Large-time asymptotics for nonlinear diffusions: the initial-boundary value problem

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Abstract

In this paper we investigate the large-time behavior of solutions to the first initial-boundary value problem for the non-linear diffusion $u_t = (u^m)_{xx}$, $m > 0$. In particular, we prove exponential decay of $u(x,t)$ towards its own steady state in L^1-norm for long times and we give an explicit upper bound for the rate of decay. The result is based on a new application of entropy estimates, and on detailed lower bounds for the entropy production in this situation.

Keywords: asymptotic behavior, entropy dissipation, degenerate parabolic equation, diffusion equation.

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1 Introduction

In recent years, entropy dissipation methods (also called *entropy-entropy production* methods) have been successfully applied to reckon decay rates towards equilibrium of weak solutions to Cauchy problems for nonlinear second and fourth-order parabolic equations and their systems [1, 2, 8, 9, 10, 13, 18].

However only rarely, at least to our knowledge, such strategies have been applied to the study of initial-boundary value problems with non-homogeneous or non-periodic boundary conditions [17].

The main reason for this fact relies in the lack of mass conservation, which is one of the main ingredients to handle the problem and to obtain L^1-estimates from the entropy decay by means of Csiszár-Kullback inequalities [3, 12].

An entropy dissipation method to handle almost general Dirichlet boundary conditions will be developed in this paper. Our results show that entropy methods can be fruitfully applied also to initial-boundary value problems with non-homogeneous Dirichlet boundary conditions.

In more details, we will be concerned with the large-time behavior of the solution to the initial–boundary value problem

$$u_t(x,t) = [u^m(x,t)]_{xx} \qquad (x,t) \in \Omega \times (0,+\infty) \tag{1}$$

$$u(-L,t) = \varphi_-, \quad u(L,t) = \varphi_+ \tag{2}$$

$$u(x,0) = u_0(x) \tag{3}$$

in $\Omega = (-L, L)$, $L > 0$, with $m > 0$. The initial data $u_0(x)$ and the boundary conditions φ_+ and φ_- are non-negative real constants satisfying the hypotheses stated in the following definition:

Definition 1.1 *The initial and boundary conditions of Problem* (1)-(3) *are said to be* admissible *if and only if*

1. *φ_+ and φ_- are non-negative and $(\varphi_+ + \varphi_-) > 0$;*

2. *$u_0(x) \in C([-L, L])$ is a non-negative function which satisfies the compatibility conditions $u_0(-L) = \varphi_-$ and $u_0(L) = \varphi_+$.*

While the large-time behavior of solutions to the Cauchy problem for the nonlinear equation (1) was intensively studied from many years now (cfr. the excellent survey paper of Vázquez [19] and the references therein), the behavior of solutions to the initial-boundary value problem with non-homogenous Dirichlet boundary conditions was less studied (see [4, 7]; results on the logarithmic nonlinearity can however be found in [6]). Other related results in bounded domains deserve to be referred in. A complete study of the asymptotic behavior of solutions to the initial-Dirichlet problem with homogeneous boundary condition has been recently presented by Vázquez [20]. However, the techniques used there are different from ours. The question of the possible application of the entropy method to the study of the problem treated in [20] is to our knowledge open in higher dimensions. In dimension one, the analysis of the asymptotic behavior with homogeneous boundary conditions (namely when $\varphi_+ = \varphi_- = 0$) can be performed, by means of the entropy dissipation method, in a rather standard way. We focus therefore our attention only to the case when at least one of the boundary conditions does not vanish.

As we will discuss later on, our method works in the whole range $m > 0$, covering both the fast diffusion equation ($m < 1$) and the porous media equation ($m > 1$).

In order to study the large-time behavior, we shall analyze the time-decay of a suitable relative entropy, where the word "relative" means *relative to the steady state* of Equation (1)-(3).

The method introduced in this work, useful to handle second-order diffusions in one dimension, cannot be easily generalized due to a main fact. The entropy associated to the steady state is explicit as the steady state is;

in the general case (higher dimensions or higher order diffusions), the steady states are not explicit nor the entropy functionals. Therefore, the application of the entropy-entropy production method based on the knowledge of the derivative in time of the functional becomes rather difficult.

The results on the large-time behavior will be reached by steps.

The existence and uniqueness theorem for equation (1)-(3) will be stated in Section 2.

