

LONG-TIME BEHAVIOR FOR A NONLINEAR FOURTH-ORDER PARABOLIC EQUATION

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ABSTRACT. We study the asymptotic behavior of solutions of the initial-boundary value problem, with periodic boundary conditions, for a fourth-order nonlinear degenerate diffusion equation with a logarithmic nonlinearity. For strictly positive and suitably small initial data we show that a positive solution exponentially approaches its mean as time tends to infinity. These results are derived by analyzing the equation verified by the logarithm of the solution.

1. INTRODUCTION

We investigate the long-time behavior for solutions $f = f(t, x)$ of the initial periodic-boundary value problem

$$(1.1) \quad f_t = -(f(\log f)_{xx})_{xx},$$

$$(1.2) \quad f(0, x) = f_0(x) \in H_+^1(S^1),$$

where $x \in X$ and $t \in \mathbb{R}^+$, with $X = S^1$ parameterized by a variable x satisfying $0 \leq x \leq 1$. Equation (1.1), which can be equivalently written as

$$f_t = -f_{xxxx} + \left(\frac{f^2}{f} \right)_{xx},$$

arises as a scaling limit in the study of interface fluctuations in a certain spin system [6], and also models the electron concentration in a quantum semiconductor device with zero temperature and negligible electric field [7].

The initial periodic-boundary value problem for (1.1) was first studied by Bleher, Lebowitz and Speer in [1]. They proved local in time existence of positive solutions for strictly positive initial data and global existence (in time) for “small” initial data. Moreover, in this last case they were able to show convergence without rate to the constant steady state $\int_{S^1} f_0$.

Following the numerical investigation done in [3], one can conjecture exponential convergence towards the constant steady state for global (in time) solutions of the initial-boundary value problem (1.1)-(1.2) with periodic boundary conditions (see

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also [2, 5, 9]). Indeed, this conjecture has recently been proved in [9] by different techniques for (1.1) with the boundary conditions introduced in [7, 8].

Concerning the Cauchy problem for (1.1), it was remarked [2, 5] that (1.1) is a particular case of a class of fourth order diffusion equations which admit self-similar solutions. The self-similar profiles for (1.1) can be defined by a minimization problem involving the physical entropy and thus given by modified heat kernels. It is formally argued in [2, 5] and numerically investigated in [3] that this property should provide an algebraic decay in L^1 -norm of solutions of (1.1) towards the corresponding self-similar profile.

In this work, we show that under suitable smallness assumptions on the initial data, the equilibration rate of solutions of (1.1) with periodic boundary conditions is exponential. Instead of working directly with (1.1)-(1.2), we rather consider the initial value problem in terms of $\log f$. Setting $\alpha(t, x) = \log f(t, x)$, $\alpha = \alpha(t, x)$ then satisfies the equation

$$(1.3) \quad \alpha_t = -(\alpha_{xxxx} + 2\alpha_x \alpha_{xxx} + \alpha_{xx} \alpha_x^2 + \alpha_{xx}^2) = -e^{-\alpha} (e^\alpha \alpha_{xx})_{xx}.$$

The initial value problem (1.1)-(1.2) is in this way translated into the initial value problem (1.3) with the initial condition

$$(1.4) \quad \alpha(0, x) = \alpha_0(x) = \log f_0(x).$$

Equation (1.3) plays an important role in the analysis of (1.1)-(1.2). In fact, all Lyapunov functionals studied in [1] can be easily recovered in this framework. Moreover, the form (1.3) allows us to recognize conditions under which the initial value problem (1.1)-(1.2) stabilizes exponentially.

It is remarkable that (1.1) can be equivalently written in many ways, which can be fruitfully used to get different results. While the form (1.3) to our knowledge has never been used before, in [1] was considered the equation

$$w_t = -w_{xxxx} + \frac{w_{xx}^2}{w},$$

obtained by setting $f = w^2$.

Before we state the main results of this work, we summarize the existence results given by Bleher, Lebowitz and Speer in [1], since they are the starting point of our analysis.

Theorem 1.1 ([1, Theorem 4.2]). *Suppose that $f_0 \in H_+^1(S^1)$. Then the following hold:*

- (a) **Local existence and uniqueness of a mild solution:** *For some $T > 0$ there exists a unique mild solution $f = f(t, x)$ of the initial value problem (1.1)-(1.2), such that $f \in C_+([0, T]; H^1(S^1))$.*
- (b) **Regularity of mild solutions:** *If $f \in C_+([0, T]; H^1(S^1))$ is a mild solution of the initial value problem (1.1)-(1.2), then $f \in C((0, T]; H^r(S^1))$ for all r ; moreover, f is a classical solution of (1.1)-(1.2) in $H^r(S^1)$ for any r .*

Let $I = [0, T(f_0))$ be the maximal interval where the solution of the initial value problem (1.1)-(1.2) exists.

Theorem 1.2 ([1, Theorem 5.1]). *Let $f_0 \in H_+^1(S^1)$. Let $f \in C_+([0, T(f_0)); H^1(S^1))$ be the mild solution of the initial value problem (1.1)-(1.2) defined on a maximal half-open interval. If $T(f_0) < \infty$, then $h = \lim_{t \nearrow T(f_0)} f(t)$ exists in C^1 , but the limiting function h vanishes at least at one point of S^1 .*

Theorems 1.1 and 1.2 show that preservation of strict positivity of the solution implies global existence. This property is at the basis of our main results.

