

## OVER-POPULATED TAILS FOR CONSERVATIVE-IN-THE-MEAN INELASTIC MAXWELL MODELS

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**ABSTRACT.** We introduce and discuss spatially homogeneous Maxwell-type models of the nonlinear Boltzmann equation undergoing binary collisions with a random component. The random contribution to collisions is such that the usual collisional invariants of mass, momentum and energy do not hold point-wise, even if they all hold in the mean. Under this assumption it is shown that, while the Boltzmann equation has the usual conserved quantities, it possesses a steady state with power-like tails for certain random variables. A similar situation occurs in kinetic models of economy recently considered by two of the authors [24], which are conservative in the mean but possess a steady distribution with Pareto tails. The convolution-like gain operator is subsequently shown to have good contraction/expansion properties with respect to different metrics in the set of probability measures. Existence and regularity of isotropic stationary states is shown directly by constructing converging iteration sequences as done in [8]. Uniqueness, asymptotic stability and estimates of overpopulated high energy tails of the steady profile are derived from the basic property of contraction/expansion of metrics. For general initial conditions the solutions of the Boltzmann equation are then proved to converge with computable rate as  $t \rightarrow \infty$  to the steady solution in these distances, which metricizes the weak convergence of measures. These results show that power-like tails in Maxwell models are obtained when the point-wise conservation of momentum and/or energy holds only globally.

**1. Introduction.** In this paper, we introduce and discuss the possibility to obtain steady solutions with power-like tails starting from conservative molecular systems described by the Boltzmann equation with Maxwell-type collision kernels. The starting point of our model is to consider binary collisions that result in a linear combination of an inelastic collision and a random contribution. As we shall see,

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the random addition to the post-collision velocities can only increase the mean of the collisional energy, and, among other things, it gives the possibility to construct a binary collision that preserves (in the mean) mass, momentum and energy. Our model is closely related to a kinetic model for economics introduced by Pareschi and two of the present authors [18]. There, the random contribution to the collision (trade) was introduced to take into account the returns of the market.

Inelastic Maxwell models were introduced by Bobylev, Gamba and one of the authors in 2000 [6]; see also [23] for the one dimensional case. Maybe the most interesting result (absent in the elastic case) is the existence of self-similar solutions in the homogeneous cooling problem and the non-Maxwellian behavior of these solutions, which displays power-like decay for large velocities. It was conjectured in [20] and later proved in [8], [11], [2] that such solutions represent intermediate asymptotics for a wide class of initial data. Other results concerned with self-similar solutions in the theory of the classical (elastic) Boltzmann equation for Maxwell molecules were also recently published in [7], [8]. In light of these results, it looks clear that in many aspects both elastic and inelastic Maxwell models must be studied from a unified point of view. As observed by Bobylev, Cercignani and Gamba in [9], [10], an interesting question arises in connection with power-like tails for high velocities. Is it possible to observe a similar effect, i.e., an appearance of power-like tails from initial data with exponential tails, in a system of particles interacting according to laws of classical mechanics without energy loss? In [12] Bobylev and Gamba gave a partial answer to this question by showing that, under a certain limiting procedure, such behavior can in principle be observed if one considers a mixture of classical Maxwell gases. More precisely, self-similar solutions converging towards Maxwellian equilibrium were proved to have power-like tails once normalized by the equilibrium.

In this paper, we will try to elucidate the same question, starting from a somewhat different point of view. Our starting point will be a suitable modification to the homogeneous Boltzmann equation for the inelastic Maxwell molecules introduced in Ref. [6], in such a way that the usual conservations of mass, momentum and energy in the binary collisions still continue to hold in the mean sense. The scaled-in-time inelastic Boltzmann equation introduced in [6] reads

$$\frac{\partial f}{\partial t} = Q_e(f, f). \quad (1)$$

Here,  $f(v, t)$  is the density for the velocity space distribution of the molecules at time  $t$ , while  $Q_e(f, f)$  is the inelastic Boltzmann collision operator, which contains the effects of binary collisions of grains. As usual in this context, the collision operator  $Q_e(f, f)$  is more easily treated if expressed in weak form. This corresponds to writing, for every suitable test function  $\varphi$ ,

$$(\varphi, Q_e(f, f)) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(v)f(w) [\varphi(v^*) - \varphi(v)] dv dw d\sigma. \quad (2)$$

In (2),  $v^*$  is the outgoing velocity corresponding to a particle in the collision defined by the incoming velocities  $v, w$  and the angular parameter  $\sigma \in S^2$ :

$$\begin{aligned} v^* &= \frac{1}{2}(v + w) + \frac{1 - e}{4}(v - w) + \frac{1 + e}{4}|v - w|\sigma, \\ w^* &= \frac{1}{2}(v + w) - \frac{1 - e}{4}(v - w) - \frac{1 + e}{4}|v - w|\sigma. \end{aligned} \quad (3)$$

The parameter  $0 \leq e \leq 1$  represents the restitution coefficient.

In the model we consider, this restitution coefficient will be chosen as a random variable that can be interpreted from a physical point of view as the stochasticity in the microscopic process of collision due to the randomness of the grains' geometry and the mechanical properties of the medium. We will show in the next section that this random behavior in restoring energy leads to a precise form of the energy gain term that differs from the usually chosen diffusion term, the so-called "thermal bath". This new form of "thermal bath" is thus related to the process generated by the randomness of the granular media. We prove that this particular thermal bath yields equilibrium states with power law tails.