Section 3 deals with the study of the entropy functional for the same initial-boundary value problem. The sub-linear case (i. e. $0 < m \leq 1$) and the super-linear case ($m > 1$) lead to different relative entropies, and are treated in a slightly different way. It is interesting to remark that, in the limit $m \to 1$, the entropy functionals of both the super-linear case and the sub-linear cases coincide, but the rate of convergence obtained in the linear case as limit of the nonlinear case is worse than the rate obtained from a direct inspection.

The case of strictly positive initial-boundary conditions is considered in Section 4, where the rate of decay of the solution towards the steady state is discussed, both in the sub-linear case and in the super-linear one.

Finally, the general case of non-negative initial-boundary conditions, obtained by removing the limitation on the data, is studied in Section 5.

2 Main properties of the solution

We will start the study of our problem by discussing the well posedness of equation (1)-(3) and the main properties of the solution. Following [14], the precise characterization of a weak solution to problem (1)-(3) is obtained by the following definition:

Definition 2.1 *A function $u(x,t)$ defined on $\bar{\Omega} \times [0,T]$ is a weak solution of Problem* (1)-(3) *if and only if:*

1. *$u(x,t)$ is real, non-negative and continuous on $\bar{\Omega} \times [0,T]$;*

2. *$u(-L,t)=\varphi_-$ and $u(L,t)=\varphi_+$ for all $t\in[0,T]$;*

3. *$u^m(x,t)$ has a square-integrable generalized derivative with respect to x in $\Omega\times[0,T]$;*

4. *$u(x,t)$ satisfies the identity*

$$\int_0^T\int_\Omega [(u^m)_x\phi_x - u\phi_t]\,dxdt = \int_\Omega \phi(x,0)u_0(x)\,dx, \tag{4}$$

for all $\phi\in C(\bar\Omega\times[0,T])\cap H^1(\Omega\times(0,T))$ which vanish for $|x|=L$ and for $t=T$.

The main result of this section is due to Gilding [14, 15, 16]. We remark that, as far as the well-posedness of (1)-(3) is concerned, more recent contributions are present in the literature. We refer in particular to the paper of Bénilan and Touré [5], where well-posedness is investigated in the framework of the nonlinear semigroup theory. Gilding proved (in a more general form) the following theorem.

Theorem 2.1 *Let us consider the initial-boundary value problem* (1)-(3) *with admissible (in the sense of Definition 1.1) non-negative initial data $u(x,0)=u_0(x)\in C(\bar\Omega)$ and boundary conditions $u(-L,t)=\varphi_-$ and $u(L,t)=\varphi_+$.*

Then Problem (1)-(3) *has exactly one and only one non-negative weak solution $u(x,t)$.*

Moreover, $u(x,t)$ is a classical solution in a neighborhood of any point $(x_0,t_0)\in\bar\Omega\times[0,T]$ where $u(x_0,t_0)>0$.

Finally, if we denote with $u_1(x,t)$ and $u_2(x,t)$ two solutions of problem (1)-(3) *with initial and boundary conditions $u_{0,1}(x)$, $\varphi_-^{(1)}$, $\varphi_+^{(1)}$ and $u_{0,2}(x)$, $\varphi_-^{(2)}$, $\varphi_+^{(2)}$ respectively,*

$$u_{0,1}(x)\le u_{0,2}(x) \text{ for all } x\in\bar\Omega$$

and

$$\varphi_-^{(1)} \le \varphi_-^{(2)} \qquad \varphi_+^{(1)} \le \varphi_+^{(2)},$$

imply $u_1(x,t) \le u_2(x,t)$ *for all* $(x,t) \in \bar{\Omega} \times [0,T]$.

The stationary state for problem (1)-(3) can be easily found. We have the following

Lemma 2.1 *Problem* (1)-(2) *admits one and only one stationary solution. This solution is given by*

$$\bar{u}^m(x) = ax + b, \tag{5}$$

where

$$a = \frac{1}{2L}[\varphi_+^m - \varphi_-^m], \quad b = \frac{1}{2}[\varphi_+^m + \varphi_-^m].$$

Proof: The stationary solution of Equation (1)-(3) satisfies

$$\bar{u}^m(x)_{xx} = 0 \quad \text{in } \Omega \tag{6}$$

$$\bar{u}(-L) = \varphi_-, \quad \bar{u}(L) = \varphi_+. \tag{7}$$

Integration of (6), together with the use of boundary conditions (7) gives (5). □

Obviously, when $\varphi_+ = \varphi_- = 0$, we have that $\bar{u}(x) = 0$ for all $x \in \bar{\Omega}$. Since this case can be easily solved directly, from now on we will suppose that at least one of the boundary conditions φ_+ or φ_- does not vanish.