In what follows, let us denote by \bar{f} the mean value of f in S^1 ,

$$\bar{f} = \int_{S^1} f(x) dx.$$

We prove

Theorem 1.3. *Let $f_0 \in H^1_+(S^1)$ be such that*

$$(H) \quad \int_{S^1} \frac{f_x^2(0, x)}{f^2(0, x)} dx < 12,$$

and let $f(t, x)$ be the solution of (1.1)-(1.2). Then

- (a) *the initial value problem (1.1)-(1.2) has a unique global in time solution;*
- (b) $\int_{S^1} \frac{f_x^2(t, x)}{f^2(t, x)} dx$ *converges exponentially to 0, as $t \rightarrow \infty$;*
- (c) $\|f(t) - \bar{f}_0\|_{H^1(S^1)}$ *converges exponentially to 0, as $t \rightarrow \infty$.*

By standard Sobolev inequalities, the previous result implies that for any $p \in [1, 2]$, $\|f\|_{L^p(S^1)}$ converges exponentially to \bar{f}_0 . If $1 \leq p \leq 4/3$ we will show that this exponential convergence to equilibrium follows independently of assumption (H), provided the initial data are such that global existence can be guaranteed. Precisely we prove

Theorem 1.4. *Let f be a global solution of the initial value problem (1.1)-(1.2). Then, for $1 \leq p \leq 4/3$*

$$(1.5) \quad 0 \leq \int_{S^1} f^p dx - \bar{f}_0^p \leq \left(\int_{S^1} f_0^p dx - \bar{f}_0^p \right) e^{-Kt},$$

where $K = 64\pi^4(p-1)/p$.

Classical Csiszar–Kullback inequalities [10],

$$(1.6) \quad \left(\int_{S^1} |f - \bar{f}| dx \right)^2 \leq \frac{4\bar{f}^{2-p}}{p(p-1)} \left(\int_{S^1} f^p dx - \bar{f}^p \right), \quad 1 < p \leq 2,$$

will then show exponential convergence in $L^1(S^1)$ of the solution towards the steady state with the explicit rate $K/2$.

The paper is organized as follows. In Section 2, by using (1.3), we recover the monotonicity in time of the Lyapunov functionals obtained in [1] by different methods. Section 3 is devoted to the proof of the large-time results for small positive initial data (Theorem 1.3) and to the exponential L^p -decay (Theorem 1.4).

2. ENTROPY FUNCTIONALS REVISITED

The existence of various functionals $\Psi(f, f_x)$, which behave monotonically in time when f is the solution of (1.1)-(1.2), was already discussed in [1]. In this section we show that the monotonicity of these Lyapunov functionals can easily be obtained as a consequence of (1.3). Throughout the section we will mainly work formally, and we will restrict ourselves to the case $X = S^1$. The results however can be rigorously justified for classical solutions of our problem, even on the whole line \mathbb{R} , provided f and its derivatives decay fast enough at infinity to guarantee a correct integration by parts.

In addition to (1.3),

$$\alpha_t = -(\alpha_{xxxx} + 2\alpha_x \alpha_{xxx} + \alpha_{xx} \alpha_x^2 + \alpha_{xx}^2) = -e^{-\alpha} (e^\alpha \alpha_{xx})_{xx},$$

a second equation will be widely used in this section,

$$(2.1) \quad (\alpha_x)_t = -(\alpha_{xxxx} + 2\alpha_x \alpha_{xxx} + 4\alpha_{xx} \alpha_{xxx} + \alpha_x^2 \alpha_{xx} + 2\alpha_x \alpha_{xx}^2).$$

Equation (1.3) can be obtained in two lines. In fact

$$\alpha_t = \log f_t = \frac{f_t}{f} = -\frac{(f(\log f)_{xx})_{xx}}{f} = -\left(\alpha_{xx} \frac{f_{xx}}{f} + 2\alpha_x \alpha_{xxx} + \alpha_{xxxx}\right)$$

and

$$\alpha_{xx} = \left(\frac{f_x}{f}\right)_x = \frac{f_{xx}}{f} - \left(\frac{f_x}{f}\right)^2, \quad \frac{f_{xx}}{f} = \alpha_{xx} + (\alpha_x)^2.$$

For the second equality in (1.3) we only use $\alpha = \log f$ and consequently $e^\alpha = f$. Equation (2.1) is obtained from (1.3) by differentiating with respect to the x -variable.

2.1. Functionals depending of α . The functionals $\int_{S^1} f^p dx$, $\int_{S^1} f \log f dx$ and $\int_{S^1} \log f dx$ can easily be rewritten in terms of $\log f$. They are nothing but particular cases of functionals depending of α , $\int_{S^1} \phi(\alpha) dx$. Therefore, for $\alpha(t, x)$, the solution to our problem, we begin by finding the expression of the time-derivative of $\int_{S^1} \phi(\alpha(t, x)) dx$.

Proposition 2.1. *Let $\phi \in C^\infty(\mathbb{R})$ and $\alpha = \log f$, where f is a (positive) classical solution of (1.1)-(1.2). Then*

$$(2.2) \quad \begin{aligned} \frac{d}{dt} \int_{S^1} \phi(\alpha) dx &= \int_{S^1} \alpha_{xx}^2 [\phi'(\alpha) - \phi''(\alpha)] dx \\ &+ \frac{1}{3} \int_{S^1} \alpha_x^4 [\phi''(\alpha) - 2\phi'''(\alpha) + \phi^{(iv)}(\alpha)] dx, \end{aligned}$$

where prime denotes differentiation of $\phi(\cdot)$ with respect to its argument.

