# A central limit theorem for solutions of the porous medium equation

G. Toscani*

March 11, 2004

**Abstract**

We study the large–time behavior of the second moment (energy) $E(t) = \frac{1}{2}\int |x|^2 v(x,t)dx$ for the flow of a gas in a $N$-dimensional porous medium with initial density $v_0(x) \geq 0$. The density $v(x,t)$ satisfies the nonlinear degenerate parabolic equation $v_t = \Delta v^m$ where $m > 1$ is a physical constant. Assuming that $\int (v_0^m(x) + |x|^{2+\delta} v_0(x))dx < \infty$ for some $\delta > 0$, we prove that $E(t)$ behaves asymptotically, as $t \to \infty$, like the energy $E_B(t)$ of the Barenblatt-Pattle solution $B(|x|,t)$. This is shown by proving that $E(t)/E_B(t)$ converges to 1 at the (optimal) rate $t^{-2/(N(m-1)+2)}$. A simple corollary of this result is a central limit theorem for the scaled solution $E(t)^{N/2} v(E(t)^{1/2} x, t)$.

**2000 AMS subject classification.** 35K55, 35K60, 35K65, 35B40.

**Key words.** Nonlinear parabolic equations, porous medium equation, long-time behavior of solutions, central limit problem.

**Running title.** Central Limit Problem for the Porous Medium Equation

## 1 Introduction.

This paper is intended to study the large–time behavior of the second moment (energy) of solutions to the porous medium equation. As we shall briefly discuss in the following, the knowledge of the time evolution of the energy in a nonlinear diffusion equation is of paramount importance to reckon the intermediate asymptotics of the solution itself when the similarity is missing. Thus, the present study can be considered as a first step in the validation of a more general conjecture on the large–time asymptotics of a general diffusion equation.

The flow of gas in an $N$-dimensional porous medium equation is classically described by the solution to the Cauchy problem

$$\frac{\partial v}{\partial t} = \Delta v^m, \qquad (x \in \mathbb{R}^N, t > 0), \tag{1}$$

$$v(x, t = 0) = v_0(x) \geq 0, \qquad (x \in \mathbb{R}^N) \tag{2}$$

*Department of Mathematics, University of Pavia, via Ferrata 1, 27100 Pavia, ITALY.

The function $v$ represents the density of the gas in the porous medium and $m > 1$ is a physical constant. To emphasize the analogies between the central limit theorem of probability theory and the intermediate asymptotics problem, without loss of generality, in the rest of the paper we will always assume that $v_0(x)$ is a centered probability density with finite second moment. The solution to equation (1) satisfies mass and momentum conservations, so that

$$\int_{\mathbb{R}^N} v(x,t)\, dx = 1\,; \quad \int_{\mathbb{R}^N} x v(x,t)\, dx = 0\,; \quad t \geq 0. \tag{3}$$

Let us define the energy $E(t)$ of a solution $v(x,t)$ as its second moment

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^2 v(x,t) dx. \tag{4}$$

Then, $E(t)$ increases in time, and its evolution is given by the nonlinear law

$$\frac{dE(t)}{dt} = N \int_{\mathbb{R}^N} v^m(x,t)\, dx \geq 0, \tag{5}$$

which is not explicitly integrable unless $m = 1$. It is well-known that equation (1) admits a family of self-similar solutions (in a weak sense) called Barenblatt-Pattle solutions given by

$$B_c(|x|,t) = (c + \lambda t)^{-N/\lambda} \left( C^2 - \frac{(m-1)}{2m} |x|^2 (c+\lambda t)^{-2/\lambda} \right)_+^{\frac{1}{m-1}} \tag{6}$$

where the constant $c \geq 0$,

$$\lambda = \lambda(N,m) = N(m-1) + 2, \tag{7}$$

and the constant $C$ is fixed by mass conservation

$$\int_{\mathbb{R}^N} B(|x|,t)\, dx = 1, \quad t \geq 0\ .$$

The study of the large–time behavior of the solution to (1), and its convergence towards the self–similar solution (6) is a classical problem that has been intensively studied from many years now. This behavior has been described in dimension 1 in [18, 19, 26], in a $L^\infty$-setting. These results have been subsequently generalized to the case $N > 1$ [12]. The authors identify the limit object and prove convergence without rates. Their method, which relies in showing local uniform convergence of the scaled solution in expanding sets, works for compactly supported solutions, but not if the data are more general (for instance if the data belong to $L^1$). The full proof with data in $L^1$ was done recently by Vázquez in [27]. This paper collects in addition most of the existing results, and represents an excellent survey to the asymptotic problem. Among other results, it is remarked that the assumption of finite moments is crucial to obtain convergence with rate. In fact, it is proved that for general data in $L^1$ no rate can exist.

In [6, 10, 22], a different approach led to new results in the whole space. This methods requires the initial data possess a sufficiently high number of moments (typically $2+\delta$ for $\delta > 0$), and gives an explicit rate of convergence. Instead of working on (1) directly, it was investigated the auxiliary problem of the asymptotic decay towards a steady state of solutions to the (nonlinear) Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} = \operatorname{div}(xu + \nabla u^m), \qquad (x \in \mathbb{R}^N, \tau > 0), \tag{8}$$

$$u(x, t = 0) = u_0(x) \geq 0, \qquad (x \in \mathbb{R}^N) \tag{9}$$

The link between equations (1) and (8) relies in a time dependent scaling which transforms the porous medium equation (1) into the Fokker–Planck equation (8). As it can be easily seen, the kinetic equation (8) has a unique compactly supported equilibrium state $u_\infty(x)$ of mass 1, which coincides with the similarity solution $B_1(|x|, 1)$, namely the Barenblatt–Pattle solution (6) with $c = 1$ evaluated at time $t = 0$. Hence, any result on time asymptotics of the Fokker-Planck like equation, by application of this time-dependent scaling, translates into a result on the time asymptotics of the porous medium equation.