3 The entropy functional

In order to prove the exponential decay of the solution of Problem (1)-(3), we will consider an auxiliary problem and use an entropy monotonicity approach.

We suppose from now on and until the end of Section 4 that both the initial and boundary conditions are not allowed to take the value zero. If it were the case, we would consider a modified problem, obtained by lifting up the data in a suitable way, as we will see in Section 5.

Let us define $f = u/\bar{u}$; by the hypotheses on φ_+ and φ_-, $\bar{u}(x) \neq 0$ for all $x \in \Omega$, so that f is always well defined.

It is immediate to show that f satisfies the initial-boundary value problem:

$$\bar{u}\, f_t = 2a(f^m)_x + (ax+b)(f^m)_{xx} \tag{8}$$

$$f(-L,t) = 1, \quad f(L,t) = 1 \tag{9}$$

$$f(x,0) = u_0(x)/\bar{u}(x) \tag{10}$$

in Ω.

It will be clear later on that it is convenient to consider separately the two cases $0 < m \leq 1$ and $m > 1$ respectively. Both cases, however, can be treated by studying the time decay of a suitable entropy, defined by the convex functional:

$$H(f) = \int_\Omega \Phi(f)\bar{u}dx, \tag{11}$$

where the convex function $\Phi(\,\cdot\,)$, which is defined in the next subsections, depends on the case we are considering.

3.1 The sub-linear case

In the sub-linear case (i. e. $0 < m \leq 1$) we consider the convex function $\Phi(\,\cdot\,)$ given by

$$\Phi(f) = \left[\frac{1}{m(2-m)(3-m)} f^{3-m} - \frac{1}{m(2-m)} f + \frac{1}{m(3-m)}\right]. \tag{12}$$

Note that the linear part of $\Phi(\,\cdot\,)$ is chosen so that $\Phi(1) = \Phi'(1) = 0$.

We prove the following lemma:

Lemma 3.1 *Let $u_0(x)$, φ_-, φ_+ be admissible initial-boundary conditions, and moreover be such that φ_-, $\varphi_+ > 0$ and $u_0(x)$ is strictly positive for all*

$x \in \Omega$. Then, if $0 < m \leq 1$, the functional $H(f)$, with $\Phi(f)$ given by (12), is an entropy for Problem (8)-(10), that is

$$\frac{dH(f)}{dt} = -I(f),$$

where the (nonnegative) entropy production $I(f)$ is

$$I(f) = \int_\Omega \bar{u}^m \left(f_x\right)^2 \, dx \tag{13}$$

Proof: Let us multiply both sides of Equation (8) by $\Phi'(f)$. Integrating over Ω with respect to x, we obtain

$$\int_\Omega \bar{u}\Phi'(f)\frac{\partial f}{\partial t}\, dx = \int_\Omega \Phi'(f)(ax+b)(f^m)_{xx}\, dx.$$

Let us integrate by parts the right-hand side. Thanks to the boundary conditions (9) and to the property $\Phi'(1) = 0$ we obtain

$$\frac{d}{dt}\int_\Omega \bar{u}\Phi(f)dx = -\int_\Omega \left[\bar{u}^m\Phi'(f)\right]_x (f^m)_x\, dx =$$

$$= -m\int_\Omega \bar{u}^m\Phi''(f)f^{m-1}\left(f_x\right)^2 \, dx.$$

Since $m\Phi''(f)f^{m-1} = 1$, the lemma is fully proven. □

3.2 The super-linear case

In the super-linear case (which corresponds to the choice $m > 1$), we introduce the function $\Phi(\,\cdot\,)$ given by

$$\Phi(f) = \left[\frac{m + f^{m+1}}{m+1} - f\right]. \tag{14}$$

As in the previous case, $\Phi(1) = \Phi'(1) = 0$.