Such over-populated tails in distributions at equilibrium arise in other contexts. We shall present similar results on the large time behavior of collisional kinetic theory applied to economic modelling. In this framework, the kinetic variable represents the wealth of agents and the collision operator describes the evolution of the wealth distribution through exchanges. We refer to [18], [24] and references therein for a mathematical presentation of these models closely related to so called "econo-physics". In such models, the equations between pre- and post-collisional values involve some randomness that is related to the stochasticity of the market that provides random returns.

In the remainder of this paper, we will study in detail the large time behavior of the solution of the Boltzmann equation involving such a stochastic process. We show that the validity (at a macroscopic level) of the classical collision invariants is enough to guarantee convergence towards a steady profile, but not enough to reach a Maxwellian-like profile. In fact, we will show that there is a class of random perturbations of the coefficient of restitution such that the steady state possesses power-like tails.

A crucial role in our analysis is played by the weak norm convergence, which is obtained by further pushing the development of a method first used in [21] to control the exponential convergence of Maxwellian molecules in certain weak norms. This will be done by using the fact that the nonlinear operator in the Boltzmann equation (see (9)) can be expressed in Fourier variables in a simple closed form using Bobylev's identity [5]. Estimates of the evolution of the Wasserstein distance [32], [29], [30] between solutions will be presented for the economic and the inelastic model since they give complementary information with respect to the results in [24]. Concerning this second aspect, we will take advantage of the recent analysis of Bolley and Carrillo [14], [17] of the inelastic Boltzmann equation for Maxwell molecules. From this analysis, we will obtain the uniqueness and asymptotic stability of stationary states for this model. Finally, the appearance of power-like tails for the asymptotically stable stationary states will be discussed for both models, giving explicit examples of random variables producing this behavior.

The paper is organized as follows: in section 2 we detail the collisional models for both granular media and economy applications including random coefficients in the relations between pre- and post-collisional variables. In section 3, we recall the main properties of probability metrics. In section 4, we investigate large time behavior of the solution of the kinetic economy model and section 5 is devoted to large time behavior of stochastic granular media.

Lastly, let us summarize the two main results of this paper: first, we give some insight into conditions for a collision operator to lead to power-law tails (conservatism in mean being some kind of necessary condition); second, we propose a new form

for the thermal bath with a physically relevant origin (the restitution coefficient taking into account the randomness of granular media).

**2. Modelling issues and diffusion approximation.** Let us present the proposed stochastic granular model (with a random restitution coefficient) and its diffusion limit and then recall briefly the similar analysis for the economy model following [18].

**2.1. Stochastic granular media.** Considering the weak formulation (2), easy computations show that  $(\varphi(v), Q_e(f, f)) = 0$  whenever  $\varphi(v) = 1$  and  $\varphi(v) = v$ , while  $(\varphi(v), Q_e(f, f)) < 0$  if  $\varphi(v) = v^2$ . This corresponds to conservation of mass and momentum, and, respectively, to loss of energy for the solution to equation (1). For this reason, if we fix the initial data to be a centered probability density function, the solution will remain centered at any subsequent time  $t > 0$ . The loss of energy in a single collision with a constant restitution coefficient  $e$  is given by

$$|v'|^2 + |w'|^2 = |v|^2 + |w|^2 - \frac{1 - e^2}{4} (|v - w|^2 - |v - w|(v - w) \cdot \sigma). \quad (4)$$

The previous formula is the key to our modification of the collisions. Let us replace the constant coefficient of restitution  $e$  with a stochastic coefficient of restitution  $\tilde{e}$ , such that for a given random variable  $\eta$

$$\tilde{e} = e + \eta, \quad \text{with } \langle \eta \rangle = 0 \quad \text{and} \quad \langle \eta^2 \rangle = \beta^2. \quad (5)$$

In (5) and in the rest of the paper,  $\langle \cdot \rangle$  denotes the mathematical expectation of the real-valued random variable  $\eta$ , i.e., integration against a measure  $\mu$ . For obvious physical reasons, the random variable  $\eta$  has to be chosen to satisfy  $\eta \geq -e$ , in order to guarantee that the (random) coefficient of restitution  $\tilde{e} \geq 0$ . Using  $\tilde{e}$  instead of  $e$  in (3) gives that the momentum is *conserved in average* for a suitable choice of the variance. In fact, since

$$\langle |v'|^2 + |w'|^2 \rangle = |v|^2 + |w|^2 - \frac{1 - e^2 - \beta^2}{4} (|v - w|^2 - |v - w|(v - w) \cdot \sigma), \quad (6)$$

by choosing the variance  $\beta^2 = 1 - e^2 > 0$ , we obtain

$$\langle |v'|^2 + |w'|^2 \rangle = |v|^2 + |w|^2. \quad (7)$$

We will call a collision process (or equivalently a random cross section) satisfying (7) *conservative in the mean*. Let us remark that condition (7) *cannot be satisfied if  $\tilde{e}$  takes only values less than 1*, since in that case  $\tilde{e}^2$  remains also less than 1 and so does its average  $\langle \tilde{e}^2 \rangle = e^2 + \beta^2 < 1$ . The main idea behind this is that particles can even gain energy in collisions even though the total energy is conserved in the mean.