Proof. Differentiation with respect to time gives

$$\frac{d}{dt} \int_{S^1} \phi(\alpha) dx = \int_{S^1} \phi'(\alpha) \alpha_t dx = - \int_{S^1} \phi'(\alpha) [\alpha_{xxxx} + 2\alpha_x \alpha_{xxx} + \alpha_{xx} \alpha_x^2 + \alpha_{xx}^2] dx.$$

Integrating by parts, we can reduce the order of the derivatives with respect to x of α in the first three terms. We obtain

$$\begin{aligned} - \int_{S^1} \phi'(\alpha) \alpha_{xxxx} dx &= \int_{S^1} \phi''(\alpha) \alpha_x \alpha_{xxx} dx = - \int_{S^1} [\phi'''(\alpha) \alpha_x^2 \alpha_{xx} + \phi''(\alpha) \alpha_{xx}^2] dx \\ &= - \int_{S^1} \left[\phi'''(\alpha) \frac{(\alpha_x^3)_x}{3} + \phi''(\alpha) \alpha_{xx}^2 \right] dx = \int_{S^1} \phi^{(iv)}(\alpha) \frac{\alpha_x^4}{3} dx - \int_{S^1} \phi''(\alpha) \alpha_{xx}^2 dx. \end{aligned}$$

Moreover

$$\begin{aligned} -2 \int_{S^1} \phi'(\alpha) \alpha_x \alpha_{xxx} dx &= 2 \int_{S^1} [\phi''(\alpha) \alpha_x^2 \alpha_{xx} + \phi'(\alpha) \alpha_{xx}^2] dx \\ &= 2 \int_{S^1} \left[\phi''(\alpha) \frac{(\alpha_x^3)_x}{3} + \phi'(\alpha) \alpha_{xx}^2 \right] dx = -2 \int_{S^1} \phi'''(\alpha) \frac{\alpha_x^4}{3} dx + 2 \int_{S^1} \phi'(\alpha) \alpha_{xx}^2 dx. \end{aligned}$$

Finally

$$-\int_{S^1} \phi'(\alpha) \alpha_{xx} \alpha_x^2 dx = -\int_{S^1} \phi'(\alpha) \frac{(\alpha_x^3)_x}{3} dx = \int_{S^1} \phi''(\alpha) \frac{\alpha_x^4}{3} dx.$$

Collecting these identities we obtain (2.2). \square

The proposition allows us to immediately find some of the functionals which behave monotonically in time. In fact, for any given ϕ it is enough to check the sign of the expressions $\phi'(\alpha) - \phi''(\alpha)$ and $\phi''(\alpha) - 2\phi'''(\alpha) + \phi^{(iv)}(\alpha)$. The first example of functional which behaves monotonically is clearly obtained setting $\phi(\alpha) = \alpha$. In this case $\phi' \equiv 1$, while $\phi'' \equiv \phi''' \equiv \phi^{(iv)} \equiv 0$ and (2.2) becomes

$$(2.3) \quad \frac{d}{dt} \int_{S^1} \log f dx = \int_{S^1} \alpha_{xx}^2 dx.$$

Thus we have

Corollary 2.2. *Let f be a (positive) classical solution of (1.1)-(1.2). Then, $\int_{S^1} \log f(t, x) dx$ is nondecreasing in time.*

Let us now set $\phi(\alpha) = \exp(p\alpha)$, $p > 0$, which implies $\phi(\alpha) = f^p$. We recover the following:

Corollary 2.3. *Let f be a (positive) classical solution of (1.1)-(1.2). Then*

$$(2.4) \quad \frac{d}{dt} \int_{S^1} f^p dx = p(1-p) \left[\int_{S^1} e^{p\alpha} \alpha_{xx}^2 dx + \frac{p(1-p)}{3} \int_{S^1} e^{p\alpha} \alpha_x^4 dx \right].$$

Furthermore, $\int_{S^1} f^p dx$ is nondecreasing for $0 < p < 1$, nonincreasing for $1 < p \leq \frac{3}{2}$ and constant for $p = 1$.

Proof. The sign of the right-hand side of (2.4) is clearly positive when $0 < p < 1$, while it is equal to zero when $p = 1$. To conclude the proof we have only to check that the right-hand side of (2.4) is nonpositive for $1 < p \leq \frac{3}{2}$. To this aim, consider that, for all $p \geq 0$, the following inequality holds:

$$(2.5) \quad \int_{S^1} e^{p\alpha} \alpha_{xx}^2 \geq \frac{p^2}{9} \int_{S^1} e^{p\alpha} \alpha_x^4 dx.$$

In fact, for any constant $d \in \mathbb{R}$,

$$(2.6) \quad 0 \leq \int_{S^1} e^{p\alpha} [\alpha_{xx} + d\alpha_x^2]^2 dx.$$

Expanding the square, integrating by parts the integral containing the product $\alpha_x \alpha_{xx}$, and using analogous arguments as given in the proof of Proposition 2.1, we obtain the inequality

$$(2.7) \quad \int_{S^1} e^{p\alpha} \alpha_{xx}^2 \geq \left(\frac{2}{3} dp - d^2 \right) \int_{S^1} e^{p\alpha} \alpha_x^4 dx.$$

The function $u = u(d) = \frac{2}{3} dp - d^2$ attains the maximum value at $d = p/3$, where $u(p/3) = p^2/9$. Since inequality (2.7) holds for any $d \in \mathbb{R}$, we can take $d = p/3$, which shows (2.5).

For $p > 1$, $p(1-p) < 0$. Thus, to show that for a given p the right-hand side of (2.4) is nonpositive is equivalent to showing that

$$(2.8) \quad \int_{S^1} e^{p\alpha} \alpha_{xx}^2 dx \geq \frac{p(p-1)}{3} \int_{S^1} e^{p\alpha} \alpha_x^4 dx.$$

By (2.5) this inequality holds if $p^2/9 \geq p(p-1)/3$, which gives $p \leq 3/2$. We point out that under the assumption $f_0 \neq \bar{f}_0$ we obtain $\frac{d}{dt} \int_{S^1} f^p dx < 0$, since in this case inequality (i) is strict. \square

Remark 2.4. The proof of Corollary 2.3 is quite different from the proof in [1]. In [1] $\frac{d}{dt} \int_{S^1} f^p dx$ is shown to be the integral of a semi-definite quadratic form provided $1 < p < 3/2$. In addition, (2.4) can be further analyzed to get explicit decay-rates of the L^p -norm of the solution.

To end up, we remark that a third possible choice is $\phi(\alpha) = \alpha e^\alpha$. In this way, we recover the monotonicity in time of the functional $\int_{S^1} f \log f dx$.