We prove the following lemma:

Lemma 3.2 *Let $u_0(x)$, φ_-, φ_+ be admissible initial-boundary conditions, and moreover be such that φ_-, $\varphi_+ > 0$ and $u_0(x)$ is strictly positive for all $x \in \Omega$.*

Then, if $m > 1$, the functional

$$H(f) = \int_\Omega \Phi(f)\bar{u}dx, \tag{15}$$

where $\Phi(f)$ is given in (14)*, is an entropy for Problem* (8)-(10)*, that is*

$$\frac{dH(f)}{dt} = -I(f),$$

and the (nonnegative) entropy production is

$$I(f) = \int_\Omega \bar{u}^m (f^m)_x^2 \, dx. \tag{16}$$

Proof: Let us multiply both sides of Equation (8) by $\Phi'(f) = (f^m - 1)$ and integrate over Ω with respect to x.

$$\int_\Omega \bar{u}\Phi'(f)\frac{\partial f}{\partial t}\, dx = \int_\Omega \Phi'(f)\left[2a(f^m)_x + (ax+b)(f^m)_{xx}\right] dx.$$

Integrating by parts the right-hand side, taking into account the boundary conditions (9) and the property $\Phi'(1) = 0$ we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega \bar{u}\Phi(f)dx &= 2a\int_\Omega (f^m - 1)(f^m)_x \, dx + \int_\Omega \bar{u}^m (f^m - 1)(f^m)_{xx}\, dx \\ &= -\int_\Omega (f^m)_x[\bar{u}^m(f^m - 1)]_x \, dx = -\int_\Omega \bar{u}^m (f^m)_x^2 \, dx, \end{aligned} \tag{17}$$

which is the thesis of the lemma. □

4 Convergence for strictly positive initial data

In order to investigate the entropy decay rate, we look now for lower bounds to the entropy production functional

$$I(f) = \int_\Omega \bar{u}^m (f^n)_x^2 \, dx,$$

with $m > 0$ and $n \geq 1$. In order to achieve this goal, we prove the following

Lemma 4.1 *Let f belong to $H^1(\Omega)$, and let $\bar{u}(x)$ be the stationary solution of Problem* (6)-(7)*. Then, for any $m > 0$ and $n \geq 1$,*

$$\int_\Omega \bar{u}^m \left[(f^n)_x\right]^2 \, dx \geq \frac{1}{16\, L^2} J(\varphi_+, \varphi_-) \left(\int_\Omega |u - \bar{u}| \, dx \right)^2 ,$$

where

$$J(\varphi_+, \varphi_-) = \begin{cases} \dfrac{1}{mL} \left[\dfrac{\varphi_+^m - \varphi_-^m}{\varphi_+^2 - \varphi_-^2} \right] & \text{if } \varphi_+ \neq \varphi_-, \\ \\ \dfrac{1}{2L} \varphi^{m-2} & \text{if } \varphi_+ = \varphi_- = \varphi. \end{cases}$$

Proof: Let us first suppose that $\bar{u}^m$ is non decreasing. The opposite case can be treated in the same way.

Given any function $g \in H^1(\Omega)$, such that $g(L) = 1$,

$$\begin{aligned} \|\bar{u}(g-1)\|_{L^1} &\leq \int_{-L}^{L} \bar{u}(x) \int_x^L |g'(y)| \, dy \, dx \\ &\leq \int_{-L}^{L} \bar{u}(x)^{(2-m)/2} \int_x^L \bar{u}(y)^{m/2} |g'(y)| \, dy \, dx \\ &\leq \int_{-L}^{L} \bar{u}(x)^{(2-m)/2} \left(\int_x^L \bar{u}(y)^m |g'(y)|^2 \, dy \right)^{1/2} (L-x)^{1/2} \, dx \\ &\leq \int_{-L}^{L} \bar{u}(x)^{(2-m)/2} \left(\int_x^L \bar{u}(y)^m |g'(y)|^2 \, dy \right)^{1/2} (L-x)^{1/2} \, dx \\ &\leq \|\bar{u}^{m/2} g'\|_{L^2} \left(\int_{-L}^{L} \bar{u}(x)^{2-m} \, dx \right)^{1/2} \left(\int_{-L}^{L} (L-x) \, dx \right)^{1/2} . \end{aligned} \tag{18}$$

For any $m > 0$, the function $\bar{u}^{2-m} = (ax+b)^{2/m-1}$ is integrable on Ω. If we denote

$$J(\varphi_+, \varphi_-) = \left(\int_\Omega (ax+b)^{2/m-1} \, dx \right)^{-1} ,$$

we have in fact that

$$J(\varphi_+, \varphi_-) = \begin{cases} \dfrac{1}{mL} \left[\dfrac{\varphi_+^m - \varphi_-^m}{\varphi_+^2 - \varphi_-^2} \right] & \text{if } \varphi_+ \neq \varphi_- \\ \\ \dfrac{1}{2L} \varphi^{m-2} & \text{if } \varphi_+ = \varphi_- = \varphi, \end{cases}$$

