WEALTH REDISTRIBUTION IN CONSERVATIVE LINEAR KINETIC MODELS WITH TAXATION

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Abstract. We introduce and discuss kinetic models for wealth distribution which include both taxation and uniform redistribution. The evolution of the continuous density of wealth obeys a linear Boltzmann equation where the background density represents the action of an external subject on the taxation mechanism. The case in which the mean wealth is conserved is analyzed in full details, by recovering the analytical form of the steady states. These states are probability distributions of convergent random series of a special structure, called perpetuities. Among others, Gibbs distribution appears as steady state in case of total taxation and uniform redistribution.

Keywords. Wealth and income distributions, kinetic models, Boltzmann equation.

1. Introduction

Various concepts and techniques of statistical mechanics have been fruitfully applied for years to a wide variety of complex extended systems, physical and otherwise, in an effort to understand the emergent properties appearing in them. Economics is, by far, one of the complex extended systems to which methods borrowed from statistical mechanics for particle systems have been applied [7, 8, 9, 11, 12, 19, 20, 23]. In most of the models introduced so far, the trading mechanism leaves the total mean wealth unchanged. Then, a substantial difference on the final behavior of the model (presence or not of tailed steady states) can be observed depending on the fact that binary trades are pointwise conservative, or conservative in the mean [13, 21]. The asymptotic distribution of wealth, however, depends completely on the microscopic structure of binary trades. Other kinetic models have been recently proposed, which, while maintaining the kinetic description, introduce more sophisticated rules for trading. For example, a description of the behavior of a stock price has been developed by Cordier, Pareschi and Piatecki in [10]. Further, there have been efforts to include non-microscopic effects, like global taxation (and subsequent redistribution), in recent works of Guala [18], Pianegonda, Iglesias, Abramson and Vega [22], Garibaldi, Scalas and Viarengo [16] and Bisi, Spiga and the present author [3].

The kinetic model in [3] consists in a nonlinear kinetic equation of Boltzmann type, similar to the ones introduced in [21]. The novelty was to introduce a simple taxation mechanism at the level of the single binary trade to generate a portion of the mean wealth of the society, totally redistributed to agents, to maintain the total wealth constant. The mechanism of redistribution was assumed sufficiently flexible to be able to redistribute to agents a constant amount of wealth independently of the wealth itself, or to redistribute proportionally (or inversely proportionally) to their wealth. The nonlinearity of the model, however, did not allow to obtain the resulting steady distributions explicitly, not to relate the structure of these states to the taxation and redistribution mechanism.

In what follows we will introduce a linear kinetic model, reminiscent of the taxation and redistribution mechanism introduced in [3], in which taxation and redistribution depend of the action of an external (fixed in time) subject, which is nothing but the fixed background of field particles in kinetic theory, adapted to the present situation. Indeed, the wealth distribution of the market is here driven by the *collisions* undergone with a background $M(w)$, and the wealth distribution function will be consequently governed by a single linear integro-differential Boltzmann equation [6]. The uniform redistribution of the wealth which is not restituted in a single trade with the

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background, is uniformly redistributed among agents. Consequently, the total mean wealth is maintained constant in time, and the wealth distribution is shown to converge towards a steady profile, which depends both of the background density and the redistribution operator. The mathematical description will take advantage from the linearity of the model, and from the possibility to make use of the Fourier transform version of the kinetic equation [4]. Particular cases lead to an explicit expression of the steady profile. In particular, a total taxation plus a uniform redistribution is responsible of convergence towards Gibbs distribution. In all the other cases, the steady states are shown to be probability distributions of convergent series of a special structure, called perpetuities. Steady states of this structure are known to occur in the realm of insurance and finance as a sum of discounted payment streams (cfr. [2, 17] and the references therein).

The paper is organized as follows. In the next section we introduce the model which is described by a linear Boltzmann equation with a transport term, and its main features are discussed in some detail. Mathematical aspects related to the well posedness of the problem and to the uniqueness of the steady states are analyzed in Section 3. The representation of these states as perpetuities, as well as particular exactly solvable cases are discussed in Section 4.

2. Linear kinetic models with redistribution

The goal of the forthcoming kinetic model is to describe the evolution of the distribution of wealth in a population of agents by means of *microscopic* interactions with a fixed background. In this picture each trade is interpreted as an interaction where a fraction of the money moves to the background, which at the same time can restitute a fraction of its money. In order to mimic taxation, the balance of the mean wealth is negative, and in absence of redistribution, the mean wealth of the population decreases. Hence, a uniform redistribution operator is introduced into the model [3], to keep the mean wealth constant. In consequence of these trade rules, it is expected to obtain for large time a stationary wealth distribution $f_\infty(v)$.