Corollary 2.5. *Let f be a (positive) classical solution of (1.1)-(1.2). Then*

$$(2.9) \quad \frac{d}{dt} \int_{S^1} f \log f dx = - \int_{S^1} \alpha_{xx}^2 e^\alpha dx \leq 0.$$

Proof. If $\phi(\alpha) = \alpha e^\alpha$, then

$$\phi'(\alpha) = e^\alpha(1+\alpha), \quad \phi''(\alpha) = e^\alpha(2+\alpha), \quad \phi'''(\alpha) = e^\alpha(3+\alpha), \quad \phi^{(iv)}(\alpha) = e^\alpha(4+\alpha).$$

Therefore,

$$\phi'(\alpha) - \phi''(\alpha) = -e^\alpha, \quad \phi''(\alpha) - 2\phi'''(\alpha) + \phi^{(iv)}(\alpha) = 0.$$

\square

Remark 2.6. Unlikely, from (2.2) it seems difficult to recover other functionals which vary monotonically in time. The natural choice $\phi(\alpha) = \alpha^q e^{p\alpha}$ gives no new functionals (in addition to the previously known ones corresponding to $q = 1, p = 0$, $q = 1, p = 1$ and $q = 0, p < 1$ or $q = 0, 1 < p \leq 3/2$).

2.2. Functionals depending on α and α_x . We will consider in this section functionals which depend both on the derivative f_x and jointly on f and its derivative f_x . As a main example in the first class we will study the evolution in time of the functional $\int_{S^1} \alpha_x^2 dx$, from which we will obtain under suitable smallness assumptions on the initial value both positivity of the solution and exponential convergence to the constant steady state (see next section).

An important example of functionals which depend jointly on f and its derivative f_x is furnished by the Fisher information,

$$I(f) = \int_{S^1} \frac{f_x^2}{f} dx.$$

In this class, we will again recover all the functionals which were considered in [1].

We begin our study by considering functionals depending only on α_x .

Proposition 2.7. *Let $\phi \in C^\infty(\mathbb{R})$. Then*

$$(2.10) \quad \begin{aligned} \frac{d}{dt} \int_{S^1} \phi(\alpha_x) dx &= \int_{S^1} \phi''(\alpha_x) (\alpha_x^2 \alpha_{xx}^2 - \alpha_{xxx}^2) dx - \int_{S^1} \phi'''(\alpha_x) \alpha_x \alpha_{xx}^3 dx \\ &+ \frac{1}{3} \int_{S^1} \phi^{(iv)}(\alpha_x) \alpha_x^4 dx. \end{aligned}$$

Proof. We have

$$\begin{aligned} & \frac{d}{dt} \int_{S^1} \phi(\alpha_x) dx = \int_{S^1} \phi'(\alpha_x)(\alpha_x)_t dx \\ & = - \int_{S^1} \phi'(\alpha_x) (\alpha_{xxxx} + 2\alpha_x \alpha_{xxx} + 4\alpha_{xx} \alpha_{xx} + \alpha_x^2 \alpha_{xxx} + 2\alpha_x \alpha_{xx}^2) dx, \end{aligned}$$

where we used (2.1). Various integration by parts permits us to reduce the order of the derivatives and show (2.10). Since the computations are very similar to those we did in Proposition 2.1, we leave them to the reader. \square

The most important example is given by the choice $\phi(r) = r^2$. In this case we obtain

Corollary 2.8. *Let f be the solution of (1.1)-(1.2). Then*

$$(2.11) \quad \frac{1}{2} \frac{d}{dt} \int_{S^1} \alpha_x^2 dx = - \int_{S^1} \alpha_{xxx}^2 dx + \int_{S^1} \alpha_x^2 \alpha_{xx}^2 dx.$$

Lastly, we consider functionals depending jointly on α and α_x . In particular, we will study the evolution in time of functionals $\phi(\alpha, \alpha_x) = \phi_1(\alpha)\phi_2(\alpha_x)$, where $\phi_1(s) = e^s$ and $\phi_2(s) = s^2$. This class includes, among others, the Fisher information. The monotonicity in time of this functional follows easily in our framework.

Proposition 2.9. *Let f be a solution of (1.1)-(1.2). Then*

$$\frac{d}{dt} \int_{S^1} \frac{f_x^2(t)}{f(t)} dx = -2 \int_{S^1} e^\alpha (\alpha_x \alpha_{xx} + \alpha_{xxx})^2 dx,$$

which implies that the Fisher information

$$I(f)(t) = \int_{S^1} \frac{f_x^2(t, x)}{f(t, x)} dx$$

is nonincreasing in time.

Proof. We use the second equality in (1.3).

$$\begin{aligned} & \frac{d}{dt} \int_{S^1} \frac{f_x^2(t)}{f(t)} dx = \frac{d}{dt} \int_{S^1} e^\alpha \alpha_x^2 dx = \int_{S^1} e^\alpha (\alpha_x^2 \alpha_t + 2\alpha_x \alpha_{xt}) dx \\ & = \int_{S^1} [e^\alpha \alpha_x^2 - (2e^\alpha \alpha_x)_x] \alpha_t dx = \int_{S^1} [e^\alpha \alpha_x^2 - 2e^\alpha (\alpha_x^2 + \alpha_{xx})] \alpha_t dx \\ & = \int_{S^1} e^\alpha (\alpha_x^2 + 2\alpha_{xx}) e^{-\alpha} (e^\alpha \alpha_{xx})_{xx} dx = - \int_{S^1} e^\alpha (\alpha_x^2 + 2\alpha_{xx})_x (\alpha_x \alpha_{xx} + \alpha_{xxx}) dx. \end{aligned}$$

This concludes the proof. \square

Remark 2.10. The previous proof can be extended to show that the functionals

$$I^p(f) = \int_{S^1} [f_x^2(t, x)/f(t, x)]^p dx$$

are nonincreasing in time if $1 \leq p \leq 3/2$. In terms of α , these functionals in fact take the form $\int_{S^1} e^{p\alpha} [\alpha_x^2]^p dx$, which allows explicit computations. In this case however, the proof in terms of α does not develop essential simplifications with respect to the proof given in [1].