Leaving to the forthcoming section the detailed computations, the time–dependent scaling is given by the relation

$$u(x, \tau(t)) = \alpha^N(t) v(\alpha(t)x, t), \tag{10}$$

where

$$\alpha(t) = (1 + \lambda t)^{1/\lambda}\ ; \quad \tau(t) = \log \alpha(t). \tag{11}$$

This choice of $\alpha(t)$ and $\tau(t)$ implies $\alpha(0) = 1$ and $\tau(0) = 0$. Considering that $\tau(t) \to \infty$ as $t \to \infty$, the convergence results of $u(x, \tau)$ towards $u_\infty(x)$ proven in [6] can be rephrased as a convergence result of $\tilde{v}(x, t) = \alpha^N(t) v(\alpha(t)x, t)$ towards the compactly supported function $B_1(|x|, 1)$. Let us remark that, $\tilde{v}(x, t = 0) = v_0(x)$, and in addition mass and momentum of $\tilde{v}(x, t)$ are conserved, while its energy is not. In analogy with the central limit problem, however, it would be more natural to investigate the large–time behavior of the solution to (1) normalized to have constant mass, momentum and energy,

$$\bar{v}(x, t) = E(t)^{N/2} v(E(t)^{1/2} x, t)\,.$$

The main problem in looking for the large–time behavior of $\bar{v}(x, t)$ is related to the fact that the energy $E(t)$ is not known explicitly. Despite the large amount of research in the field , to our knowledge, the time evolution of the energy of the solution has never been dealt with in details. Since the energy $E_B(t)$ of the Barenblatt solution (6) behaves like $t^{2/\lambda}$, and the intermediate asymptotics of the solution is represented by the Barenblatt $B(\cdot, t)$, it is reasonable to conjecture that the energy $E(t)$ behaves like $t^{2/\lambda}$ for large times. The goal of this note will be to make this result precise. Our main result is that for large–times $E(t)$ behaves like $\alpha^2(t)$, which is proportional to the energy of the Barenblatt solution at the same time $t$. We will prove in fact that $E(t)/E_B(t)$ converges to 1 at an explicit (optimal) rate.

A corollary of this result is the validity of a central limit theorem for the scaled solution $\bar{v}(x, t)$, namely the $L^1$–convergence of $\bar{v}(x, t)$ towards the Barenblatt $B(|x|)$ with unit mass, zero momentum and unit energy, with an explicit convergence rate.

Our main result is a consequence of the analysis of [6], where the convergence towards equilibrium for the (nonlinear) Fokker–Planck equation was concluded using the time monotonicity of the physical entropy. This entropy method has been fruitfully applied to general linear and nonlinear Fokker-Planck type equations [2, 7]. All these results show exponential convergence towards equilibrium in relative entropy. Then, exponential convergence in $L^1$ follows by Csiszar-Kullback type inequalities [9, 20].

We are confident that this type of central limit theorem holds for a general diffusion equation

$$\frac{\partial v}{\partial t} = \Delta \Phi(v), \qquad (x \in \mathbb{R}^N, t > 0), \tag{12}$$

$$v(x,t=0) = v_0(x) \geq 0, \qquad (x \in \mathbb{R}^N) \tag{13}$$

at least when the function $\Phi(r), r > 0$ is a convex function. In this general case, in fact, we can not take advantage from the time rescaling (the existence of similarity variables), and the large–time behavior of the solution is largely unknown in details. Partial results have been obtained recently for functions $\Phi(r)$ which behave like a power close to $r = 0$ [3]. A deeper insight into the conjecture has been recently done in [4]. There, for the first time, the intermediate asymptotics of the solution to (12) for a very large class of non-homogeneous nonlinearities $\phi$ for which long time asymptotics cannot be characterized by self-similar solutions has been described. The existence of a universal asymptotic profile has been characterized by fixed points of certain maps in probability measures spaces, endowed with the Euclidean Wasserstein distance. In dimension 1 the validity of the same conjecture has been analyzed from a numerical point of view in [13]. By means of a simple explicit numerical scheme which is based on an alternative formulation of the problem using the pseudo-inverse of the solution to (12), numerical evidence of the validity of the central limit theorem for various diffusion equations has been derived. Among others, the large-time behavior of the (rescaled) solution to the Buckley-Leverett doubly degenerated equation [1], given by

$$\Phi(v) = \frac{2v^2}{2v^2 + (1-v)^2},$$

has been numerically identified.

The organization of the paper is as follows. Section 2 is devoted to some preliminary results concerning the linear case of the heat equation. In section 3 we report the essential results from paper [6] which are needed for the study of the time evolution of the energy. The evolution of the energy, and its rate of growth are contained in Section 4, which is devoted in addition to the proof of the central limit theorem for the solution of equation (1).

## 2 Central limit theorem for the heat equation

The aim of this section is to focus on the analogies between the convergence towards the Gaussian law in the central limit theorem [15, 16] and the large-time asymptotic of the solution to the heat equation. In particular, we will emphasize the role of the energy (the second moment) in this limit procedure to extend it to nonlinear diffusion equations. In its simplest version, the central limit theorem for a Gaussian random variable, with law $\omega$, consists of finding the set of probability density functions $f$ such that, when dealing with the sum

$$S_n = X_1 + ... + X_n - \sum_{i=1}^{n} m_i \tag{14}$$

of the independent, identically distributed random variables $X_i$ with common density function $f$ of finite variance $\sigma$, the density $\bar{f}_n$ of the (scaled) sum $\mathcal{S}_n = S_n/n^{1/2}$ converges to $\omega$. In (14) $m_i$ denotes the mathematical expectation of the variable $X_i$. By construction, it is clear that the density $f_n(x)$ of $S_n$, for all $n \geq 1$ satisfies

$$\int_{\mathbb{R}} f_n(x)\, dx = 1\,; \quad \int_{\mathbb{R}} x f_n(x))\, dx = 0 \quad \int_{\mathbb{R}} x^2 f_n(x))\, dx = n\sigma. \tag{15}$$

Hence, like in (3), mass and momentum are conserved when $n$ varies, while the second moment is increasing with $n$. This behavior of the densities simply expresses the fact that the random variable $S_n$ is *diffusing* on the real line. To obtain information on the behavior of $f_n(x)$ for large values of $n$, the key idea in the central limit theorem is to consider the limit of the density of $\mathcal{S}_n$, which is nothing but $\bar{f}_n(x) = \sqrt{n} f_n(\sqrt{n}x)$, namely $f_n(x)$ scaled with respect to the square root of the variance (second moment). With this rescaling, the second moment of $\mathcal{S}_n$ is normalized to $\sigma$.