The study of the time-evolution of the wealth distribution among individuals in a simple economy, together with a reasonable explanation of the formation of tails in this distribution has been recently achieved by means of kinetic collision-like models in [21] (see also [13]). The time evolution of the wealth distribution $f(v,t)$ is based on the assumption of collision-like trade events. Let $M(v),\ v \geq 0$ denote the probability distribution of wealth of the (fixed) background. We will assume moreover that $M(v)$ has a bounded mean wealth, so that

$$\int_{\mathbb{R}_+} M(v)\, dv = 1\,; \quad \int_{\mathbb{R}_+} vM(v)\, dv = m_B \tag{2.1}$$

The interaction with the background will be described by a trade in which the agent with wealth v comes out with wealth

$$v^* = (1-\epsilon)v + \delta w \tag{2.2}$$

In (2.2) $0 < \epsilon \leq 1$ and $\delta \leq \epsilon$ are positive constants. In consequence of the interaction (2.2), each agent will assign a percentage $(1-\epsilon)v$ of his wealth v to the external background, which on the other hand will restitute part of its wealth w to the agent. It is clear that the structure of the background will induce a certain taxation policy. For example, if we assume that the background is a random variable uniformly distributed on the interval $(0, a)$, where $a > 0$ is a fixed constant, if the wealth v of the agent is bigger than a, the agent will come out of the interaction with a wealth v^* less than v. Hence, interactions with the background are favorable to agents with small wealth.

In a suitable scaling, the effect of collisions can be quantitatively described by a Boltzmann-like equation which is fruitfully written in weak form. It corresponds to say that the solution $f(v,t)$ satisfies, for all smooth functions $\phi(v)$

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\, dv =$$

$$\int_{\mathbb{R}_+^2} \big(\phi(v^*) - \phi(v)\big) f(v,t)M(w)\, dv\, dw + (\epsilon\, m(t) - \delta\, m_B)\int_{\mathbb{R}_+} f(v,t)\phi'(v)\, dv. \tag{2.3}$$

The right-hand side of equation (2.3) consists of two parts. The first one is the linear interaction operator, while the second is a transport operator which is responsible of a (uniform) redistribution among agents (cfr. [3] for details). Note that (2.3) implies that $f(v,t)$ remains a probability density if it so initially

$$\int_{\mathbb{R}_+} f(v,t)\,dv = \int_{\mathbb{R}_+} f_0(v)\,dv = 1. \tag{2.4}$$

Moreover, on the basis of (2.2), choosing $\phi(v) = v$ shows that also the total mean wealth is preserved in time

$$m(t) = \int_{\mathbb{R}_+} vf(v,t)\,dv = \int_{\mathbb{R}_+} vf_0(v)\,dv = m(0). \tag{2.5}$$

Consequently, without loss of generality, in what follows, we assign both to the background and to the initial density a unit mean

$$\int_{\mathbb{R}_+} vM(v)\,dv = \int_{\mathbb{R}_+} vf_0(v)\,dv = 1. \tag{2.6}$$

In addition, we assume further that the background density has some moment bigger than one bounded, typically the second

$$\int_{\mathbb{R}_+} v^2 M(v)\,dv = \sigma^2 < +\infty \tag{2.7}$$

By (2.6), the Boltzmann equation assumes the simpler form

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\,dv = \\ &\int_{\mathbb{R}_+^2} \big(\phi(v^*) - \phi(v)\big) f(v,t)M(w)\,dv\,dw + (\epsilon - \delta)\int_{\mathbb{R}_+} f(v,t)\phi'(v)\,dv.\end{aligned} \tag{2.8}$$

Note that, since by construction $\delta \le \epsilon$, part of the money that comes out from the trades with the background is redistributed uniformly. Thank to (2.7), if the initial density f_0 has bounded second moment, it holds

$$\frac{d}{dt}\int_{\mathbb{R}_+} v^2 f(v,t)\,dv = -(2\epsilon - \epsilon^2)\int_{\mathbb{R}_+} v^2 f(v,t)\,dv + \delta^2\sigma^2 + 2\epsilon(1-\delta).$$