To conclude this section we remark that the time monotonicity of the various functionals considered in [1] can be viewed in a unified picture by using (1.3) to study their evolution. In addition, we can show in this way that the convex functional (2.11) has a monotone in time evolution for some suitably small initial data. As we will show in the next section, the study of the time evolution of this functional permits us to obtain a new result of exponential convergence towards equilibrium for the solution of (1.1)-(1.2), with an explicitly computable rate.

3. CONVERGENCE TO EQUILIBRIUM

In this section we will deal with the problem of finding conditions on the initial values that guarantee the positivity of the solution on (1.1)-(1.2) and consequently its global existence. As a by-product of our analysis, we obtain exponential convergence to equilibrium.

For the initial-boundary value problem (1.1)-(1.2) the equilibrium states $f_\infty > 0$ are given by constants, since by (2.3) these states are such that

$$(\log f_\infty)_{xx} = 0,$$

and thus

$$f_\infty(x) = e^{C_3x+C_4}.$$

Then, the periodic boundary conditions imply that f_∞ must be constant ($f_\infty(x) = e^{C_4}$). On the other hand, since the equilibrium solution must have mass equal to \bar{f}_0 , one concludes that $f_\infty(x) = \bar{f}_0$.

Theorems 1.1 and 1.2 give us a way to prove global existence of the solution for the initial value problem (1.1)-(1.2). On the basis of these theorems, it is in fact enough to show that the solution remains strictly positive to infer that this solution exists for all times. Therefore, we focus on conditions on the initial value that guarantee strict positivity of the solution. Before going further, we recall some standard Sobolev inequalities and embeddings with optimal constants which will be used in the rest of the paper.

Lemma 3.1. *If $g \in W^{1,1}$ and $\bar{g} = 0$, then*

$$(3.1) \quad \|g\|_{C^0} \leq \frac{1}{2} \|g_x\|_{L^1},$$

$$(3.2) \quad \|g\|_{C^0} \leq \frac{1}{2\sqrt{3}} \|g_x\|_{L^2},$$

$$(3.3) \quad \|g\|_{L^2} \leq \frac{1}{2\pi} \|g_x\|_{L^2}.$$

In the following proposition we recover a condition that guarantees that the functional $\int_{S^1} \alpha_x^2(t) dx$ is nonincreasing in time. This condition will be enough to obtain both global existence and exponential convergence towards equilibrium, with an explicit rate.

Proposition 3.2. *Let the initial value f_0 satisfy $f_0 \neq \bar{f}_0$ and condition (H). Then, the solution of the initial value problem (1.1)-(1.2) $f \in C(I, H^1(S^1)_+)$, and*

$$\int_{S^1} \left(\frac{f_x}{f} \right)^2 (t) dx < 12,$$

for all $t \in I$.

Proof. We apply (3.1) to $\alpha_{xx}(t)$. For every $t \in I$, we obtain in this way

$$(3.4) \quad |\alpha_{xx}(t)|^2 \leq \frac{1}{12} \int_{S^1} \alpha_{xxx}^2(t) \, dx.$$

Using this bound into (2.9) gives

$$(3.5) \quad \frac{1}{2} \frac{d}{dt} \int_{S^1} \alpha_x^2(t) \, dx \leq - \left(\int_{S^1} \alpha_{xxx}^2(t) \, dx \right) \left(1 - \frac{1}{12} \int_{S^1} \alpha_x^2(t) \, dx \right).$$

We conclude the proof by a connection argument. Let

$$F(t) = \int_{S^1} \left(\frac{f_x}{f} \right)^2(t, x) \, dx$$

and

$$A = \{t \in I \text{ such that } F(s) < 12, \forall s \in [0, t]\}.$$

A is not empty since

$$1 - \frac{1}{12} \int_{S^1} \alpha_x^2(0) \, dx > 0$$

by assumption (H). Therefore $0 \in A$. A is obviously an open set, since F is continuous.

Let $t^* \in \bar{A}$. If $t^* \leq t_n$ for some n , $t^* \in A$ and we finish the proof. Then, let $\{t_n\}$ be an increasing sequence in I , such that $\{t_n\} \rightarrow t^*$, $t^* \in I$. Thus, $\forall n$ $F(t_n) < 12$.

Condition $f_0 \neq \bar{f}_0$ implies (by the uniqueness of solution) that $f(t) \neq \bar{f}_0$, $\forall t \in I$. Thus,

$$\int_{S^1} \alpha_{xxx}^2(s) \, dx > 0,$$

since, otherwise $f(t) = \bar{f}_0$ in contradiction with the previous statement. Hence, for all n , $\frac{d}{dt} \int_{S^1} \alpha_x^2(s) \, dx < 0$ $\forall s \in [0, t_n]$ since in this interval $F(s) < 12$. Therefore for all n , F is decreasing in $[0, t_n]$ and since $\{t_n\} \rightarrow t^*$ we conclude that $F(t^*) < F(0) < 12$.

We proved in this way that $t^* \in A$ and that A is a closed set. Thus $A = I$. \square

We are now in a position to prove our first main theorem, namely Theorem 1.3. We remark that we can skip the initial condition, $f_0 = \bar{f}_0$, since in this case $f(t) = \bar{f}_0$, $t \geq 0$ and there is nothing to prove.