From the physical point of view, this assumption of energy-gain particle collisions may seem strange. We will show in the sequel that this energy input can be interpreted as a sort of thermal bath. Particles are immersed in a medium that produces this random change in the strength of their relative velocity. We will argue, based on a derivation of a Fokker-Planck approximation, that this random component in the collision operator can be approximated by a second-order differential operator whose diffusion matrix depends on the second moments of the solution  $f$  itself and the random variable  $\eta$  (see [18], [25] for a similar approach in one dimension).

This idea allows us to consider a new class of Maxwell-type models, from now on called *conservative in the mean*, which are obtained from (post-collision) velocities given by

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-\tilde{e}}{4}(v-w) + \frac{1+\tilde{e}}{4}|v-w|\sigma, \\ w' &= \frac{1}{2}(v+w) - \frac{1-\tilde{e}}{4}(v-w) - \frac{1+\tilde{e}}{4}|v-w|\sigma, \end{aligned} \quad (8)$$

where  $\tilde{e}$  is the random coefficient of restitution defined in (5), and  $\beta^2 = 1 - e^2$ . The corresponding Boltzmann equation reads

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f, f) = \langle Q_{\tilde{e}}(f, f) \rangle, \quad (9)$$

and its corresponding weak form is

$$(\varphi, \tilde{Q}_e(f, f)) = \frac{1}{4\pi} \left\langle \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(v)f(w) [\varphi(v') - \varphi(v)] dv dw d\sigma \right\rangle. \quad (10)$$

In view of our choice of the random contribution to the coefficient of restitution, we now have  $(\varphi(v), \tilde{Q}_e(f, f)) = 0$  whenever  $\varphi(v) = 1, v, |v|^2$ , that is, the classical collision invariants of the elastic Boltzmann equation.

**2.2. Formal diffusive asymptotics.** Before entering into the study of the large-time behavior of the Boltzmann equation (9), we shall present here some formal arguments that hopefully clarify the action of the random restitution coefficient in the collision mechanism (8).

To this end, following the same method as in [27], letting  $(v', w')$  denote the post-collision velocities (8) in our random collision with  $(v^*, w^*)$  as post collision velocities defined by the classic inelastic collision (3), we can split the velocities into their deterministic and random parts

$$v' = v^* + \eta\Delta(u, \sigma), \quad w' = w^* - \eta\Delta(u, \sigma), \quad (11)$$

where we let  $u = v - w$  and

$$\Delta(u, \sigma) = \frac{1}{4}(|u|\sigma - u).$$

Let us consider a Taylor expansion of  $\varphi(v')$  around  $\varphi(v^*)$  up to second order in  $\eta$ . Thanks to (11) we get

$$\varphi(v') = \varphi(v^*) + \eta (\nabla\varphi(v^*) \cdot \Delta(u, \sigma)) + \frac{1}{2}\eta^2 \sum_{i,j} \frac{\partial^2 \varphi(v^*)}{\partial v_i^* \partial v_j^*} \Delta_i \Delta_j + \dots \quad (12)$$

Thus, taking the mean of the expansion (12), and using the property  $\langle \eta \rangle = 0$ , we get

$$\langle \varphi(v') \rangle = \varphi(v^*) + \frac{1}{2}\beta^2 \sum_{i,j} \frac{\partial^2 \varphi(v^*)}{\partial v_i^* \partial v_j^*} \Delta_i \Delta_j + \dots \quad (13)$$

Truncating the expansion (13) after the second-order term and inserting (13) into (10), we conclude

$$\begin{aligned} (\varphi, \tilde{Q}_e(f, f)) &\simeq (\varphi, Q_e(f, f)) + (\varphi, D_e(f, f)) \\ &= (\varphi, Q_e(f, f)) + \frac{\beta^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \sum_{i,j} \frac{\partial^2 \varphi(v^*)}{\partial v_i^* \partial v_j^*} \Delta_i \Delta_j f(v)f(w) dv dw d\sigma. \end{aligned} \quad (14)$$

While the first term in (14)  $Q_e(f, f)$  is the classical inelastic Boltzmann collision operator, the second term  $D_e(f, f)$  needs to be further analyzed.

Denoting by  $(^*v, ^*w)$  the pre-collision velocities in the inelastic collision, and taking into account the fact that the Jacobian of the transformation  $d^*v d^*w$  into  $dv dw$  for a constant restitution coefficient is equal to  $e^{-1}$ , one obtains

$$\begin{aligned}
(\varphi, D_e(f, f)) &= \frac{\beta^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \sum_{i,j} \frac{\partial^2 \varphi(v^*)}{\partial v_i^* \partial v_j^*} \Delta_i \Delta_j f(v) f(w) dv dw d\sigma \\
&= \frac{\beta^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{e} \sum_{i,j} \frac{\partial^2 \varphi(v)}{\partial v_i \partial v_j} \Delta_i^* \Delta_j^* f(^*v) f(^*w) dv dw d\sigma \\
&= \frac{\beta^2}{8\pi} \int_{\mathbb{R}^3} \left[ \sum_{i,j} \frac{\partial^2 \varphi(v)}{\partial v_i \partial v_j} \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{e} \Delta_i^* \Delta_j^* f(^*v) f(^*w) dw d\sigma \right] dv \\
&= \int_{\mathbb{R}^3} \varphi(v) \left[ \frac{\beta^2}{8\pi} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{e} \Delta_i^* \Delta_j^* f(^*v) f(^*w) dw d\sigma \right] dv.
\end{aligned} \tag{15}$$