Hence, (18) implies

$$\int_\Omega \bar{u}(x)^m |g'(x)|^2\, dx \ge \frac{J(\varphi_+,\varphi_-)}{2L^2} \|\bar{u}(g-1)\|^2_{L^1}. \tag{19}$$

Next, if $n \ge 1$, we have $|r^n - 1| \ge |r-1|$ for $r \ge 0$. Consequently,

$$\frac{J(\varphi_+,\varphi_-)}{2L^2}\|\bar{u}(f-1)\|^2_{L^1} \le \frac{J(\varphi_+,\varphi_-)}{2L^2}\|\bar{u}(f^n-1)\|^2_{L^1} \le \int_\Omega \bar{u}(x)^m \left[(f^n(x))_x\right]^2\, dx \tag{20}$$

□

Remark 4.1 *We remark that $J(\varphi_+,\varphi_-)$ is monotone decreasing with respect to m. This is consistent with the fact that we expect a faster convergence towards equilibrium in the fast diffusion case.*

Remark 4.2 *The proof of Lemma* 4.1*, which is nothing but the proof of a weighted Poincaré inequality, takes essential advantage from the simple form of the steady state $\bar{u}$. Of course, other proofs can be found in the literature, and the constant could be improved. Recent results on weighted Poincaré inequalities are due to Chua and Wheeden* [11]*.*

In order to obtain a proof of the exponential convergence of the solution $u(x,t)$ to Problem (1)-(3) towards its stationary state $\bar{u}(x)$ when $u_0(x) > 0$, it remains to show that the relative entropy satisfies an inequality of Csiszár-Kullback type [12]. This result is contained into the following lemma:

Lemma 4.2 *Let $f(x,t)$ be the solution of Problem* (8)-(10)*, $\bar{u}(x)$ the solution of Problem* (6)-(7) *. Let*

$$\xi(m) = \min_{0\le\alpha\le 1} \frac{1-\alpha}{\Phi(\alpha)} + \frac{2}{\Phi''(\alpha)}, \tag{21}$$

where Φ is given by (12) *if $m < 1$, or by* (14) *if $m \ge 1$. Then the following Csiszár-Kullback-type inequality holds:*

$$\int_\Omega \Phi(f)\bar{u}\, dx \ge \frac{\xi(m)}{\int_\Omega \bar{u}\, dx}\left(\int_\Omega |u-\bar{u}|\, dx\right)^2. \tag{22}$$

Proof:

First, we note that the function

$$\Psi(r) = \frac{\Phi(r)}{1-r}, \quad 0 \le r \le 1,$$

is non-increasing. Since

$$\Psi'(r) = \frac{\Phi'(r)}{1-r} + \frac{\Phi(r)}{(1-r)^2} = \frac{1}{(1-r)^2}\left[\Phi'(r)(1-r) + \Phi(r)\right],$$

it is enough to show that the function $z(r) = \Phi'(r)(1-r) + \Phi(r)$ is non-positive in the set $0 \le r \le 1$. This follows easily considering that $\Phi(1) = 0$, while the convexity of Φ implies

$$z'(r) = \Phi''(r)(1-r) \ge 0.$$

Fix $\alpha \in (0,1)$. Since Ψ is non-increasing, and

$$\int_{\{f<\alpha\}} \bar{u}|f-1|\,dx \le \left(\int_{\{f<\alpha\}} \bar{u}|f-1|\,dx\right)^{1/2}\left(\int_\Omega \bar{u}\,dx\right)^{1/2},$$

we obtain

$$\int_{\{f<\alpha\}} \bar{u}\Phi(f)\,dx \ge \Psi(\alpha)\int_{\{f<\alpha\}} \bar{u}|f-1|\,dx \ge \frac{\Psi(\alpha)}{\|\bar{u}\|_{L^1}}\left(\int_{\{f<\alpha\}} \bar{u}|f-1|\,dx\right)^2. \tag{23}$$