Theorem 3.3. *Let the initial data f_0 satisfy $f_0 \in H_+^1(S^1)$ and condition (H). If $f(t, x)$ is the solution of (1.1)-(1.2), then*

- (a) $T(f_0) = \infty$;
- (b) $\int_{S^1} f_x^2(t, x) / f^2(t, x) \, dx$ converges exponentially to 0, when t tends to infinity, and the following bound holds:

$$\int_{S^1} \frac{f_x^2(t, x)}{f^2(t, x)} \, dx \leq M_1 e^{-M_2 t},$$

where $M_1 = 12 \left(12 / \int_{S^1} \alpha_x^2(0) \, dx - 1 \right)^{-1} > 0$ and $M_2 = 2(2\pi)^4$;

- (c) $\|f(t) - \bar{f}_0\|_{H^1(S^1)}$ converges exponentially to 0, when t tends to infinity, and for a given constant $C = C(f_0)$ the following bound holds:

$$(3.6) \quad \|f(t) - \bar{f}_0\|_{H^1(S^1)}^2 \leq C M_1 e^{-M_2 t}.$$

In (3.6)

$$C = \left(\frac{1}{4\pi^2} + 1 \right) \exp \left(2 + 2 \max \left\{ \left| \log \bar{f}_0 \right|, \left| \int_{S^1} \log f_0 \right| \right\} \right).$$

Proof. The proof of (a) follows by a contradiction argument. Suppose that $T(f_0) < \infty$. By Theorem 1.2, this implies that the limiting function $h = \lim_{t \nearrow T(f_0)} f(t)$ vanishes at some point.

By Lemma 3.1 applied to $\log f(t)$, for all $t \in I$ we obtain

$$(3.7) \quad \left\| \log f(t) - \int_{S^1} \log f(t) dx \right\|_{L^\infty(S^1)} \leq \frac{1}{2\sqrt{3}} \|(\log f(t))_x\|_{L^2(S^1)} < 1.$$

The last inequality is a consequence of Proposition 3.2. Thus,

$$(3.8) \quad \|\log f(t)\|_{L^\infty(S^1)} < 1 + \left| \int_{S^1} \log f(t) dx \right|.$$

The right side of this inequality is bounded since Jensen's inequality, applied to the convex function $-\log r$, yields

$$(3.9) \quad -\log \int_{S^1} f(t) dx \leq -\int_{S^1} \log f(t) dx.$$

Corollary 2.2 implies

$$(3.10) \quad \int_{S^1} \log f_0 dx \leq \int_{S^1} \log f(t) dx \leq \log \int_{S^1} f(t) dx = \log \int_{S^1} f_0 dx.$$

Hence, $\|\log f(t)\|_{L^\infty(S^1)}$ is uniformly bounded in $t \in I$, which is in contradiction with the fact that h vanishes at some point in S^1 .

Next, Lemma 3.1 gives the bounds

$$\alpha_{xx}^2 \leq \frac{1}{12} \int_{S^1} \alpha_{xxx}^2 dx$$

and

$$(2\pi)^4 \int_{S^1} \alpha_x^2 dx \leq \int_{S^1} \alpha_{xxx}^2 dx.$$

In this way, we obtain from (2.11) the inequality

$$(3.11) \quad \frac{1}{2} \frac{d}{dt} \int_{S^1} \alpha_x^2 dx \leq -(2\pi)^4 \int_{S^1} \alpha_x^2 dx \left[1 - \frac{1}{12} \int_{S^1} \alpha_x^2 dx \right].$$

Proposition 3.2 in fact guarantees that $1 - \frac{1}{12} \int_{S^1} \alpha_x^2 dx > 0$. Let us denote $y(t) = \int_{S^1} \alpha_x(t, x)^2 dx$. Then we can rewrite (3.11) as the logistic differential inequality

$$(3.12) \quad \frac{dy}{dt} \leq -2(2\pi)^4 y \left(1 - \frac{1}{12} y \right).$$

Setting $z = ye^{2(2\pi)^4 t}$, (3.12) changes to

$$(3.13) \quad \frac{dz}{dt} \leq \frac{2(2\pi)^4}{12} z^2 e^{-2(2\pi)^4 t}.$$

Integrating the differential inequality (3.13) and going back to original variables we find

$$(3.14) \quad \begin{aligned} \int_{S^1} \alpha_x^2 dx &\leq e^{-2(2\pi)^4 t} \left(\frac{1}{\int_{S^1} \alpha_x^2(0) dx} - \frac{1}{12} + \frac{e^{-2(2\pi)^4 t}}{12} \right)^{-1} \\ &= 12e^{-2(2\pi)^4 t} \left(\frac{12}{\int_{S^1} \alpha_x^2(0) dx} - 1 + e^{-2(2\pi)^4 t} \right)^{-1}. \end{aligned}$$

Once again using the bound $\int_{S^1} \alpha_x^2(0) dx < 12$, we finally get

$$(3.15) \quad \int_{S^1} \alpha_x^2 dx \leq 12e^{-2(2\pi)^4 t} \left(\frac{12}{\int_{S^1} \alpha_x^2(0) dx} - 1 \right)^{-1}.$$

Inequality (3.15) shows (b).

Finally, to prove (c) we use Lemma 3.1 to obtain

$$(3.16) \quad \int_{S^1} |f(t, x) - \bar{f}|^2 dx \leq \frac{1}{4\pi^2} \int_{S^1} f_x^2(t, x) dx.$$

Next,

$$(3.17) \quad \int_{S^1} f_x^2(t, x) dx = \int_{S^1} \frac{f_x^2(t, x)}{f^2(t, x)} f^2(t, x) dx \leq \sup_{x \in [0, 1]} f^2(t, x) \left(\int_{S^1} \frac{f_x^2(t, x)}{f^2(t, x)} dx \right).$$

Thus

$$\|f(t) - \bar{f}_0\|_{H^1(S^1)} \leq \left(\frac{1}{4\pi^2} + 1 \right) \sup_{x \in [0, 1]} f^2(t, x) \left(\int_{S^1} \frac{f_x^2(t, x)}{f^2(t, x)} dx \right).$$