Hence, the second moment of the solution remain bounded, and

$$\int_{\mathbb{R}_+} v^2 f(v,t)\,dv \le \max\left\{\int_{\mathbb{R}_+} v^2 f_0(v)\,dv; \frac{2\epsilon(1-\delta) + \delta^2\sigma^2}{2\epsilon - \epsilon^2}\right\} \tag{2.9}$$

The two extremal cases, corresponding to $\delta = 0$ ($\delta = \epsilon$ respectively) give rise to a noticeable simplification. If $\delta = 0$, the outgoing wealth does not depend of the background M, and the Boltzmann equation simplifies to

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\,dv = \int_{\mathbb{R}_+} \big(\phi(v^*) - \phi(v)\big) f(v,t)\,dv + \epsilon\int_{\mathbb{R}_+} f(v,t)\phi'(v)\,dv. \tag{2.10}$$

In this case (2.10) does not depend on the background, and the post-interaction wealth is simply

$$v^* = (1-\epsilon)v. \tag{2.11}$$

The Boltzmann equation (2.10) represents in this case the evolution of a wealth density in which a constant part of the wealth is taken away in each interaction (taxation) and the same portion of wealth is redistributed uniformly among agents.

If on the contrary $\delta = \epsilon$, the redistribution operator in (2.8) disappears, and the Boltzmann equation simplifies to

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\,dv = \int_{\mathbb{R}_+^2} \big(\phi(v^*) - \phi(v)\big) f(v,t)M(w)\,dv\,dw. \tag{2.12}$$

In (2.12) the post-interaction wealth is now

$$v^* = (1-\epsilon)v + \epsilon w. \tag{2.13}$$

Denote by $\mathcal{M}_0$ the space of all probability measures in $\mathbb{R}_+$, and for $\alpha \geq 1$ by

$$\mathcal{M}_\alpha = \left\{ \mu \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |v|^\alpha \mu(dv) < +\infty \right\}, \tag{2.14}$$

the space of all Borel probability measures of finite moment of order α, equipped with the topology of the weak convergence of measures.

A weak solution to the initial value problem for equation (2.8), with initial probability density $f_0(v) \in \mathcal{M}_\alpha, \alpha \geq 1$ is a probability density $f \in C^1(\mathbb{R}_+, \mathcal{M}_\alpha)$ which satisfies (2.8) for $t > 0$ and all smooth functions ϕ, and the initial condition is realized as

$$\lim_{t\to 0^+} \int_{\mathbb{R}_+} \phi(v) f(v,t)\, dv = \int_{\mathbb{R}_+} \phi(v) f_0(v)\, dv \tag{2.15}$$

for all ϕ.

3. Preliminary results

As it is usually done for this type of kinetic models, most of the analytical properties of the solution are obtained by means of Fourier analysis. From the kinetic theory of rarefied gases it is well-known that Boltzmann-like equations are often conveniently studied in Fourier space, see e.g. [4]. This is particularly true for the model under consideration. In fact, the weak formulation (2.8) is equivalent to the Fourier transformed equation

$$\frac{\partial \widehat{f}(\xi,t)}{\partial t} = \widehat{f}((1-\epsilon)\xi, t)\hat{M}(\delta\xi) - \widehat{f}(\xi,t) - i(\epsilon-\delta)\xi\widehat{f}(\xi,t), \tag{3.1}$$

where $\widehat{f}(\xi,t)$ is the Fourier transform of $f(v,t)$,

$$\widehat{f}(\xi,t) = \int_{\mathbb{R}_+} e^{-i\xi v}\, f(v,t)\, dv.$$

In (3.1) we used the fact that $M(v)$ is a probability density function, so that $\hat{M}(0) = 1$. The initial conditions (2.6) turn into

$$\widehat{f_0}(0) = 1 \quad \text{and} \quad \widehat{f_0}'(0) = -\, i.$$

The existence of a solution to equation (3.1) can be seen easily using the same methods available for the linear Boltzmann equation [6]. In order to show uniqueness of the solution, we use the idea from [15]. It is proven that the operator on the right-hand side of (3.1) is Lipschitz continuous in a suitable metric.

Introduce for arbitrary $s > 0$ the *Fourier metric* on $\mathcal{M}_\alpha$ by

$$d_s(f,g) = \sup_{\xi\in\mathbb{R}} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s}. \tag{3.2}$$

Write $s = m + \beta$, where m is an integer and $0 \leq \beta < 1$. Two functions f and g have a finite distance, $d_s(f,g) < \infty$, if and only if their moments up to order m agree.