Considering now a diffusion equation, the classical counterpart of the central limit problem is the linear heat equation

$$\frac{\partial v}{\partial t} = \Delta v \qquad (x \in \mathbb{R}^N, t > 0), \tag{16}$$

$$v(x, t = 0) = v_0(x) \geq 0, \qquad (x \in \mathbb{R}^N), \tag{17}$$

where the initial energy is finite, and $v_0$ satisfies conditions (3). In this case $E(t)$ increases linearly in time,

$$\frac{dE(t)}{dt} = N \int_{\mathbb{R}^N} v(x,t)\, dx = N, \tag{18}$$

so that, if $E(0) = NE_0$, $E(t) = N(E_0 + t)$. Due to linearity, the large–time behavior of the solution can be easily recovered by passing to Fourier transforms. If

$$\widehat{v}(\xi, t) = \int_{\mathbb{R}^N} e^{-i\xi \cdot x}\, v(x,t)\, dx,$$

equation (16) for $\widehat{v}$ takes the form

$$\frac{\partial \widehat{v}}{\partial t} = -|\xi|^2 \widehat{v}, \qquad (\xi \in \mathbb{R}^N, t > 0) \tag{19}$$

$$\widehat{v}(\xi, t = 0) = \widehat{v}_0(\xi), \qquad (\xi \in \mathbb{R}^N). \tag{20}$$

Equation (19) is explicitly solvable to give

$$\widehat{v}(\xi, t) = \widehat{v}_0(\xi) \exp\left\{-|\xi|^2 t\right\}.$$

To simplify notations, given any probability density function $f$, in the following $f_\beta(x)$ will denote the probability density

$$f_\beta(x) = \beta^{N/2} f(\beta^{1/2} x), \quad \beta > 0. \tag{21}$$

Now, consider that the Fourier transform of $v_{E(t)}(x,t)$ is $\widehat{v}_{E(t)}(\xi, t) = \widehat{v}(\xi/E(t)^{1/2}) = \widehat{v}(\xi/\sqrt{N(E_0+t)})$. Thus,

$$\widehat{v}_{E(t)}(\xi, t) = \widehat{v}_0 \left( \frac{\xi}{\sqrt{N(E_0+t)}} \right) \exp\left\{ -|\xi|^2 \frac{t}{N(E_0+t)} \right\} \tag{22}$$

and, since $\widehat{v}(0) = 1$,

$$\widehat{v}_{E(t)}(\xi, t) \longrightarrow_{t\to\infty} \widehat{v}_0\,(0) \exp\left\{ -\frac{|\xi|^2}{N} \right\} = \exp\left\{ -\frac{|\xi|^2}{N} \right\} = \widehat{\omega}_{N/2}(\xi), \tag{23}$$

where

$$\omega(x) = \left(\frac{1}{2\pi}\right)^{N/2} \exp\left\{ -\frac{|x|^2}{2} \right\}$$

is the Gaussian density of mass 1, momentum zero and energy $N/2$. This enlightens the analogies of the central limit theorem with the large–time asymptotic of the solution to the heat equation. The previous limit procedure requires the sequence $v_{E(t)}(x,t)$ normalized to satisfy

$$\int_{\mathbb{R}^N} v_{E(t)}(x,t)\,dx = 1\,; \quad \int_{\mathbb{R}^N} x v_{E(t)}(x,t)\,dx = 0; \quad \frac{1}{2}\int_{\mathbb{R}^N} |x|^2 v_{E(t)}(x,t)\,dx = 1; \quad t \geq 0. \tag{24}$$

On the other hand, the standard way to recover the intermediate asymptotic of the solution to a diffusion equation with initial density $v_0$ only requires that the self–similar solution has the same mass as $v_0$. Let us recall briefly this procedure. By means of the time–dependent rescaling (10), where now

$$\alpha(t) = (1+2t)^{1/2}\ ; \quad \tau(t) = \log \alpha(t),$$

one shows that, whenever $v(x,t)$ is a solution to the heat equation $u(x,\tau)$ solves the linear Fokker-Planck equation

$$\frac{\partial u}{\partial \tau} = \operatorname{div}(xu + \nabla u) \tag{25}$$

The scaling is such that the initial data for both the heat equation and the Fokker–Planck equation coincide. The Fokker–Planck equation has a unique steady state of unit mass, given by $u_\infty(x) = \omega(x)$. We remark that the second moment of $v_0$ is in general different from that of $u_\infty$. Under suitable conditions on the initial data $v_0$, (essentially finite energy and entropy) it was proved in [5] that the solution to (25) converges towards $u_\infty(x)$ in $L^1(\mathbb{R}^N)$ as $\tau \to \infty$ with an exponential decay rate, that is

$$\|u(x,\tau) - \omega(x)\|_{L^1(\mathbb{R}^N)} \leq Ce^{-\tau}\ , \qquad \tau \geq 0\ .$$

Now, the time-dependent scaling implies that any result about the asymptotic behavior of $u(x,\tau)$ translates into a result about the asymptotic behavior of $v(x,t)$ and vice versa. Hence

$$\|v(x,t) - \omega_{1/\alpha^2(t)}(x)\|_{L^1(\mathbb{R}^N)} \leq \frac{C}{\sqrt{2t+1}}\ ; \qquad t \geq 0$$

and that this bound is sharp. Since

$$E_\omega(t) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^2 \omega_{1/\alpha^2(t)}(x)\,dx = \frac{1}{2}N(2t+1) = \frac{N}{2}\alpha^2(t), \tag{26}$$

it follows that

$$\lim_{t\to\infty} \frac{E(t)}{E_\omega(t)} = 1, \tag{27}$$

and

$$\left|\frac{E(t)}{E_\omega(t)} - 1\right| = \frac{2E_0+1}{2t+1}. \tag{28}$$

The limit property (27) allows to obtain easily the convergence rate in the central limit theorem for the heat equation. Due to the properties of the Gaussian law, we obtain easily, for any constant $\beta > 0$

$$\int_{\mathbb{R}^N} |\omega(x) - \omega_\beta(x)| \leq \left|\left(\frac{1}{2\pi\beta}\right)^{N/2} - \left(\frac{1}{2\pi}\right)^{N/2}\right| \int_{\mathbb{R}^N} e^{-x^2/(2\beta)}\,dx +$$

$$\left(\frac{1}{2\pi\beta}\right)^{N/2}\int_{\mathrm{IR}^N}\left|e^{-x^2/(2\beta)}-e^{-x^2/2}\right|\,dx = 2\left|\beta^{N/2}-1\right| \tag{29}$$