On the other hand, since both $\Phi(1) = 0$ and $\Phi'(1) = 0$, by the monotonicity of Φ'' we have

$$\int_{\{f\ge\alpha\}} \bar{u}\Phi(f)\,dx = \int_{\{f\ge\alpha\}} \bar{u}\frac{(f-1)^2}{2}\Phi''(1+\vartheta(f)(f-1))\,dx$$

$$\ge \frac{\Phi''(\alpha)}{2}\int_{\{f\ge\alpha\}} \bar{u}(f-1)^2\,dx \ge \frac{\Phi''(\alpha)}{2\|\bar{u}\|_{L^1}}\left(\int_{\{f\ge\alpha\}} \bar{u}|f-1|\,dx\right)^2. \tag{24}$$

Grouping together inequalities (23) and (24) we finally obtain

$$\|u-\bar{u}\|_{L^1}^2 \le 2\left(\int_{\{f<\alpha\}} \bar{u}|f-1|\,dx\right)^2 + 2\left(\int_{\{f\ge\alpha\}} \bar{u}|f-1|\,dx\right)^2$$

$$2\|\bar{u}\|_{L^1}\left(\frac{1-\alpha}{\Phi(\alpha)}+\frac{2}{\Phi''(\alpha)}\right)\int_\Omega \bar{u}\Phi(f)\,dx. \tag{25}$$

Optimizing with respect to α we conclude the proof. □

The previous results are at the basis of the following theorem:

Theorem 4.3 *Let $u(x,t)$ be the solution of problem (1)-(3), where $u_0(x)$, φ_- and φ_+ are strictly positive, and $H(u_0/\bar{u})$ is bounded. Then $u(x,t)$ decays exponentially fast towards the stationary solution $\bar{u}(x)$ in L^1-norm, and the following bound holds*

$$\left(\int_\Omega |u-\bar{u}|\,dx\right)^2 \le \frac{\int_\Omega \bar{u}\,dx}{\xi(m)} H(u_0/\bar{u}) \exp\left[-\frac{\int_\Omega \bar{u}\,dx}{16\,L^2\xi(m)}\left(\int_\Omega \bar{u}^{2/m-1}\,dx\right)^{-1} t\right]. \tag{26}$$

Proof: In consequence of Lemmas 3.1, 3.2, 4.1, and 4.2 we deduce that

$$\left(\int_\Omega |u-\bar{u}|\,dx\right)^2 \le$$

$$\frac{\int_\Omega \bar{u}\,dx}{\xi(m)}\left[H(u_0/\bar{u}) - \frac{1}{16\,L^2}\left(\int_\Omega \bar{u}^{2/m-1}\,dx\right)^{-1}\int_0^t\left(\int_\Omega |u-\bar{u}|\,dx\right)^2 dt'\right]. \tag{27}$$

Gronwall's lemma then implies (26).

□

5 Non negative initial-boundary conditions

In this section we briefly sketch the procedure to adopt when the initial and boundary conditions are non-negative only, provided that the stationary solution $\bar{u}(x)$ is not identically zero on $\bar{\Omega}$. This corresponds to suppose that $(\varphi_+ + \varphi_-) > 0$.

We consider two cases. In the first one, the boundary conditions are strictly positive, but the initial data can vanish somewhere on Ω. In the second one we admit the possibility of a vanishing boundary condition in $x = -L$ or in $x = L$.

We note that, while in the former case the result is a simple corollary of Theorem 4.3, in the latter the regularity of the initial data plays an important role.

5.1 Non-negative initial data

Let $u(x,t)$ be the solution of Equation (1)-(3), with admissible non-negative initial datum $u(x,0)$, and boundary conditions $u(-L,t) = \varphi_-$ and $u(L,t) = \varphi_+$ for all $t > 0$, such that $u_0(-L) = \varphi_-$ and $u_0(L) = \varphi_+$, where φ_+ and φ_- are strictly positive.

To prove the exponential decay also in this case, we lift up the initial data of a small quantity, depending on a parameter $\varepsilon > 0$ and vanishing as $\varepsilon \to 0$, and then pass to the limit.