The metric (3.2) has been introduced in [15] to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. There, the case $s = 2 + \beta$ with $\beta > 0$ was considered. Further applications of d_s to various kinetic models in economy can be found in [13, 14]. In order to show Lipschitz-continuity, let us define

$$\hat{g}(\xi,t) = \widehat{f}(\xi,t) \exp\{i(\epsilon-\delta)\xi t\}. \tag{3.3}$$

Then it is shown easily that, whenever $\widehat{f}$ satisfies (2.8), $\hat{g}$ satisfies

$$\frac{\partial \widehat{g}(\xi,t)}{\partial t} + \hat{g}(\xi,t) = \hat{g}((1-\epsilon)\xi,t)\hat{M}(\delta\xi)\exp\{i(\epsilon-\delta)\epsilon\xi t\}. \tag{3.4}$$

Therefore, if $\hat{g}_1(\xi,t)$ and $\hat{g}_2(\xi,t)$ are two solutions to equation (3.4) corresponding to the initial data $\hat{g}_{0,1}(\xi)$ and $\hat{g}_{0,2}(\xi)$, it holds

$$\frac{\partial}{\partial t}\frac{\hat{g}_1(\xi,t)-\hat{g}_2(\xi,t)}{|\xi|^s}+\frac{\hat{g}_1(\xi,t)-\hat{g}_2(\xi,t)}{|\xi|^s}=$$

$$=\frac{\hat{g}_1((1-\epsilon)\xi,t)-\hat{g}_2((1-\epsilon)\xi,t)}{|\xi|^s}\hat{M}(\delta\xi)\exp\left\{i(\epsilon-\delta)\epsilon\xi t\right\}.$$

On the other hand, since $|\hat{M}(\xi|\leq 1$

$$\sup_{\xi\in\mathbb{R}}\left|\frac{\hat{g}_1((1-\epsilon)\xi,t)-\hat{g}_2((1-\epsilon)\xi,t)}{|\xi|^s}\hat{M}(\delta\xi)\exp\left\{i(\epsilon-\delta)\epsilon\xi t\right\}\right|\leq$$

$$\sup_{\xi}\left|\frac{\hat{g}_1((1-\epsilon)\xi,t)-\hat{g}_2((1-\epsilon)\xi,t)}{|\xi|^s}\right|=(1-\epsilon)^s d_s(g_1(t),g_2(t)). \tag{3.5}$$

Thanks to (3.5),

$$\left|\frac{\partial}{\partial t}\frac{\hat{g}_1(\xi,t)-\hat{g}_2(\xi,t)}{|\xi|^s}+\frac{\hat{g}_1(\xi,t)-\hat{g}_2(\xi,t)}{|\xi|^s}\right|\leq(1-\epsilon)^s d_s(g_1(t),g_2(t)),$$

that, by Gronwall's inequality implies

$$d_s(g_1(t),g_2(t))\leq d_s(g_{0,1},g_{0,2})\exp\left\{\left((1-\epsilon)^s-1\right)t\right\}. \tag{3.6}$$

Considering that, by definition

$$d_s(g_1(t),g_2(t))=d_s(f_1(t),f_2(t)) \tag{3.7}$$

while $g(v,t=0)=f(v,t=0)=f_0(v)$, we proved

Theorem 3.1. *Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation (2.8), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (2.6). Let $s>0$ be such that $d_s(f_{1,0},f_{2,0})$ is finite. Then, for all times $t\geq 0$ the d_s-distance of f_1 and f_2 decays exponentially in time, and*

$$d_s(f_1(t),f_2(t))\leq\exp\left\{\left((1-\epsilon)^s-1\right)t\right\}d_s(f_{1,0},f_{2,0}). \tag{3.8}$$

We remark that in the present case, since the solution to equation (2.8) conserves both mass and momentum, the natural setting to apply Theorem 3.1 is to choose initial data with bounded first moment, and, consequently to set $s=1+\beta$, where $0\leq\beta<1$. Theorem 3.1 has an important consequence.