Therefore

$$\|v_{E(t)}(x,t)-\omega_{N/2}(x)\|_{L^1(\mathrm{IR}^N)} = \left\|v(x,t)-\omega_{N/(2E(t))}\,(x)\right\|_{L^1(\mathrm{IR}^N)}$$

$$\leq \left\|v(x,t)-\omega_{1/\alpha^2(t)}\,(x)\right\|_{L^1(\mathrm{IR}^N)} + \left\|\omega_{1/\alpha^2(t)}\,(x)-\omega_{N/(2E(t))}\,(x)\right\|_{L^1(\mathrm{IR}^N)}$$

$$\leq \frac{C}{\sqrt{2t+1}} + \left\|\omega_{E(t)/E_\omega(t)}\,(x)-\omega\,(x)\right\|_{L^1(\mathrm{IR}^N)} \leq$$

$$\frac{C}{\sqrt{2t+1}} + \left|\left(\frac{E(t)}{E_\omega(t)}\right)^{N/2}-1\right|. \tag{30}$$

Finally, considering that $E(t)/E_\omega(t) \leq C_0 = \max\{1, 2E_0\}$ we obtain

$$\left|\left(\frac{E(t)}{E_\omega(t)}\right)^{N/2}-1\right| \leq \left|\frac{E(t)}{E_\omega(t)}-1\right| NC_0^N = NC_0^N\,\frac{2E_0+1}{2t+1}. \tag{31}$$

Substituting into (30) we get the decay of $v_{E(t)}(x,t)$ towards the normalized Gaussian law in the $L^1$-setting with a sharp rate

$$\|v_{E(t)}(x,t)-\omega_{N/2}(x)\|_{L^1(\mathrm{IR}^N)} \leq \frac{C}{\sqrt{2t+1}} + NC_0^N\,\frac{2E_0+1}{2t+1}. \tag{32}$$

**Remark 2.1** We remark that condition (24) are quite natural for the Fokker-Planck equation (25), where the usual requirements for the particle density, in addition to the positivity, are exactly the initial bounds on mass and energy, while the same condition is in principle not so natural to study the large–time behavior of the solution to the heat equation.

The convergence result (32) enlightens a basic difference between the classical central limit theorem of probability theory and the corresponding one for the linear diffusion equation. In the former the rate of convergence to the Gaussian law depends strictly on the number of extra moments which are bounded for the $X_i$ random variables. This phenomenon is well-known, and it takes usually the name of Berry–Esseen bound. Recent results on these bounds for smooth densities can be found in [14]. On the contrary, in the latter case the $L^1$–convergence of the (scaled) solution of the heat equation towards the Gaussian law holds with a minimal rate which depends only on the rate of growth of the similarity variable, and not on the number of extra moments bounded at time $t = 0$ .

**Remark 2.2** In (29) we obtained a simple inequality for the $L^1$ difference of two Gaussian densities one of which scaled of a fixed positive constant. This bound is more general, and it can be derived for any other density $f(|x|)$, provided the function $f(r), r \geq 0$ is non-increasing with respect to its argument. This bound is more general, and it can be derived for any other density $f(|x|)$, provided the function $f(r), r \geq 0$ is non-increasing with respect to its argument. In the proof of (29), in fact, we used only this monotonicity property. For this reason, the same bound holds for the Barenblatt–Pattle solutions (6).

# 3 Entropy estimates

We recall here the results obtained in [6] which are necessary to obtain precise estimates on the behavior in time of the energy of the porous medium equation. To this aim, we outline briefly the strategy of the aforementioned paper, from which the convergence towards the similarity solution is derived. As briefly discussed in the introduction, the main argument is to transform the diffusion equation into a kinetic–like equation of Fokker-Planck type (8). This is achieved by using the change of variables (10) (11). Since we will use heavily the properties of this scaling in the next section, we will describe now it in some details. Starting from the time dependent scaling

$$v(x,t) = \frac{1}{\alpha^N(t)} u\left(\frac{x}{\alpha(t)}, \tau(t)\right),$$

and setting $x' = x/\alpha$, one obtains

$$\frac{\partial}{\partial t} v(x,t) = -\frac{\alpha'}{\alpha^{N+1}} \nabla \cdot \left[x' u(x',\tau)\right] + \frac{\tau'}{\alpha^N} \frac{\partial}{\partial \tau} u(x',\tau),$$

and

$$\Delta_x v^m(x,t) = \frac{1}{\alpha^{Nm+2}} \Delta_{x'} u^m(x',\tau).$$

The resulting equation for $u(x',\tau)$ has time–independent coefficients if both $\alpha(t)$ and $\tau(t)$ satisfy

$$\frac{\tau'(t)}{\alpha^N(t)} = \frac{\alpha'(t)}{\alpha^{N+1}(t)}$$

and

$$\frac{\alpha'(t)}{\alpha^{N+1}(t)} = \frac{1}{\alpha^{Nm+2}(t)}.$$

Thus $\alpha(t)$ is required to satisfy the ordinary differential equation

$$\alpha'(t) = \frac{1}{\alpha^{N(m-1)+1}(t)}. \tag{33}$$

If we impose the initial conditions $\alpha(0) = 1$ and $\tau(0) = 0$ we obtain finally

$$\alpha(t) = \left[1 + (N(m-1)+2)\, t\right]^{1/(N(m-1)+2)}, \quad \tau(t) = \log \alpha(t). \tag{34}$$

This shows how to pass from equation (1) to equation (8).