We denote by $u_\varepsilon(x,t)$ the solution of the lifted problem, where the initial data is

$$u_\varepsilon(x,0) = u_{0,\varepsilon}(x) = u_0(x) + \varepsilon \cos\left(\frac{\pi}{2L}x\right).$$

The corresponding stationary solutions remains unchanged. Thanks to Theorem 4.3,

$$\|u_\varepsilon(x,t) - \bar{u}(x)\|_1^2 \leq$$
$$\frac{\int_\Omega \bar{u}\,dx}{\xi(m)} H(u_{0,\varepsilon}/\bar{u}) \exp\left[-\frac{\int_\Omega \bar{u}\,dx}{16\,L^2\xi(m)}\left(\int_\Omega \bar{u}^{2/m-1}\,dx\right)^{-1} t\right].$$

In the sub-linear case,

$$H(u_{0,\varepsilon}/\bar{u}) = \int_\Omega \left[\frac{1}{m(2-m)(3-m)} f_\varepsilon^{3-m} - \frac{1}{m(2-m)} f_\varepsilon + \frac{1}{m(3-m)}\right] \bar{u}\,dx,$$

with $f_\varepsilon = u_{0,\varepsilon}(x)/\bar{u}$.

Since f_ε decreases as $\varepsilon \to 0^+$, and the integrand is continuous and well defined on $\bar{\Omega}$, we can apply the monotone convergence theorem to obtain

$$\lim_{\varepsilon\to 0^+} \int_\Omega f_\varepsilon^{3-m} \bar{u}\,dx = \int_\Omega \left[\frac{u_0(x)}{\bar{u}}\right]^{3-m} \bar{u}\,dx.$$

The same argument shows that also

$$\lim_{\varepsilon\to 0^+}\int_\Omega f_\varepsilon \bar{u}\,dx = \int_\Omega u_0\,dx.$$

Hence,

$$\lim_{\varepsilon\to 0} H(u_{0,\varepsilon}/\bar{u}) = H(u_0/\bar{u}).$$

The super-linear case can be equally treated. We proved

Theorem 5.1 *Let $u(x,t)$ be the solution of problem (1)-(3), where the initial data are admissible, φ_- and φ_+ are strictly positive, and $H(u_0/\bar{u})$ is bounded. Then $u(x,t)$ decays exponentially fast towards the stationary solution $\bar{u}(x)$ in L^1-norm, and bound (26) holds.*

5.2 Non-negative initial-boundary conditions

More involved is the case when we allow that a boundary condition can be zero, which is the most general situation taken into account in the paper.

In what follows, we will suppose that the vanishing boundary conditions is in $x = -L$ (i. e. $\varphi_- = 0$); the other case is specular.

Denote by $u(x,t)$ the solution of Equation (1)-(3), with admissible initial data $u(x,0) = u_0(x) \geq 0$ and boundary conditions $u(-L,t) = 0$ and $u(L,t) = \varphi_+$ for all $t > 0$, such that $u_0(-L) = 0$ and $u_0(L) = \varphi_+$, where $\varphi_+ > 0$.

Given a positive constant $\varepsilon < \varphi_+$, we introduce a lifted problem where $\varphi_- = \varepsilon$. The corresponding stationary solutions are given by

$$\bar{u}^m(x) = ax + b \quad \text{and} \quad \bar{u}^m_\varepsilon(x) = a_\varepsilon x + b_\varepsilon$$

respectively, where

$$a_\varepsilon = \frac{1}{2L}[\varphi_+^m - \varepsilon^m], \quad b_\varepsilon = \frac{1}{2}[\varphi_+^m + \varepsilon^m].$$

Moreover, we lift the initial value $u_0(x)$ in the following way

$$u_{0,\varepsilon}(x) = \bar{u}_\varepsilon(x), \quad \text{if} - L \leq x < -L + \varepsilon\,;$$

$$u_{0,\varepsilon}(x) = u_\varepsilon^*(x), \quad \text{if} - L + \varepsilon \le x < -L + \varepsilon + \varepsilon^{m+1};$$

In the above formula $0 \le u_\varepsilon^*(x) \le C$ is any function in $C(\bar{\Omega})$ which connects continuously $\bar{u}_\varepsilon(-L+\varepsilon)$ with $u_0(-L+\varepsilon+\varepsilon^{m+1})$. Finally

$$u_{0,\varepsilon}(x) = u_0(x), \quad \text{if} - L + \varepsilon + \varepsilon^{m+1} \le x \le L\,;$$