Corollary 3.2. *Let $f(v,t)$ be a weak solution of the Boltzmann equation* (2.8)*, which has initially finite moments up to order* 2*. Then f converges exponentially fast in d_s to a steady state f_∞,*

$$d_s(f(t),f_\infty)\leq d_s(f_0,f_\infty)\exp\left\{\left((1-\epsilon)^s-1\right)t\right\}. \tag{3.9}$$

In addition f_∞ has mean wealth equal to 1*, and it is the only steady state with this mean wealth.*

To prove Corollary 3.2, choose some $s\in(1,2)$. The initial datum of f possesses a moment of order $2>s$. Then the solution $f(v,t)$ has the Cauchy-property w.r.t. $t\geq 0$, i.e., for each $\epsilon>0$, there exists a time $T(\epsilon)$ such that

$$d_s\big(f(t_1),f(t_2)\big)\leq\epsilon \text{ for all } t_1,t_2\geq T(\epsilon). \tag{3.10}$$

To prove this, let $t_1>t_2\geq 0$ be arbitrary, and write $t_1=t_2+n\Delta$ with some $\frac{1}{2}\leq\Delta\leq 1$ and $n\in\mathbb{N}$. If $r=1-(1-\epsilon)^s$, by the triangle inequality and the contraction estimate from Theorem

3.1,

$$\begin{aligned} d_s\big(f(t_1), f(t_2)\big) &\le \sum_{k=0}^{n-1} d_s\big(f(t_2+(k+1)\Delta), f(t_2+k\Delta)\big) \\ &\le \sum_{k=0}^{n-1} e^{-r(t_2+k\Delta)} d_s\big(f(\Delta), f(0)\big) \\ &\le \frac{e^{-rt_2}}{1-e^{-r\Delta}} d_s\big(f(\Delta), f_0\big) \le e^{-rt_2} \sup_{1/2\le \Delta' \le 1} \frac{d_s\big(f(\Delta'), f_0\big)}{1-e^{-r\Delta'}}. \end{aligned}$$

As $f(t)$ is continuous in the d_s-metric, the last supremum is finite, which implies (3.10). In addition, the first moment of $f(v,t)$ equals 1 for all $t \ge 0$. The Cauchy property together with this uniform bound on moments is enough to conclude convergence of $f(v,t)$ to a limit $f_\infty(v)$ in d_s as $t \to +\infty$, see [5]. This limit is a probability density with first moment less or equal to 1. Notice that $d_s\big(f(t), f_0\big)$ is always finite since $s < 2$ and the first moment (mean wealth) is conserved under evolution. Moreover, by (2.9), the moment of order s remains uniformly bounded. It follows that $f(t)$ converges in d_s to a limit distribution $f_\infty(v)$, which is normalized and has the same first moment as the $f(t)$.

This convergence implies that f_∞ is a steady state for the kinetic equation (2.8). Indeed, denote by $f_\infty(t)$ the solution to (2.8) with initial datum f_∞, then Theorem 3.1 gives

$$\begin{aligned} d_s\big(f_\infty(t), f_\infty\big) &\le d_s\big(f_\infty(t), f(t+T)\big) + d_s\big(f(t+T), f_\infty\big) \\ &\le e^{-rt} d_s\big(f_\infty, f(T)\big) + d_s\big(f(t+T), f_\infty\big). \end{aligned}$$

The last expression can be made arbitrarily small by choosing T large enough, so that $f_\infty(t) = f_\infty$ for all $t \ge 0$. In fact, f_∞ is the *only* steady state with the respective value of the first moment; for if f'_∞ is another steady state with the same first moment, then $d_s(f_\infty, f'_\infty)$ is finite, and so, invoking Theorem 3.1 again,

$$d_s(f_\infty, f'_\infty) \le e^{-r} d_s(f_\infty, f'_\infty),$$

which forces $f_\infty = f'_\infty$.

4. Steady states as perpetuities

The result of Corollary 3.2 gives both the existence and uniqueness of a stationary profile for the Boltzmann equation (2.8). This profile satisfies the equation

$$\widehat{f}_\infty((1-\epsilon)\xi)\hat{M}(\delta\xi) - \widehat{f}_\infty(\xi) - i(\epsilon-\delta)\xi\widehat{f}_\infty(\xi) = 0. \tag{4.11}$$

Equation (4.11) can be rewritten as

$$\widehat{f}_\infty(\xi) = \frac{\hat{M}(\delta\xi)}{1+i(\epsilon-\delta)\xi}\widehat{f}_\infty((1-\epsilon)\xi), \tag{4.12}$$

which, due to the fact that $f_\infty(0) = 1$, recursively gives

$$\widehat{f}_\infty(\xi) = \prod_{k=0}^{\infty} \frac{\hat{M}\left(\delta(1-\epsilon)^k\xi\right)}{1+i(\epsilon-\delta)(1-\epsilon)^k\xi}. \tag{4.13}$$