It is known that the Cauchy problem for (8) is well-posed for any initial data $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$. Given any probability density function $f$, let $H(f)$ define the entropy functional

$$H(f) = \int_{\mathbb{R}^N} \left[\frac{|x|^2}{2} f + \frac{1}{m-1} f^m\right] dx \tag{35}$$

This Lyapunov functional was considered first by Newman and Ralston (see [21, 24]). From now on, we restrict ourselves to initial values $u_0$ is such that $H(u_0)$ is bounded. In this case, it can be easily proven that $H(u(\tau))$ is not increasing along the solution to equation (8), and

$$\frac{d}{d\tau} \int_{\mathbb{R}^N} \left(\frac{|x|^2}{2} u + \frac{1}{m-1} u^m\right) dx = -\int_{\mathbb{R}^N} u \left| x + \frac{m}{m-1} \nabla u^{m-1}\right|^2 dx \leq 0 \tag{36}$$

A direct inspection shows that equation (8) has a compactly supported stationary solution of mass equal to one defined by

$$B(|x|) = B_1(|x|,0) = \left(C^2 - \frac{(m-1)}{2m}|x|^2\right)_+^{\frac{1}{m-1}}. \qquad (37)$$

It is interesting to remark that both energy and entropy of $B(|x|)$ are linked to each other, and can be explicitly expressed in terms of the constant $C$ which determines the mass. This follows from the fact that, in the region where $B(|x|) > 0$, we have the identity

$$\frac{|x|^2}{2}B(|x|) = -\frac{x}{2}\cdot\nabla B(|x|)^m,$$

which, combined with

$$\int_{\mathbb{R}^N} B(|x|)^m\,dx = \int_{\mathbb{R}^N}\left(C^2 - \frac{m-1}{2m}|x|^2\right)B(|x|)\,dx = C^2 - \frac{m-1}{m}\int_{\mathbb{R}^N}\frac{|x|^2}{2}B(|x|)\,dx,$$

shows that

$$E_B = \int_{\mathbb{R}^N}\frac{|x|^2}{2}B(|x|)\,dx = -\int_{\mathbb{R}^N}\frac{x}{2}\cdot\nabla B(|x|)^m\,dx =$$
$$\frac{N}{2}\int_{\mathbb{R}^N} B(|x|)^m\,dx = \frac{N}{2}C^2 - \frac{N}{2}\frac{m-1}{m}E_B.$$

Thus, we obtain

$$E_B = \frac{N(m-1)+2m}{Nm}C^2, \qquad (38)$$

and

$$H(B) = \frac{N(m-1)+2}{N(m-1)}E_B. \qquad (39)$$

The relative entropy $H(u|B)$ is easily found by the position

$$H(u|B) = H(u) - H(B) = \int_{\mathbb{R}^N}\frac{|x|^2}{2}\left(u(x) - B(|x|)\right) + \frac{1}{m-1}\left(u^m(x) - B^m(|x|)\right)dx \qquad (40)$$

Since $H(B)$ is a constant, (36) shows that $H(u|B)$ is non-increasing.

It is remarkable that $H(u|B) \geq 0$ [25]. Moreover, it was proven in [6] that the relative entropy decays to zero exponentially, under weak assumptions on the initial values. We have

**Theorem 3.1** [6] *Let $u(x,\tau)$ be the solution to the initial value problem for the Fokker-Planck equation (8), corresponding to the initial condition $0 \leq u_0(x)$ such that $H(u_0|B)$ is finite. If in addition $|x|^{2+\delta}u_0(x) \in L^1(\mathbb{R}^N)$ for some $\delta > 0$, then, for all $\tau \geq 0$ the relative entropy $H(u(\tau)|B)$ is exponentially decreasing, with*

$$H(u(\tau)|B) \leq H(u_0|B)e^{-2\tau}. \qquad (41)$$

**Remark 3.2** In [6] theorem 3.1 was proven under some additional assumptions on the initial value $u_0$, namely under the assumption of strict positivity and boundedness. These conditions can be easily relaxed by a standard density argument [27]. Consider an initial data $0 \leq u_0(x) \in L^1(\mathbb{R}^N)$ with $H(u_0) < \infty$. Let

$$u_0^\epsilon = (\omega_\epsilon * u_0) + \epsilon e^{-|x|^2}$$

for any $\epsilon > 0$, where $\omega_\epsilon$ is a regularizing sequence. Now, $u_0^\epsilon$ satisfies all the hypotheses of Theorems 3.1 and 3.2 in [6]. Using now the $L^1$ contraction property of the solutions of (8) one obtains that

$$\|u^\epsilon(\tau) - u(\tau)\|_{L^1(\mathbb{R}^N)} \le \|u_0^\epsilon - u_0\|_{L^1(\mathbb{R}^N)}.$$

which implies $L^1$-convergence of $u^\epsilon(\tau)$ towards $u(\tau)$ for all $\tau > 0$. By lower semi–continuity of the convex entropy,

$$H(u(\tau)|B) \le H(u^\epsilon(\tau)|B),$$

while $H(u_0^\epsilon) \to H(u_0)$.

**Remark 3.3** The condition on the existence of additional moments for the initial value is essential in order to identify the energy of the steady state. This hypothesis, however, is typical of the proof of the central limit theorem, and can be substituted only by analogous conditions like the one by Lindeberg [15].

In the next Section, we will show how to extract from Theorem 3.1 information on the time evolution of the energy of the solution to equation (1). Our result will follow from a simple remark, which enlightens the meaning of the physical entropy, if written in terms of the solution to (1), instead of in terms of the solution to (8).

# 4 Large–time behavior of the energy

The time scaling (34) which links the solution $v(x,t)$ of equation (1) to the solution $u(x,\tau)$ of equation (8) has been chosen such that both $\tau(0) = 0$, and the initial data $v_0(x)$ and $u_0(x)$ coincide. Moreover, by (5) and (10) we deduce

$$\int_{\mathbb{R}^N} u^m(x,\tau)\, dx = \int_{\mathbb{R}^N} \frac{1}{\alpha(t)^N} u^m\left(\frac{x}{\alpha(t)}, \tau\right) dx = \alpha^{N(m-1)}(t) \cdot$$

$$\cdot \int_{\mathbb{R}^N} \left[\frac{1}{\alpha(t)^N} u\left(\frac{x}{\alpha(t)}, \tau\right)\right]^m dx = \alpha^{N(m-1)}(t) \int_{\mathbb{R}^N} v^m\,(x,t)\; dx = \frac{\alpha^{N(m-1)}(t)}{N}\frac{dE(t)}{dt}, \tag{42}$$

and

$$\int_{\mathbb{R}^N} \frac{|x|^2}{2} u\,(x,\tau)\; dx = \int_{\mathbb{R}^N} \frac{|x|^2}{2\alpha^2(t)} \frac{1}{\alpha(t)^N} u\left(\frac{x}{\alpha(t)}, \tau\right) dx = \frac{1}{\alpha^2(t)} E(t). \tag{43}$$