This shows that, at least for small inelasticity, the random part of the collision corresponds to a correction given by the nonlinear diffusion operator  $D_e(f, f)(v)$ , where

$$D_e(f, f)(v) = \frac{\beta^2}{8\pi} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{e} \Delta_i^* \Delta_j^* f(^*v) f(^*w) dw d\sigma. \tag{16}$$

Different expressions of the operator (16) can be recovered owing to the definition of  $\Delta$ . For the purposes of the present paper, however, we simply remark that, choosing the test function  $\varphi(v) = |v|^2$ , direct computations show that the correction  $D_e(f, f)$  is such that

$$\begin{aligned}
(|v|^2, D_e(f, f)) &= \frac{\beta^2}{64\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} ||u|\sigma - u|^2 f(v) f(w) dv dw d\sigma \\
&= \frac{1}{4} \beta^2 \left[ \int_{\mathbb{R}^3} |v|^2 f(v) dv - \left| \int_{\mathbb{R}^3} v f(v) dv \right|^2 \right].
\end{aligned} \tag{17}$$

This reveals the fundamental fact that the diffusion operator produces a growth of the second moment proportional to the second moment itself. This action is clearly different from the action of a linear diffusion operator (a thermal bath), which induces a growth of the second moment proportional to the mass. This supports the fact that the Boltzmann equation (9) can produce fat tails.

**2.3. Simple economy market modelling.** In one dimension of the “velocity” variable, a similar construction leads to kinetic models for wealth redistribution [18], [24]. In this case, the variable  $v \in \mathbb{R}_+$  represents the wealth of the agents, binary collisions are trades between agents, and the (eventual) power-like tails of the steady distribution of wealth are known in the pertinent literature as Pareto tails. Due to the fact that the variable is in  $\mathbb{R}_+$ , the possible conserved quantities reduce to mass and momentum. In [18] the collision mechanism is given by

$$v' = (1 - \lambda)v + \lambda w + \eta v; \quad w' = \lambda v + (1 - \lambda)w + \eta^* w \tag{18}$$

where  $0 \leq \lambda \leq 1$  represents the constant saving rate and  $\eta$  and  $\eta^*$  are random variables with law given by a measure  $\mu(s)$  of zero mean, variance  $\beta^2$  and support in  $[-\lambda, +\infty)$ . In this way, for all realizations of the random variable we have  $\eta \geq -\lambda$  and wealths after trading are well defined i.e., remain nonnegative. This is the so-called *no debt* condition. In this context, the Boltzmann equation (9) is replaced by

$$(\varphi, \tilde{Q}_\lambda(f, f)) = \left\langle \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(v)f(w) [\varphi(v') - \varphi(v)] dv dw \right\rangle. \quad (19)$$

Here, we use the notation

$$\langle h \rangle := \int_{-\lambda}^{\infty} h(s) d\mu(s).$$

The unique possible collision invariants of the one-dimensional Boltzmann equation are obtained for  $\varphi(v) = 1$  and  $\varphi(v) = v$ .

The weak formulation of the Boltzmann equation can also be rewritten

$$\int_{\mathbb{R}_+} \varphi(v) \tilde{Q}_\lambda(f, f) dv = \frac{1}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(v)f(w) \langle \varphi(v') + \varphi(w') - \varphi(v) - \varphi(w) \rangle dv dw. \quad (20)$$

In (20) the wealth variables  $v, w$  are nonnegative quantities, and the collision mechanism is given by (18). A one-dimensional Boltzmann type equation of the form

$$\frac{\partial f}{\partial t} = \tilde{Q}_\lambda(f, f) \quad (21)$$

based on the binary interaction given in (18) has been considered in [18], [24] and we refer to them for a deeper discussion of the model. Without loss of generality, we can fix the initial density  $f_0(v)$  with finite second moment, with the normalization condition

$$m(t) := \int_{\mathbb{R}_+} vf(v, t) dv = \bar{m}, \quad (22)$$

since by choosing  $\varphi(v) = v$ , (20) shows that  $m(t) = m(0)$  for all  $t \geq 0$ .

As in section 2.2, one splits the collision mechanisms into a deterministic inelastic part and the random part:

$$v' = v^* + \eta v; \quad w' = w^* + \eta^* w$$

where  $v^*, w^*$  are deterministic wealth (corresponding to inelastic collision with constant restitution coefficient  $(1 - \lambda)$ )

$$v^* = (1 - \lambda)v + \lambda w; \quad w^* = \lambda v + (1 - \lambda)w.$$

A formal Taylor expansion similar to (2.2), in the limit for  $\lambda$  and  $\eta$  small, leads to a drift term for the difference between  $(v, w)$  and  $(v^*, w^*)$  and a diffusion term proportional to the variance  $\beta^2$ .

$$\varphi(v') = \varphi(v^*) + \eta v \partial_v \varphi(v^*) + \frac{1}{2} \eta^2 v^2 \partial_v^2 \varphi(v^*) + \dots$$