This formula completely identifies the (unique) steady state to equation (2.8). Let us describe in some detail both the structure and the properties of $f_\infty(v)$. By direct inspection, it can be easily verified that, for a given constant $\alpha > 0$ the function

$$\hat{G}_\alpha(\xi) = \frac{1}{1+i\alpha\xi}, \qquad \xi \in \mathbb{R} \tag{4.14}$$

is the Fourier transform of the Gibbs density ($G_1 = G$) of mean α

$$G_\alpha(v) = \frac{1}{\alpha}\exp\left\{-\frac{v}{\alpha}\right\}, \quad v \ge 0; \qquad G_\alpha(v) = 0 \quad v < 0. \tag{4.15}$$

Consequently, if Z is a random variable with law f_∞, the law of Z, defined in (4.12), is a distributional fixed point of the equation

$$Z =^d \delta X + (\epsilon - \delta)Y + (1-\epsilon)Z, \tag{4.16}$$

where as usual the variables X, Y and Z are assumed to be independent each other. In (4.16) the random variable X is distributed according to $M(v)$, while Y is distributed according to the Gibbs distribution G. The law of Z is often called perpetuity due to its occurrence in the realm of insurance and finance as a sum of discounted payment streams (cfr. [2, 17] and the references therein). As discussed in [2], it is quite difficult to extract properties on perpetuities in general. What is proven is that such distributions are of pure type, namely degenerate, absolutely continuous, or continuously singular. Moreover, necessary and sufficient criteria are presented in [2] for the finiteness of polynomial as well as exponential moments.

In our case, due to the fact that we know the laws of the random variables X and Y, one can extract more information about regularity and behavior at the point $v = 0$.

The simplest case corresponds to the extremal choice $\delta = 0, \epsilon = 1$, which leads to the equation (2.10) with $\epsilon = 1$. This case corresponds to a total taxation (all the wealth put into the trade moves to the background), and a uniform redistribution. Setting $\delta = 0$ and $\epsilon = 1$ into (4.12) gives

$$\widehat{f}_\infty(\xi) = \frac{1}{1+i\xi}, \tag{4.17}$$

and the steady state is a the Gibbs distribution of unit mean

$$f_\infty(v) = G(v). \tag{4.18}$$

Choosing now $\delta = 0, \epsilon < 1$, which leads to the equation (2.10), we obtain from (4.13) the expression

$$\widehat{f}_\infty(\xi) = \prod_{k=0}^{\infty} \frac{1}{1+i\epsilon(1-\epsilon)^k\xi}, \tag{4.19}$$

namely an infinite convolution of independent Gibbs distributions of means $m_k = \epsilon(1-\epsilon)^k$. In this case regularity of the steady state follows by showing that it belongs to the Sobolev spaces of high degree

$$\|g\|^2_{\dot{H}^\eta(\mathbb{R})} = \int_{\mathbb{R}} |\xi|^{2\eta} |\hat{g}(\xi)|^2 \, d\xi$$

with $\eta > 0$. In fact, since

$$|\hat{G}_\alpha(\xi)| = \left| \frac{1}{1+i\alpha\xi} \right| = \left(\frac{1}{1+\alpha^2|\xi|^2} \right)^{1/2}, \tag{4.20}$$

it follows that, for a given $N \in \mathbb{N}$

$$\left| \prod_{k=0}^{N} \frac{1}{1+i\epsilon(1-\epsilon)^k\xi} \right| = \left(\prod_{k=0}^{N} \frac{1}{1+\epsilon^2(1-\epsilon)^{2k}|\xi|^2} \right)^{1/2} \le \left(\frac{1}{1+c_{N,\epsilon}|\xi|^{2N+2}} \right)^{1/2}. \tag{4.21}$$

In (4.21) the positive constant $c_{N,\epsilon}$ is given by

$$c_{N,\epsilon} = \prod_{k=0}^{N} \epsilon^2(1-\epsilon)^{2k}.$$

Consequently, for any given $N \in \mathbb{N}$,

$$|\widehat{f}_\infty(\xi)|^2 \le \frac{1}{1+c_{N,\epsilon}|\xi|^{2N+2}}, \tag{4.22}$$

and this shows that f_∞ belongs to $\dot{H}^N(\mathbb{R})$. This result guarantees that $f_\infty(v=0) = 0$. In fact, since $f_\infty(v) \in \dot{H}^N$, with $N > 1/2$, by the Sobolev imbedding [1], this regularity is enough to guarantee that the steady state is a bounded and continuous function. Since the trade mechanism (2.2) is such that the post-trade wealth is nonnegative, while the (transport) redistribution operator moves the wealth density on the right, a solution density which is initially distributed on $\mathbb{R}_+$ remains distributed on $\mathbb{R}_+$ at any subsequent time, so that $f(v,t) = 0$ if $v < 0$. The same property thus holds for the steady state f_∞. On the other hand, since $f_\infty(v)$ is continuous in $v=0$, the condition $f_\infty(v=0)=0$ follows.