Thus, for any time $\tau > 0$ we can express the entropy in terms of the energy of $u(x,t)$,

$$H(u(\tau)) = \frac{1}{\alpha^2(t)} E(t) + \frac{\alpha^{N(m-1)}(t)}{N(m-1)} \frac{dE(t)}{dt}. \tag{44}$$

Expression (44) can be written in a simpler way owing to (33). A direct computation shows that

$$H(u(\tau)) = \frac{1}{N(m-1)} \frac{d}{dt}\left[E(t)\alpha^{N(m-1)}(t)\right] \tag{45}$$

Now, let us recall that the relative entropy (40) is bounded from below by zero, and thanks to the result of theorem 3.1, it satisfies the upper bound (41). Putting them together, we obtain that, under the hypotheses of Theorem 3.1, for any time $t > 0$ we get the bounds

$$H(B) \le \frac{1}{N(m-1)}\frac{d}{dt}\left[E(t)\alpha^{N(m-1)}(t)\right] \le H(B) + H(u_0|B)\frac{1}{\alpha^2(t)}, \tag{46}$$

where the right bound in (46) is a consequence of the definition of $\tau(t)$ given in (34). Integrating with respect to time we get the inequalities

$$N(m-1)H(B)t \le E(t)\alpha^{N(m-1)}(t) - E_0 \le N(m-1)H(B)t + \int_0^t \frac{ds}{\alpha^2(s)}. \tag{47}$$

Now, consider that, by definition of $\alpha(t)$,

$$\int_0^t \frac{ds}{\alpha^2(s)} = \frac{N(m-1)+2}{N(m-1)}\left[\alpha^{N(m-1)}(t) - 1\right]$$

Hence (47) implies

$$0 \le \frac{E(t)}{\alpha^2(t)} - \frac{E_0}{\alpha^{N(m-1)+2}} - \frac{N(m-1)t}{\alpha^{N(m-1)+2}(t)}H(B) \le \frac{N(m-1)+2}{N(m-1)}\left[\frac{\alpha^{N(m-1)}(t)-1}{\alpha^{N(m-1)+2}}\right] \tag{48}$$

by definition, $\alpha^{N(m-1)+2}(t)$ grows like $t$, as $t \to \infty$, and

$$\frac{N(m-1)t}{\alpha^{N(m-1)+2}(t)} = \frac{N(m-1)}{N(m-1)+2}\left[1 - \frac{1}{\alpha^{N(m-1)+2}(t)}\right].$$

Substituting into (48), and using (39) we finally obtain

$$C(t) \le \frac{E(t)}{\alpha^2(t)} - E_B \le C(t) + D(t), \tag{49}$$

where, thanks to (39),

$$C(t) = [E_0 + E_B]\,\frac{1}{1 + (N(m-1)+2)t}, \tag{50}$$

and

$$D(t) = \frac{N(m-1)+2}{N(m-1)}\left[\frac{\alpha^{N(m-1)}(t)-1}{\alpha^{N(m-1)+2}(t)}\right] \tag{51}$$

One can easily argue that both $A(t)$ and $B(t)$ decay with time, and the worse decay rate is given by the first term in (51). Since $\alpha^2(t)E_B$ is the energy of the Barenblatt solution $B_1(|x|,t)$, defined in (6), we proved

**Theorem 4.1** *Let $u(x,\tau)$ be the solution to the initial value problem for the porous medium equation (1), corresponding to the initial condition $0 \le v_0(x)$ such that $H(v_0|B)$ is finite. If in addition $|x|^{2+\delta}v_0(x) \in L^1(\mathbb{R})$ for some $\delta > 0$, then, for all $t \ge 0$ the energy $E(t)$ of the solution is increasing at the same rate of the energy $E_B(t)$ of the self–similar solution $B_1(|x|,t)$, and*

$$\lim_{t\to\infty} \frac{E(t)}{E_B(t)} = 1. \tag{52}$$

*Moreover, the following decay holds*

$$\left|\frac{E(t)}{E_B(t)} - 1\right| = O\left(\frac{1}{t^{2/[N(m-1)+2]}}\right) \tag{53}$$

**Remark 4.2** If $u_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N)$ with $|x|^{2+\delta} u_0(x) \in L^1(\mathbb{R}^N)$ for some $\delta > 0$, the convergence of $E(t)/\alpha^2(t)$ to $E_B$ can be easily concluded from Theorem 3.1 of [6]. The $L^1$-convergence of $u(x,\tau)$ towards $B(|x|)$, thanks to the Csiszar-Kullback inequality, follows from the convergence to zero of the relative entropy. Moreover, a uniform bound in time for the $2+\delta$-moment of $u(x,t)$ holds in this case. Hence, by (43),

$$\int_{\mathbb{R}^N} \frac{|x|^2}{2} u(x,\tau)\, dx = \frac{E(t)}{\alpha^2(t)} \to_{t\to\infty} E_B$$

On the other hand, owing only to $L^1$ or $L^\infty$ convergence rates, make it possible only to obtain a rate for the quotient of the energies that depends on $\delta$. A way to show this, is the following

$$\left| \int_{\mathbb{R}^N} \frac{|x|^2}{2} \left[u(x,\tau) - B(|x|)\right] dx \right| \le \int_{|x|\le R} \frac{|x|^2}{2} |u(x,\tau) - B(|x|)|\, dx +$$

$$R^{-\delta} \int_{|x|\le R} \frac{|x|^{2+\delta}}{2} |u(x,\tau) - B(|x|)|\, dx \le \frac{R^2}{2} \|u(x,\tau) - B(|x|)\|_{L^1(\mathbb{R}^N)} + R^{-\delta} C_\delta$$

Optimizing over $R$ we obtain

$$\left| \frac{E(t)}{\alpha^2(t)} - E_B \right| \le A_\delta \|u(x,\tau) - B(|x|)\|_{L^1(\mathbb{R}^N)}^{\delta/(2+\delta)} C_\delta^{2/(2+\delta)}. \tag{54}$$

The same strategy can be applied if we know the rate of decay in $L^\infty$. In this case in fact, we can use the bound

$$\int_{|x|\le R} \frac{|x|^2}{2} |u(x,\tau) - B(|x|)|\, dx \le \frac{R^{N+1}}{2} \mathcal{B}_N \|u(x,\tau) - B(|x|)\|_{L^\infty(\mathbb{R}^N)},$$

where $\mathcal{B}_N$ is the measure of the ball of radius 1 in $\mathbb{R}^N$. Optimizing over $R$ we obtain now

$$\left| \frac{E(t)}{\alpha^2(t)} - E_B \right| \le A'_\delta \|u(x,\tau) - B(|x|)\|_{L^\infty(\mathbb{R}^N)}^{\delta/(N+1+\delta)} C_\delta^{(N+1)/(N+1+\delta)}. \tag{55}$$

Both estimates depends on the extra-moment.