Taking the average

$$\langle \varphi(v') \rangle = \varphi(v^*) + \frac{1}{2} \beta^2 v^2 \partial_v^2 \varphi(v^*) + \dots,$$

and on the other hand, the deterministic part gives

$$\varphi(v^*) = \varphi(v) + \lambda(w - v) \partial_v \varphi(v) + \frac{1}{2} \lambda^2 (v - w)^2 \partial_v^2 \varphi(v) + \dots$$

Inserting these expansions into the weak formulation of the Boltzmann equation (19) and rescaling the time gives

$$\langle (\varphi, \tilde{Q}_\lambda(f, f)) \rangle = \int_{\mathbb{R}_+^2} f(v)f(w) \left[ \lambda(w-v)\partial_v\varphi(v) + \frac{1}{2}(\lambda^2(v-w)^2 + \beta^2v^2)\partial_v^2\varphi(v) \right] dv dw.$$

More precisely, the asymptotics of the one-dimensional Boltzmann equation for wealth distribution (19) for  $\lambda$  sufficiently small, and in the limit  $\frac{\lambda}{\beta^2} \rightarrow \gamma$ , has been studied in [18]. In this so-called ‘‘continuous trading limit’’, it is proved that the solution to the Boltzmann equation converges toward the solution to the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{\gamma}{2} \frac{\partial^2}{\partial v^2} (v^2 f) + \frac{\partial}{\partial v} ((v - \bar{m})f), \quad (23)$$

which admits a unique stationary state of unit mass, given by the  $\Gamma$ -distribution

$$M_\lambda(v) = \frac{(\mu - 1)^\mu \exp\left(-\frac{\mu-1}{v}\right)}{\Gamma(\mu) v^{1+\mu}} \quad (24)$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1.$$

This stationary distribution exhibits a Pareto power law tail for large velocities. We remark that in (23) the growth of the second moment follows the same law as the Boltzmann equation (9).

**3. Quick overview of probability metrics.** In this section, we first briefly recall the main definitions and results about probability metrics and, more precisely, on Wasserstein ( $W_2$ ) and Fourier ( $d_s$ ) distances between two probability measures.

**3.1. Wasserstein distances.** Let us denote by  $\mathcal{P}(\mathbb{R}^N)$  the set of probability measures. Given two probability measures  $f, g \in \mathcal{P}(\mathbb{R}^N)$ , the Euclidean Wasserstein Distance is defined as

$$W_2(f, g) = \inf_{\Pi \in \Gamma} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |v - x|^2 d\Pi(v, x) \right\}^{1/2} \quad (25)$$

where  $\Pi$  runs over the set of transference plans  $\Gamma$ , that is, the set of joint probability measures on  $\mathbb{R}^N \times \mathbb{R}^N$  with marginals  $f$  and  $g \in \mathcal{P}(\mathbb{R}^N)$ . From a probabilistic point of view, the Wasserstein distance can be alternatively defined as

$$W_2(f, g) = \inf_{(V, X) \in \tilde{\Gamma}} \left\{ \mathbb{E} [|V - X|^2] \right\}^{1/2} \quad (26)$$

where  $\tilde{\Gamma}$  is the set of all possible couples of random variables  $(V, X)$  with  $f$  and  $g$  as respective laws. Let us denote by  $\mathcal{P}_2(\mathbb{R}^N)$  the set of probability measures with finite second moment and let us remark that  $W_2$  is finite for  $f, g \in \mathcal{P}_2(\mathbb{R}^N)$ .

The main properties of the Euclidean Wasserstein distance  $W_2$  are summarized in the following proposition. We refer to [13], [29], [31] for the proofs and further information on the connections to optimal mass transport theory.

**Proposition 3.1** ( $W_2$ -properties). *The space  $(\mathcal{P}_2(\mathbb{R}^N), W_2)$  is a complete metric space. Moreover, the following properties of the distance  $W_2$  hold:*

- i) **Optimal transference plan:** *The infimum in the definition of the distance  $W_2$  is achieved at a joint probability measure  $\Pi_o$  called an optimal transference plan satisfying:*

$$W_2^2(f, g) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |v - x|^2 d\Pi_o(v, x).$$

- ii) **Convergence of measures:** *Given  $\{f_n\}_{n \geq 1}$  and  $f$  in  $\mathcal{P}_2(\mathbb{R}^N)$ , the following three assertions are equivalent:*

- a)  $W_2(f_n, f)$  tends to 0 as  $n$  goes to infinity.  
b)  $f_n$  tends to  $f$  weakly-\* as a measure and

$$\int_{\mathbb{R}^N} |v|^2 f_n(v) dv \rightarrow \int_{\mathbb{R}^N} |v|^2 f(v) dv \text{ as } n \rightarrow +\infty.$$

- iii) **Convexity:** *Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_2(\mathbb{R}^N)$  and  $\alpha$  in  $[0, 1]$ ,*

$$W_2^2(\alpha f_1 + (1 - \alpha)f_2, \alpha g_1 + (1 - \alpha)g_2) \leq \alpha W_2^2(f_1, g_1) + (1 - \alpha)W_2^2(f_2, g_2).$$