The same conclusion can be drawn in the case in which $\delta>0$, due to the fact that inequality (4.21) still holds with a different constant

$$c_{N,\epsilon,\delta}=\prod_{k=0}^{N}(\epsilon-\delta)^2(1-\epsilon)^{2k}.$$

The case $\delta=\epsilon$ has to be treated separately. In this case, regularity of the steady state follows provided the background density satisfies

$$|\hat{M}(\xi)|\le\left(\frac{1}{1+\mu|\xi|^2}\right)^\eta, \tag{4.23}$$

for some positive constants μ and η.

As far as the background profile is concerned, the choice $\hat{M}(\xi)=e^{-i\xi}$, which corresponds to a background density which is a Dirac delta function concentrated in $v=1$, leads to exact computations. If $\delta=\epsilon$, the steady profile satisfies

$$\widehat{f}_\infty((1-\epsilon)\xi)e^{-i\epsilon\xi}-\widehat{f}_\infty(\xi)=0, \tag{4.24}$$

which implies $\widehat{f}_\infty(\xi)=e^{-i\xi}$. Therefore, the kinetic equation

$$\frac{d}{dt}\int_{\mathbb{R}_+}f(v,t)\phi(v)\,dv=\int_{\mathbb{R}_+}\big(\phi(v^*)-\phi(v)\big)f(v,t)\,dv, \tag{4.25}$$

with the post-interaction wealth

$$v^*=(1-\epsilon)v+\epsilon, \tag{4.26}$$

has the (unique) solution which converges towards a steady state in which all agents have the same wealth. We proved

Theorem 4.1. *Let $f_\infty(v)$ be the (unique) steady solution of the Boltzmann equation* (2.8)*.Then, for all $0<\epsilon<1,\delta<\epsilon$, f_∞ belongs to $\dot{H}^N(\mathbb{R})$, for all $N>0$, and $f_\infty(v=0)=0$. The same result holds if $\delta=\epsilon$, and the background density satisfies condition (4.23). In case $\epsilon=1$ and $\delta=0$, $f_\infty(v)$ equals the Gibbs distribution $G(v)$ of unit mean. Finally, if the background density is a Dirac delta function concentrated in the mean wealth $v=1$, while $\epsilon=\delta$, $f_\infty(v)$ equals the same Dirac delta function, so that the background density is globally attracting.*

Last, we examine the boundedness of moments of the steady distribution. The computations of Section 2 leading to (2.9) can be easily extended to show that the number of moments which remain bounded are driven by the number of moments of the background which are bounded. In fact, if

$$\int_{\mathbb{R}_+}v^{\bar{s}}M(v)\,dv=m_{\bar{s}}<+\infty,\qquad \bar{s}>2 \tag{4.27}$$

we obtain, for $s\le\bar{s}$

$$\frac{d}{dt}\int_{\mathbb{R}_+}v^sf(v,t)\,dv=$$

$$\int_{\mathbb{R}_+^2}\left[((1-\epsilon)v+\delta w)^s-v^s\right]f(v,t)M(w)\,dv\,dw+(\epsilon-\delta)s\int_{\mathbb{R}_+}v^{s-1}f(v,t)\,dv. \tag{4.28}$$

Let us denote by $[s]$ the entire part of s, so that $s=[s]+\nu$, where $0\le\nu<1$. Given any two constants a and b, the following inequality holds

$$(|a|+|b|)^\nu\le|a|^\nu+|b|^\nu.$$

Therefore

$$((1-\epsilon)v+\delta w)^s=((1-\epsilon)v+\delta w)^{[s]+\nu}\le((1-\epsilon)v+\delta w)^{[s]}\left([(1-\epsilon)v]^\nu+(\delta w)^\nu\right),$$

so that

$$((1-\epsilon)v+\delta w)^s - v^s \le -\left(1-(1-\epsilon)^s\right) v^s + \delta^s w^s + P_{[s]}(v,w), \tag{4.29}$$

and $P_{[s]}$ contains only powers of v and w of order less or equal than $[s]$. Substituting now formula (4.29) into (4.28), and considering that the coefficient of the s-th moment of f is negative, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}_+} v^s f(v,t)\,dv \le -\left(1-(1-\epsilon)^s\right)\int_{\mathbb{R}_+} v^s f(v,t)\,dv + \delta^s \int_{\mathbb{R}_+} w^s M(v)\,dw + M_{[s]}. \tag{4.30}$$