We must note that $\sup_{x \in [0, 1]} f^2(t, x)$ is uniformly bounded for all $t \in I$, since in the proof of (a) we showed that $\|\log f(t)\|_{L^\infty(S^1)}$ is uniformly bounded in time. If $|\log f(t, x)| \leq D$, $f^2(t, x) \leq e^{2D}$. By (3.8) and (3.10), $|\log f(t, x)|$ can be bounded by $D = 1 + \max\{|\log \bar{f}_0|, |\int_{S^1} \log \bar{f}_0|\}$. This shows (3.6), since we can define $C = (\frac{1}{4\pi^2} + 1)e^{2D}$. In order to conclude, we must only recall that $\bar{f} = \bar{f}_0$ and, as a consequence of (b), $\|f(t) - \bar{f}_0\|_{H^1(S^1)}^2$ tends exponentially to 0 when $t \rightarrow \infty$. \square

Remark 3.4. The conditions for the initial value, $f_0 \in H_+^1(S^1)$ and (H) can be replaced by $f_0 \in H^1(S^1)$, $f_0 \neq 0$, f_0 nonnegative and (H), since these hypotheses imply $f_0 > 0$. In fact, if $f_0 \neq 0$, we can consider $z \in [0, 1]$ such that $f_0(z) \neq 0$. Then, for all $x \in [0, 1]$,

$$|\log f_0(x) - \log f_0(z)| = \left| \int_z^x (\log f_0)_y dy \right| \leq \int_z^x |(\log f_0)_y| dy \leq \int_0^1 |(\log f_0)_y| dy.$$

Using Hölder inequality we obtain

$$\int_0^1 |(\log f_0)_y| dy \leq \left(\int_0^1 |(\log f_0)_y|^2 dy \right)^{1/2}.$$

Since we assume (H),

$$|\log f_0(x) - \log f_0(z)| \leq 2\sqrt{3};$$

therefore

$$|\log f_0(x)| \leq 2\sqrt{3} + |\log f_0(z)|.$$

In this way, since $f_0(z) \neq 0$, $\log f_0$ is bounded, which implies $f_0 > 0$.

Remark 3.5. The condition

$$(3.18) \quad \|f_{0,x}\|_{L^2} < \sqrt{3}\bar{f}_0$$

implies (H). This follows easily from Lemma 3.1, since

$$f_0 > \bar{f}_0 - \|f_0 - \bar{f}_0\|_{L^\infty} \geq \bar{f}_0 - \frac{1}{2\sqrt{3}}\|f_{0,x}\|_{L^2} > \frac{\bar{f}_0}{2}.$$

In this way, we obtain

$$\int_{S^1} \frac{f_{0,x}^2}{f_0^2} dx < \frac{4}{\bar{f}_0^2} \int_{S^1} f_{0,x}^2 dx = \frac{4}{\bar{f}_0^2} \|f_{0,x}\|_{L^2}^2 < 12.$$

We remark that condition (3.18) is stronger than condition (ii) given in Theorem 5.2 in [1], i.e.

$$(3.19) \quad \|f_{0,x}\|_{L^2} < \frac{4\bar{f}_0}{\sqrt{3}},$$

which implies global existence. This shows that our theorem does not provide exponential convergence to equilibrium in the whole set of initial values for which there is global existence.

By classical Sobolev imbeddings, Theorem 3.3 also shows that for $1 \leq p \leq 2$ $\|f\|_{L^p(S^1)}$ converges exponentially to $\|\bar{f}_0\|_{L^p(S^1)}$. By the previous remark, this is true only for initial data sufficiently close to the stationary solution.

In what follows, we show that effectively for some exponent p we can obtain a more general convergence result. In more detail, we will prove that if $1 \leq p \leq 4/3$ the exponential convergence of $\|f\|_{L^p(S^1)}$ to \bar{f}_0 can be shown without initially assuming hypothesis (H). To draw this conclusion, one only needs to assume global existence, which could be guaranteed if some of the conditions given in [1] hold. Classical Csiszar-Kullback inequalities [10] will then imply L^1 -convergence towards the stationary solution at explicit exponential rate.

Theorem 3.6. *Let f be a global solution of the initial value problem (1.1)-(1.2), such that for some $1 \leq p \leq 4/3$, $\|f_0\|_{L^p(S^1)}$ is bounded. Then*

$$(3.20) \quad \int_{S^1} f^p dx - \bar{f}_0^p \leq \left[\int_{S^1} f_0^p dx - \bar{f}_0^p \right] e^{-Kt},$$

where

$$K = \frac{64\pi^4(p-1)}{p}.$$

Proof. To prove the theorem, we follow ideas introduced recently in [10] to study the large-time behavior of the thin film equation. We recall that in the previous section we obtained the time decay of $\int_{S^1} f^p dx$, (2.4). We used (2.4) only to conclude that $\int_{S^1} f^p dx$ is nonincreasing if $1 \leq p \leq \frac{3}{2}$. Here we will study in more detail the absolute value of the right-hand side of (2.4), which we denote from now on the entropy production. The idea is to show that the entropy production, i.e.

$$A_p(\alpha) = p(p-1) \left[\int_{S^1} \alpha_{xx}^2 e^{p\alpha} dx + \frac{p(1-p)}{3} \int_{S^1} \alpha_x^4 e^{p\alpha} dx \right]$$

is such that, for some explicitly computable constant c ,

$$\int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx \leq cA_p(\alpha).$$

Then, various Poincaré's inequalities lead to (3.20). A direct computation shows that, for all $p > 0$, if f is a smooth function in $C^2(S^1)$, then

$$(3.21) \quad \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx = \frac{p^2}{4} \int_{S^1} e^{p\alpha} \left(\alpha_{xx}^2 - \frac{p^2}{12} \alpha_x^4 \right) dx.$$