We are now in a position to prove our main result.

**Theorem 4.3** *Let $v(x,t)$ be the solution to the initial value problem for the porous medium equation (1), corresponding to the initial condition $0 \le v_0(x)$ such that $H(v_0|B)$ is finite. If in addition $|x|^{2+\delta} v_0(x) \in L^1(\mathbb{R})$ for some $\delta > 0$, then, for all $t \ge 0$ the scaled solution $E(t)^{N/2} v(E(t)^{1/2} x, t)$ converges in $L^1$ towards $E_B^{N/2} B(E_B^{1/2}|x|)$, and the following bound holds*

$$\left\| E(t)^{N/2} v(E(t)^{1/2} x, t) - E_B^{N/2} B(E_B^{1/2}|x|) \right\|_{L^1(\mathbb{R}^N)} \le C_m \frac{1}{1+t^\gamma} + C_0 \frac{1}{1+t^{2/[N(m-1)+2]}}, \tag{56}$$

*where*

$$\gamma = \frac{1}{N(m-1)+2} \quad \text{if} \quad 1 < m \le 2; \quad \gamma = \frac{2}{m[N(m-1)+2]} \quad \text{if} \quad m > 2;$$

$$C_0 = N\left(2 + \frac{E_0}{E_B} + \frac{N(m-1)+2}{N(m-1)E_B}\right),$$

*and the constant $C_m$ can be found explicitly (cfr. [6]).*

*Proof.*

The proof is a direct consequence of Theorem 4.1 and Theorem 4.5 in [6]. In fact,

$$\left\| E(t)^{N/2} v(E(t)^{1/2}x,t) - E_B^{N/2} B(E_B^{1/2}|x|) \right\|_{L^1(\mathbb{R}^N)} = \left\| v_{E(t)}(x,t) - B_{E_B}(|x|) \right\|_{L^1(\mathbb{R}^N)} \leq$$

$$\left\| v_{E(t)}(x,t) - B_{E(t)/\alpha^2(t)}(|x|) \right\|_{L^1(\mathbb{R}^N)} + \left\| B_{E(t)/\alpha^2(t)}(|x|) - B_{E_B}(|x|) \right\|_{L^1(\mathbb{R}^N)}. \quad (57)$$

Now, the decay of the first term in (57) follows from the decay of the relative entropy and the Csiszar-Kullback inequality, since

$$\left\| v_{E(t)}(x,t) - B_{E(t)/\alpha^2(t)}(|x|) \right\|_{L^1(\mathbb{R}^N)} = \| u(x,\tau) - B(|x|) \|_{L^1(\mathbb{R}^N)}.$$

Finally, the decay of the last term in (57) comes out from the inequality (cfr. remark 2.2)

$$\int_{\mathbb{R}^N} |B(|x|) - B_\beta(|x|)| \, dx \leq 2 \left| \beta^{N/2} - 1 \right| \quad (58)$$

valid for any constant $\beta > 0$. Thanks to Theorem 4.1, we have a uniform bound in time for $E(t)/E_B(t)$, given by

$$\frac{E(t)}{E_B(t)} \leq S_0 = 2 + \frac{E_0}{E_B} + \frac{N(m-1)+2}{N(m-1)E_B}$$

Since

$$\left| \beta^{N/2} - 1 \right| \leq \left| \beta^N - 1 \right| \leq N \, |\beta - 1| \max\{1, \beta^N\},$$

choosing $\beta = \sup E(t)/E_B(t)$, we obtain the desired result.

**Remark 4.4** From the point of view of probability theory, Theorem 4.3 is nothing but a central limit theorem for the scaled solution of the porous medium equation, in which the limit density is a probability density function of beta type. Let us discuss in some details this analogy. In one-dimension of space, the general formula for the probability density function of the beta distribution is [11]

$$\beta_{p,q}(x) = \frac{(x-a)^{p-1}(b-x)^{q-1}}{B(p,q)(b-a)^{p+q+1}}, \quad a \leq x \leq b; p, q > 0. \quad (59)$$

where $p$ and $q$ are the shape parameters, $a$ and $b$ are the lower and upper bounds, respectively, of the distribution, and $B(p,q)$ is the beta function,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \, dt.$$

The case where $a = 0$ and $b = 1$ is called the standard beta distribution. It is evident that in one-dimension of the variable the Barenblatt solution (37) is a beta distribution of parameters $p = q = m/(m-1)$ and bounds $-a = b = \sqrt{2m/(m-1)}C$. Moreover, the constant $C$ can be expressed in terms of the beta function through the formula

$$C^{\frac{m+1}{m-1}} = \frac{\left(\frac{1}{2}\right)^{\frac{m+1}{m-1}} \left(\frac{m-1}{2}\right)^{\frac{m-1}{2}}}{B\left(\frac{m}{m-1}, \frac{m}{m-1}\right)}.$$

This analogy with the central limit theorem continues to hold in the case of fast diffusions, where the limit density is a $t$-distribution [11].