Since the last term $M[s]$ contains only moments of order less or equal than $[s]$, while the coefficient of the s-th moment of $f(v,t)$ is negative, the boundedness of the moment of order $[s]$ implies the uniform boundedness of the moment of order s. Starting from $s=1$ recursively we conclude. The previous analysis shows that the steady state and the background have the same number of moments. Hence, independently of the value of the taxation parameter, a background with Pareto tails produces a steady state with the same tails.

The case $\delta = 0$ leads to a completely different situation. In this case, in fact, we can prove that the steady state has exponential moments bounded. For a given ϵ, let α be such that $\epsilon\alpha < 1$. Then

$$\frac{d}{dt}\int_{\mathbb{R}_+} e^{\alpha v} f(v,t)\,dv = \int_{\mathbb{R}_+}\left(e^{\alpha(1-\epsilon)v} - e^{\alpha v}\right) f(v,t)\,dv + \epsilon\alpha \int_{\mathbb{R}_+} e^{\alpha v} f(v,t)\,dv =$$

$$\int_{\mathbb{R}_+} e^{\alpha v}\left(e^{-\epsilon\alpha v} - 1 + \epsilon\alpha\right) f(v,t)\,dv. \tag{4.31}$$

Since

$$v \ge \bar v = \frac{1}{\epsilon\alpha}\log\frac{2}{1-\epsilon\alpha}$$

implies

$$e^{-\epsilon\alpha v} \le \frac{1-\epsilon\alpha}{2},$$

we have

$$\int_{\mathbb{R}_+} e^{\alpha v}\left(e^{-\epsilon\alpha v} - 1 + \epsilon\alpha\right) f(v,t)\,dv \le \int_{v\le\bar v} e^{\alpha v}\epsilon\alpha f(v,t)\,dv + \int_{v>\bar v} e^{\alpha v}\left(e^{-\epsilon\alpha v} - 1 + \epsilon\alpha\right) f(v,t)\,dv \le$$

$$\epsilon\alpha\left(\frac{2}{1-\epsilon\alpha}\right)^{1/\epsilon} - \frac{1-\epsilon\alpha}{2}\int_{\mathbb{R}_+} e^{\alpha v} f(v,t)\,dv. \tag{4.32}$$

From (4.32) the uniform boundedness of the exponential moments follows. Hence we have

Theorem 4.2. *Let $f_\infty(v)$ be the (unique) steady solution of the Boltzmann equation (2.8). Then, for all $0 < \epsilon < 1, 0 < \delta < \epsilon$, f_∞ has the same moments of the background. If $\delta = 0$, f_∞ has exponential moments of order α, for all $\alpha < 1/\epsilon$, that is*

$$\int_{\mathbb{R}_+} e^{\alpha v} f(v,t)\,dv < +\infty, \qquad \alpha < 1/\epsilon.$$

Theorem 4.2, coupled with theorem 3.2, provides an interesting and somewhat counterintuitive result for the case described by the Boltzmann equation (2.8). This case, which corresponds to a fixed taxation of wealth, with subsequent uniform redistribution, is such that the steady profile has exponential moments of higher order in presence of small values of the taxation parameter ϵ. In other words, the steady profile is more picked around the mean wealth, and consequently closer to a Dirac delta function when the taxation is small. On the other hand, if this is the case, the relaxation time predicted by theorem 3.2 is slowed down by the presence of $1-(1-\epsilon)^s$. Therefore, if we assume that the ideal steady distribution is obtained when all agents reach the same wealth, we need to assume a very small taxation parameter. On the other hand, this will imply a very slow rate of convergence, and the steady profile can not be obtained in reasonable times.

5. Conclusions

In this paper, we introduced and discussed linear kinetic models for wealth distribution in a simple market economy, which are able to reproduce the salient features of the wealth distribution by including taxes to each trading process and redistributing the collected money among the population according to a given background. This model of redistribution allows to move the money obtained from taxation of trades in a uniform way among agents of the market. Steady states of the model are shown to be probability distributions of convergent random series of a special structure, called perpetuities. Various cases can be treated analytically, to give the main properties of the steady distributions. Also, numerical experiments indicate that in most cases, the steady profile is a unimodal density function with a unique maximum, but the rigorous proof of this fact is presently open.

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