Let $u_\varepsilon(x,t)$ be the solution of the lifted problem, with initial data $u_\varepsilon(x,0) = u_{0,\varepsilon}(x)$ and boundary conditions $u_\varepsilon(-L,t) = \varepsilon$ and $u_\varepsilon(L,t) = \varphi_+$, where $\varepsilon > 0$. Since $u_\varepsilon(x,t)$ satisfies the conditions of Theorem 5.1, we have immediately that

$$\|u_\varepsilon(x,t) - \bar{u}_\varepsilon(x)\|_1^2 \le$$
$$\frac{\int_\Omega \bar{u}_\varepsilon\,dx}{\xi(m)} H(u_{0,\varepsilon}/\bar{u}_\varepsilon) \exp\left[-\frac{\int_\Omega \bar{u}_\varepsilon\,dx}{16\,L^2\xi(m)} \left(\int_\Omega \bar{u}_\varepsilon^{2/m-1}\,dx\right)^{-1} t\right].$$

Since all integrals containing $\bar{u}_\varepsilon$ converge to the right limit as $\varepsilon \to 0$, the only problem comes out from the limit of the lifted entropy, which in the super-linear case read

$$H(u_{0,\varepsilon}/\bar{u}_\varepsilon) = \frac{1}{m+1}\int_\Omega \left(\frac{u_{0,\varepsilon}}{\bar{u}_\varepsilon}\right)^{m+1} \bar{u}_\varepsilon\,dx - \int_\Omega u_{0,\varepsilon}\,dx + \frac{m}{m+1}\int_\Omega \bar{u}_\varepsilon\,dx.$$

The linear part converges to the right limit as $\varepsilon \to 0$. For the nonlinear term, by construction we have the bounds

$$\int_{-L}^{-L+\varepsilon} \left(\frac{u_{0,\varepsilon}}{\bar{u}_\varepsilon}\right)^{m+1} \bar{u}_\varepsilon\,dx = \int_{-L}^{-L+\varepsilon} \bar{u}_\varepsilon\,dx \le \varphi_+\varepsilon,$$

and

$$\int_{-L+\varepsilon}^{-L+\varepsilon+\varepsilon^{m+1}} \left(\frac{u_{0,\varepsilon}}{\bar{u}_\varepsilon}\right)^{m+1} \bar{u}_\varepsilon\,dx \le C^{m+1}\int_{-L+\varepsilon}^{-L+\varepsilon+\varepsilon^{m+1}} \frac{1}{\bar{u}_\varepsilon^m}\,dx \le C^{m+1}\varepsilon,$$

where the last inequality follows considering that $\bar{u}_\varepsilon(x)^m \ge \varepsilon^m$. Finally, since $\bar{u}_\varepsilon \ge \bar{u}$,

$$\int_{-L+\varepsilon+\varepsilon^{m+1}}^{L} \left(\frac{u_{0,\varepsilon}}{\bar{u}_\varepsilon}\right)^{m+1} \bar{u}_\varepsilon\,dx = \int_{-L+\varepsilon+\varepsilon^{m+1}}^{L} \left(\frac{u_0}{\bar{u}}\right)^{m+1} \bar{u}\left(\frac{\bar{u}}{\bar{u}_\varepsilon}\right)^{m} dx \le$$

$$\int_{-L+\varepsilon+\varepsilon^{m+1}}^{L}\left(\frac{u_0}{\bar{u}}\right)^{m+1}\bar{u}\,dx \leq \int_{-L}^{L}\left(\frac{u_0}{\bar{u}_\varepsilon}\right)^{m+1}\bar{u}\,dx.$$

Grouping these inequalities together we obtain the bound

$$H(u_{0,\varepsilon}/\bar{u}_\varepsilon) \leq H(u_0/\bar{u}) + o(\varepsilon). \tag{28}$$

On the other hand, Fatou's lemma implies

$$H(u_0/\bar{u}) \leq \lim_{\varepsilon\to 0} H(u_{0,\varepsilon}/\bar{u}_\varepsilon).$$

Hence we proved that, for the super-linear case

$$H(u_0/\bar{u}) = \lim_{\varepsilon\to 0} H(u_{0,\varepsilon}/\bar{u}_\varepsilon) \tag{29}$$

Since the sub-linear case can be equally treated, the following theorem holds.

Theorem 5.2 *Let $u(x,t)$ be the solution of problem* (1)-(3)*, where the initial data u_0 is admissible, $\varphi_- + \varphi_+ > 0$ and $H(u_0/\bar{u})$ is bounded. Then $u(x,t)$ decays exponentially fast towards the stationary solution $\bar{u}(x)$ in L^1-norm, and bound* (26) *holds.*

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