In fact, since

$$\int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx = \int_{S^1} \left[\left(e^{\frac{p\alpha}{2}} \right)_{xx} \right]^2 dx,$$

we obtain

$$(3.22) \quad \begin{aligned} \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx &= \int_{S^1} \left[\left(\frac{p}{2} \alpha_x e^{\frac{p\alpha}{2}} \right)_x \right]^2 dx = \frac{p^2}{4} \int_{S^1} \left[\alpha_{xx} e^{\frac{p\alpha}{2}} + \frac{p}{2} \alpha_x^2 e^{\frac{p\alpha}{2}} \right]^2 dx \\ &= \frac{p^2}{4} \int_{S^1} e^{p\alpha} \left[\alpha_{xx}^2 + \frac{p^2}{4} \alpha_x^4 + p \alpha_{xx} \alpha_x^2 \right] dx = \frac{p^2}{4} \int_{S^1} e^{p\alpha} \left(\alpha_{xx}^2 - \frac{p^2}{12} \alpha_x^4 \right) dx. \end{aligned}$$

In the last step we used the identity

$$p \int_{S^1} e^{p\alpha} \alpha_x^2 \alpha_{xx} dx = \frac{p}{3} \int_{S^1} e^{p\alpha} (\alpha_x^3)_x dx = -\frac{p^2}{3} \int_{S^1} e^{p\alpha} \alpha_x^4 dx.$$

It is now immediate to compare formula (3.21) with the entropy production. We claim in fact that, for $1 < p \leq \frac{4}{3}$,

$$(3.23) \quad A_p(\alpha) \geq \frac{4(p-1)}{p} \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx.$$

By (2.4) and (3.21), to prove (3.23) is equivalent to show that

$$p(p-1) \left[\int_{S^1} \alpha_{xx}^2 e^{p\alpha} dx + \frac{p(1-p)}{3} \int_{S^1} \alpha_x^4 e^{p\alpha} dx \right] \geq \frac{4(p-1)p^2}{4} \int_{S^1} e^{p\alpha} \left(\alpha_{xx}^2 - \frac{p^2}{12} \alpha_x^4 \right) dx,$$

or, simplifying the constant,

$$\int_{S^1} \alpha_{xx}^2 e^{p\alpha} dx + \frac{p(1-p)}{3} \int_{S^1} \alpha_x^4 e^{p\alpha} dx \geq \int_{S^1} e^{p\alpha} \left(\alpha_{xx}^2 - \frac{p^2}{12} \alpha_x^4 \right) dx.$$

This of course holds when

$$-(p-1) \geq -\frac{p}{4},$$

and thus $1 \geq 3p/4$ or equivalently $\frac{4}{3} \geq p$.

To finish the proof we apply the following Poincaré inequalities (see Lemma 3.1):

$$(3.24) \quad \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_x \right]^2 dx \leq \frac{1}{4\pi^2} \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_{xx} \right]^2 dx,$$

$$(3.25) \quad \int_{S^1} \left[f^{\frac{p}{2}} - \int_{S^1} f^{\frac{p}{2}} dx \right]^2 dx \leq \frac{1}{4\pi^2} \int_{S^1} \left[\left(f^{\frac{p}{2}} \right)_x \right]^2 dx.$$

Hence we obtain from (3.23)

$$(3.26) \quad \frac{d}{dt} \int_{S^1} f^p dx \leq -\frac{64\pi^4(p-1)}{p} \int_{S^1} \left[f^{\frac{p}{2}} - \int_{S^1} f^{\frac{p}{2}} dx \right]^2 dx$$

or equivalently

$$(3.27) \quad \frac{d}{dt} \int_{S^1} f^p dx \leq -\frac{64\pi^4(p-1)}{p} \left[\int_{S^1} f^p dx - \left(\int_{S^1} f^{\frac{p}{2}} dx \right)^2 \right].$$

Since $1 \leq p \leq 4/3$, $p/2 < 1$. By Hölder's inequality we obtain

$$\int_{S^1} f^{\frac{p}{2}} dx \leq \left(\int_{S^1} f^{\frac{p}{2} \cdot \frac{2}{p}} dx \right)^{\frac{p}{2}} \left(\int_{S^1} 1^{\frac{2}{2-p}} dx \right)^{\frac{2-p}{2}} = \bar{f}_0^{p/2}$$

and

$$\bar{f}_0^p = \left(\int_{S^1} f dx \right)^p \leq \left(\int_{S^1} f^p dx \right) \left(\int_{S^1} 1^{p^*} dx \right)^{\frac{p}{p^*}} = \int_{S^1} f^p dx.$$

Therefore,

$$(3.28) \quad \frac{d}{dt} \int_{S^1} f^p dx \leq -\frac{64\pi^4(p-1)}{p} \left[\int_{S^1} f^p dx - \bar{f}_0^p \right] \leq 0.$$

Inequality (3.28) can be written as

$$(3.29) \quad \frac{d}{dt} \left(\int_{S^1} f^p dx - \bar{f}_0^p \right) \leq -\frac{64\pi^4(p-1)}{p} \left[\int_{S^1} f^p dx - \bar{f}_0^p \right],$$

which implies

$$(3.30) \quad \int_{S^1} f^p dx - \bar{f}_0^p \leq \left[\int_{S^1} f_0^p dx - \bar{f}_0^p \right] e^{-Kt},$$

where

$$K = \frac{64\pi^4(p-1)}{p}.$$

□

Remark 3.7. Theorem 3.3 remains true for other boundary conditions assuming global existence of smooth solutions, such as nonflux boundary, which are conditions which guarantee conservation of mass ($\int_{S^1} f_t dx = 0$),

$$f_x(t, 0) = f_x(t, 1) = f_{xxx}(t, 0) = f_{xxx}(t, 1) = 0.$$

In this case, the conditions are directly translated to α ,

$$\alpha_x(t, 0) = \alpha_x(t, 1) = \alpha_{xxx}(t, 0) = \alpha_{xxx}(t, 1) = 0.$$

The obvious reason for this is that we can perform the same integration by parts, and the boundary terms vanish. On the other hand, for the same reason Theorem 3.6 does not hold.

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