*As a simple consequence, given  $f, g$  and  $h$  in  $\mathcal{P}_2(\mathbb{R}^N)$ ,*

$$W_2(h * f, h * g) \leq W_2(f, g)$$

*where  $*$  stands for the convolution in  $\mathbb{R}^N$ .*

- iv) **Additivity with respect to convolution:** *Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_2(\mathbb{R}^N)$  with equal mean values,*

$$W_2^2(f_1 * f_2, g_1 * g_2) \leq W_2^2(f_1, g_1) + W_2^2(f_2, g_2).$$

**3.2. Fourier metrics.** Given  $f \in \mathcal{P}(\mathbb{R}^N)$ , its Fourier transform or characteristic function is defined as

$$\hat{f}(k) = \int_{\mathbb{R}^N} e^{-iv \cdot k} df(v).$$

Given any  $s > 0$ , the Fourier-based metric  $d_s$  is defined as

$$d_s(f, g) = \sup_{k \in \mathbb{R}_o^N} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s} \quad (27)$$

where  $\mathbb{R}_o^N = \mathbb{R}^N - \{0\}$ , for any pair of probability measures  $f, g \in \mathcal{P}(\mathbb{R}^N)$ . This metric was introduced in [21] and further used in [16], [15], [28], [22]. Only recently, various applications to the large-time behavior of the dissipative Boltzmann equation [26], [1], [2] have revealed the importance of this distance. We refer to [17] for a complete survey of this metric and the proofs of the statements below.

The metric  $d_s$  with  $s > 0$  is well-defined and finite for any two probability measures  $f, g \in \mathcal{P}_s(\mathbb{R}^N)$  with equal moments up to  $[s]$  if  $s \notin \mathbb{N}$ , or equal moments up to  $s - 1$  if  $s \in \mathbb{N}$ . The main properties of the  $d_s$  metrics relevant to the ongoing discussion are summarized in the following result:

**Proposition 3.2.** *The distances  $d_s$  with  $s > 0$  verify the following properties:*

- i) **Convexity:** *Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_s(\mathbb{R}^N)$  with equal moments up to  $[s]$  if  $s \notin \mathbb{N}$ , or equal moments up to  $s - 1$  if  $s \in \mathbb{N}$  and  $\alpha$  in  $[0, 1]$ ,*

$$d_s(\alpha f_1 + (1 - \alpha)f_2, \alpha g_1 + (1 - \alpha)g_2) \leq \alpha d_s(f_1, g_1) + (1 - \alpha)d_s(f_2, g_2).$$

- ii) **Superadditivity with respect to convolution:** *Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_s(\mathbb{R}^N)$  with equal moments up to  $[s]$  if  $s \notin \mathbb{N}$ , or equal moments up to  $s - 1$  if  $s \in \mathbb{N}$ ,*

$$d_s(f_1 * f_2, g_1 * g_2) \leq d_s(f_1, g_1) + d_s(f_2, g_2).$$

#### 4. Large time behavior for economy model.

4.1. **Evolution of Wasserstein distance.** The Boltzmann equation (21) can be rewritten as

$$\frac{\partial f}{\partial t} = \langle f_{p+\eta} * f_q \rangle - f,$$

where we use the shorthand  $f_p(v) = (1/p)f(v/p)$  with  $p = \lambda$  and  $q = 1 - \lambda$ . Here,  $f$  is extended by 0 to the whole of  $\mathbb{R}$  in the convolution. The gain operator is defined as the measure given by

$$(\varphi, \tilde{Q}_\lambda^+(f, f)) = \left\langle \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f(v) f(w) (\varphi, \delta_{(p+\eta)v+qw}) dv dw \right\rangle$$

where  $\delta_{(p+\eta)v+qw}$  is the Delta Dirac at the post-collisional velocity  $v'$  and  $(\cdot, \cdot)$  is the duality pair between continuous functions and probability measures. In probabilistic terms, the gain operator is defined as an expectation:

$$\tilde{Q}_\lambda^+(f, f) = \langle f_{p+\eta} * f_q \rangle = \mathbb{E} [\delta_{(p+\eta)V+qW}]$$

where  $V$  and  $W$  are independent random variables with law  $f$  and independent with respect to the random variable  $\eta$ . Here the expectation is taken with respect to all random variables.

Let us take two independent pairs of random variables  $(V, X)$  and  $(W, Y)$  such that  $V$  and  $W$  have law  $f_1$  and  $X$  and  $Y$  have law  $f_2$ . From the convexity of  $W_2^2$  and the independence of the pairs, it follows that

$$W_2^2(\tilde{Q}_\lambda^+(f_1, f_1), \tilde{Q}_\lambda^+(f_2, f_2)) \leq \mathbb{E} [W_2^2(\delta_{(p+\eta)V+qW}, \delta_{(p+\eta)X+qY})]$$

for any probability densities  $f_1, f_2 \in \mathcal{P}_2(\mathbb{R})$ . Now, the last term is directly computed as the Euclidean distance between the two points  $(p+\eta)V+qW$  and  $(p+\eta)X+qY$ , and thus,

$$W_2^2(\tilde{Q}_\lambda^+(f_1, f_1), \tilde{Q}_\lambda^+(f_2, f_2)) \leq \mathbb{E} [|(p+\eta)(V-X) + q(W-Y)|^2].$$

Using independence of the pairs and taking the pairs to be optimal couples for the  $W_2(f_1, f_2)$  in the probabilistic definition (26), we deduce finally the property

$$W_2^2(\tilde{Q}_\lambda^+(f_1, f_1), \tilde{Q}_\lambda^+(f_2, f_2)) \leq [((p+\eta)^2) + q^2] W_2^2(f_1, f_2).$$

