but only $\langle p^* \rangle = \langle q^* \rangle$ (equality in the mean). This new way of looking at randomness reflects heavily on the steady state of the linear Boltzmann equation (3.26), which does not concentrate along a well-defined line.

Second, the strategy of using utility functions and Hedgeworth box for the trading is not restricted to an agent-based market in which agents exchange two goods only. One can easily generalize the analysis of the previous Sections to model markets in which there are $N > 2$ types of goods to be exchanged, according to (2.17) for each binary trade.

Also, one can improve the trading by allowing people to possess different utility functions. This can be easily done for example by introducing a Cobb-Douglas utility function (2.10) in which the exponents α and β are positive random variables, subject to the constraint

$$\alpha + \beta = 1.$$

This generalization will introduce a change in the Boltzmann equation, while it will induce the same Fokker-Planck limit, with the coefficient $\alpha - \beta$ substituted by $\langle \alpha - \beta \rangle$.

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