## 5 Concluding remarks

The results of the previous section remain valid also for fast diffusion equations, provided the energy of the corresponding Barenblatt solution is bounded. Thanks to formula (39) in fact, the entropy of the steady state is bounded as soon as the energy of the steady state is. If $m < 1$, this happens when the diffusion exponent satisfies $m > N/(N+2)$. Whenever $m \leq N/(N+2)$, the entropy of the steady state is unbounded, and the relative entropy can not be split as the difference of the two entropies. In this case, convergence to the steady states still holds [17, 8], but only for initial values which leave in a suitable neighborhood of the steady state, in such a way that relative entropy is bounded. On the other hand, if we start with a initial density of finite energy, the strategy of [17, 8] does not apply, but it remains meaningful to look for the asymptotic behavior of the scaled density $\bar{v}(x,t) = E(t)^{N/2} v(E(t)^{1/2}x, t)$, which by construction is a sequence of probability densities with bounded second moment. Thus, we can immediately draw conclusions on the large–time convergence of class of probability densities $\{\bar{v}(x,t)\}_{t\geq 0}$. By virtue of Prokhorov theorem (cfr. [16]) the existence of a uniform bound on the second moment (the energy) implies that this class is tight, so that any sequence $\{v(x,t_n)\}_{n\geq 0}$ contains an infinite subsequence which converges weakly to some probability measure $v_\infty$. This limit measure has all moments of order less that two which are bounded, and for this reason it differs from the Barenblatt solution. The one–dimensional problem has been tackled recently from a numerical point of view in [13], where numerical evidence of convergence towards a universal profile for values of the diffusion constant below the critical layer $\bar{m} = 1/3$ has been found. Likewise, in [13] the numerical convergence of $\bar{v}(x,t)$ towards a universal profile has been observed for general one–dimensional diffusion equations like (12), for which it does not exist a self–similar profile. As briefly described in the introduction, the conjecture that the convergence of $\bar{v}(x,t)$ towards a universal profile which depends only on the function $\Phi$ in (12) can be proven for a large class of functions $\Phi$, has been partially solved in [4]. Concerning the asymptotic behavior of fast diffusion equations in the regime $m \leq N/(N+2)$, a similar problem where this conjecture has been recently proven concerns the large-time behavior of the solution to a Boltzmann-like equation for dissipative gases [23]. For the model introduced in this paper, two different behaviors of the solution can be observed , depending of the initial density has bounded or unbounded energy. In the latter case, convergence towards a stable law is proven if the tails of the initial density are close to the tails of the stable law. In other words, convergence towards the steady state (of infinite energy) holds only if the initial values leave in a suitable neighborhood of the steady state. On the contrary, when the initial density has finite energy, the scaled solution converges towards a universal profile, and this convergence holds as soon as some extra moments of the initial density are bounded.

**Acknowledgment:** Financial supports both from the project HYKE, "Hyperbolic and Kinetic Equations: Asymptotics, Numerics, Analysis" financed by the European Union (IHP), Contract Number HPRN-CT-2002-00282, and from the Italian MIUR project "Mathematical Problems of Kinetic Theories" are acknowledged. The author thanks J.L. Vázquez and G. Savaré for useful comments and stimulating discussions.

## References


[1] R. Ansorge, *Mathematical Models of Fluiddynamics. Modelling, Theory, Basic Numerical Facts - An Introduction* Wiley-VCH, Berlin, 2002

[2] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, **26** (2001), 43-100.

[3] P. Biler, J. Dolbeault, M.J. Esteban, Intermediate asymptotics in L1 for general nonlinear diffusion equations. *Appl. Math. Letters* **15**, (2002) 101-107.

[4] J.A. Carrillo, M. Di Francesco, G. Toscani, Intermediate asymptotics beyond homogeneity and self-similarity: long time behavior for nonlinear diffusions, (preprint) (2004)

[5] J. A. Carrillo, G. Toscani, Exponential convergence toward equilibrium for homogeneous Fokker-Planck type equations, *Math. Meth. Appl. Sci.*, **21**, (1998) 1269-1286.

[6] J. A. Carrillo, G. Toscani, Asymptotic $L^1$-decay of solutions of the porous medium equation to self-similarity, *Indiana Univ. Math. J.* **49**, (2000) 113-141.

[7] J. A. Carrillo, A. Jüngel, P. Markowich, G. Toscani, A. Unterreiter, Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities, *Monatsh. Math.* **133**, (2001) 1–82.

[8] J.A. Carrillo, J.L. Vázquez, Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations* **28** (2003), 1023–1056.

[9] I. Csiszar. Information-type measures of difference of probability distributions and indirect observations. *Stud. Sci. Math. Hung.*, **2**, (1967) : 299–318 .

[10] M. Del Pino, J. Dolbeault, Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* **81** , (2002) 847–875.

[11] M. Evans, N. Hastings, B. Peacock, *Statistical distributions. Third edition.* Wiley Series in Probability and Statistics: Texts and References Section. Wiley-Interscience, New York, 2000.

[12] A. Friedman, S. Kamin, The asymptotic behavior of gas in an N-dimensional porous medium, *Trans. Amer. Math. Soc.*, **262**, (1980) 551-563.

[13] L. Gosse, G.Toscani, Identification of asymptotic decay to self–similarity for one–dimensional filtration equations. (preprint) (2004)

[14] T. Goudon, S. Junca, G. Toscani, Fourier-based distances and Berry-Esseen like inequalities for smooth densities, *Monatsc. Mathem.*, **135** (2002) 115-136.

[15] J. Hoffmann-Jørgensen, *Probability with a view towards statistics; Vol.I*, Chapman & Hall, New York, 1994.

[16] R.G. Laha, V.K. Rohatgi, *Probability Theory*, John Wiley and Sons, New York, 1979.

[17] C. Lederman, P.A. Markowich, On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass. *Comm. Partial Differential Equations* **28** (2003), 301–332.

[18] S. Kamin, The asymptotic behaviour of the solution of the filtration equation, *Israel J. Math.*, **14**, (1973) 76-87.

[19] S. Kamin, Similar solutions and the asymptotics of filtration equations, *Arch. Rat. Mech. Anal.*, **60**, (1976) 171-183.

[20] S. Kullback, A lower bound for discrimination information in terms of variation. *IEEE Trans. Inf. The.*, **4** (1967) 126–127.

[21] W.J. Newman, A Liapunov functional for the evolution of solutions to the porous medium equation to self-similarity, I, *J. Math. Phys.*, **25**, (1984) 3120-3123.

[22] F. Otto, The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* **26** (2001), 101–174.

[23] A. Pulvirenti, G. Toscani, Asymptotic properties of the inelastic Kac model, *J. Statist. Phys.*, **114** (2004) 1453-1480.

[24] J. Ralston, A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity, II, *J. Math. Phys.*, **25**, (1984) 3124-3127.

[25] G. Toscani, Remarks on entropy and equilibrium states, *Appl. Math. Letters*, **12** (1999) 19-25.

[26] J.L. Vázquez, Asymptotic behavior and propagation properties of the one-dimensional flow of a gas in a porous medium, *Trans. Amer. Math. Soc.*, **277**, (1983) 507-527 .

[27] J.L. Vázquez, Asymptotic behaviour for the porous medium equation in the whole space, *J. Evol. Equ.* **3**, (2003) 67–118 .