Let us define, for  $s \geq 1$

$$\mathfrak{S}(s) := \langle (p+\eta)^s \rangle + q^s - 1; \tag{28}$$

then  $\mathfrak{S}(2) = \langle (p+\eta)^2 \rangle + q^2 - 1 = 2\lambda(\lambda-1) + \beta^2$ . It is not difficult to see that the convexity property of  $W_2^2$  together with the Duhamel formula for (21) and the contractive estimate of the gain operator in  $W_2$  leads to the result:

**Theorem 4.1.** *Let  $f_1(t)$  and  $f_2(t)$  be two solutions of the one dimensional Boltzmann equation (21) corresponding to initial values  $f_1^0$  and  $f_2^0$  in  $\mathcal{P}_2(\mathbb{R}^+)$ , satisfying conditions (22). Then, for all times  $t \geq 0$ ,*

$$W_2(f_1(t), f_2(t)) \leq \exp\{\mathfrak{S}(2)t\} W_2(f_1^0, f_2^0). \tag{29}$$

*If  $\beta^2 < 2\lambda(1-\lambda)$ , then  $\mathfrak{S}(2) < 0$ , and the Wasserstein metric decays exponentially to zero in time.*

**4.2. Evolution of Fourier metrics.** Analogous results for the evolution of the  $d_s$ -metric (27) have been obtained recently in [24] by a suitable generalization of results in [25]. For the detailed computations we refer to [24]. The study of the evolution of the metric (27), leading to the understanding of the large-time behavior of the solution to the kinetic equation (21), requires a fine analysis of the quantity (28). As shown for the Wasserstein metric in the previous subsection, the sign of this quantity is in fact related to the contraction properties of the metric. Moreover, as has been noted in [24], the sign of (28) is also related both to the number of moments of the solution which remain uniformly bounded in time, and to the possibility to conclude the existence and uniqueness of a steady state. The results in [24] can be briefly summarized into the following

**Theorem 4.2.** *Take  $s > 0$  with  $\mathfrak{S}(s) < \infty$  and let  $f_1(t)$  and  $f_2(t)$  be two solutions of the one dimensional Boltzmann equation (21) corresponding to initial values  $f_1^0$  and  $f_2^0$  in  $\mathcal{P}_r(\mathbb{R}^+)$ , satisfying conditions (22) with  $r = \max\{1, s\}$ . Then the following bound holds:*

$$d_s(f_1(t), f_2(t)) \leq \exp\{\mathfrak{S}(s)t\} d_s(f_1^0, f_2^0), \quad (30)$$

where  $\mathfrak{S}(s)$  is given by (28).

Also, the temporal behavior of the moments is almost completely determined by the function  $\mathfrak{S}(s)$ .

**Theorem 4.3.** *Let  $s > 1$  and  $f_0 \in \mathcal{P}_s(\mathbb{R}^+)$  with  $0 < \mathfrak{S}(s) < \infty$  and let us denote*

$$M_s^0 := \int_{\mathbb{R}^+} v^s f_0(v) dv.$$

*Then, for the weak solution to the Boltzmann equation, the following estimates hold:*

1. *If  $\mathfrak{S}(s) > 0$ , then, as  $t \rightarrow \infty$ ,*

$$\int_{\mathbb{R}^+} v^s f(v, t) dv \geq M_s^0 \exp\{\mathfrak{S}(s)t\} + o(1).$$

2. *If  $\mathfrak{S}(s) < 0$ , then the  $s$ th moment is bounded for all times. Moreover, as  $t \rightarrow \infty$ ,*

$$\int_{\mathbb{R}^+} v^s f(v, t) dv \leq M_s^0 \exp\{\mathfrak{S}(s)t\} + o(1).$$

*Here, the remainder terms  $o(1)$  converge to zero exponentially fast.*

Another important conclusion of the analysis of [24] is that the essential function  $\mathfrak{S}(s)$  does not only decide whether or not the steady state  $f_\infty$  develops a Pareto tail. In fact, the positive zero of  $\mathfrak{S}(s)$  actually determines the value of the Pareto index.

A comparison of the contraction results for the Boltzmann equation (21) shows that the contraction properties are heavily linked, through the key function (28), to the (eventual) formation of tails. While the situation for equation (21) is reasonably well understood, the corresponding analysis for the Boltzmann equation (9) deserves further investigation. We will discuss equation (9) in detail in the following section.

**5. Large time behavior for stochastic granular media.** Let us consider here the modification of the Inelastic Maxwell Model introduced in [6]

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f, f), \quad (31)$$

where the collision operator is defined weakly as

$$(\varphi, \tilde{Q}_e(f, f)) = \frac{1}{4\pi} \left\langle \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(v) f(w) [\varphi(v') - \varphi(v)] d\sigma dv dw \right\rangle. \quad (32)$$

As discussed in the introduction, the collision mechanism relies on a random coefficient of restitution,

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-\tilde{e}}{4}u + \frac{1+\tilde{e}}{4}|u|\sigma \\ w' &= \frac{1}{2}(v+w) - \frac{1-\tilde{e}}{4}u - \frac{1+\tilde{e}}{4}|u|\sigma. \end{aligned} \quad (33)$$

As before, we write  $u = v - w$ ,  $\tilde{e} = e + \eta$  and  $\eta$  is a real-valued random variable, with zero mean and variance  $\beta^2$ , given by a measure  $\mu(s)$  with support on  $[-e, \infty)$ . Here,  $\langle \cdot \rangle$  means the expectation with respect to  $\eta$ , i.e., the integral over  $\mathbb{R}$  with respect to  $\mu$ .

It is quite straightforward to check that conservation of mass and momentum remains and that

$$\langle |v'|^2 + |w'|^2 - |v|^2 - |w|^2 \rangle = 0$$

for the model that is conservative in the mean in which  $\beta^2 = 1 - e^2$ . From (32), we deduce that the temperature evolution is

$$\frac{d}{dt} \int_{\mathbb{R}^3} |v|^2 f(t, v) dv = 0,$$

and thus we deduce that  $\theta(t) = \theta(0)$  for all times  $t \geq 0$  and we will fix it to one for convenience.

**5.1. Evolution of Wasserstein distance.** Given a probability measure  $f$  on  $\mathbb{R}^3$ , the gain operator is in fact a probability measure  $\tilde{Q}_e^+(f, f)$  defined by

$$(\varphi, \tilde{Q}_e^+(f, f)) = \left\langle \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w) (\varphi, \mathcal{U}_{v,w,\eta}) dv dw \right\rangle$$

where  $\mathcal{U}_{v,w,\eta}$  is the uniform probability distribution on the sphere  $S_{v,w}$  with center  $c_{v,w} = \frac{1}{2}(v+w) + \left[\frac{1-\tilde{e}}{4} + \eta\right](v-w)$  and radius  $r_{v,w} = \frac{1+\tilde{e}}{4}|v-w|$  as in [14]. In probabilistic terms, the gain operator is defined as an expectation:

$$\tilde{Q}_e^+(f, f) = \mathbb{E}[\mathcal{U}_{V,W,\eta}]$$

where  $V$  and  $W$  are independent random variables with law  $f$  and independent of the law of  $\eta$ . As in [14], we get the following result:

**Theorem 5.1.** *Given  $f$  and  $g$  in  $\mathcal{P}_2(\mathbb{R}^3)$  with equal mean velocity, then*

$$W_2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) \leq W_2(f, g).$$

*Proof.* Let us take two independent pairs of random variables  $(V, X)$  and  $(W, Y)$  such that  $V$  and  $W$  have law  $f$  and  $X$  and  $Y$  have law  $g$ . Also, let us take two independent random variables  $\eta$  and  $\tilde{\eta}$  with law  $\mu$ . Convexity of  $W_2^2$  implies

$$W_2^2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) = W_2^2(\mathbb{E}[\mathcal{U}_{V,W,\eta}], \mathbb{E}[\mathcal{U}_{X,Y,\tilde{\eta}}]) \leq \mathbb{E}[W_2^2(\mathcal{U}_{V,W,\eta}, \mathcal{U}_{X,Y,\tilde{\eta}})] \quad (34)$$

where the expectation is taken with respect to the joint probability density in  $\mathbb{R}^{14}$  of the six random variables. Here, the independence of the pairs of random variables has been used.

As proved in [14], the  $W_2^2$  distance between the uniform distributions on the sphere with center  $O$  and radius  $r$ ,  $\mathcal{U}_{O,r}$ , and on the sphere with center  $O'$  and radius  $r'$ ,  $\mathcal{U}_{O',r'}$ , in  $\mathbb{R}^3$  is given by  $|O' - O|^2 + (r' - r)^2$ .

We now estimate the right-hand side of (34) by using the formulas for the center and radii of the spheres given in (33) to deduce

$$\begin{aligned} W_2^2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) &\leq \left\langle \frac{5 - 2\tilde{e} + \tilde{e}^2}{8} \right\rangle \mathbb{E}[|V - X|^2] + \left\langle \frac{(1 + \tilde{e})^2}{8} \right\rangle \mathbb{E}[|W - Y|^2] \\ &\quad + \left\langle \frac{1 - \tilde{e}^2}{4} \right\rangle \mathbb{E}[(V - X) \cdot (W - Y)] \end{aligned}$$

where the Cauchy-Schwartz inequality has been used.

Finally, we take both pairs  $(V, X)$  and  $(W, Y)$  as independent pairs of variables with each of them being an optimal couple for  $W_2(f, g)$  in the probabilistic definition (26) to obtain

$$W_2^2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) \leq \frac{3 + e^2 + \beta^2}{4} W_2^2(f, g) + \frac{1 - e^2 - \beta^2}{4} \mathbb{E}[(V - X) \cdot (W - Y)],$$

where the last term is zero because the random variables are independent and have equal means. Since  $\beta^2 = 1 - e^2$  in the conservative case, the result is proved.  $\square$

As a consequence of the previous property of the gain operator, we draw the following conclusion about controlling the distance between any two solutions of (31) in the conservative case.

**Theorem 5.2.** *If  $f_1$  and  $f_2$  are two solutions to (31) with respective initial data  $f_1^0$  and  $f_2^0$  in  $\mathcal{P}_2(\mathbb{R}^3)$  with zero mean velocity, then, for all  $t \geq 0$ ,*

$$W_2^2(f_1(t), f_2(t)) \leq W_2^2(f_1^0, f_2^0).$$

*Proof.* Duhamel's formula for (31) reads as

$$f_i(t) = e^{-t} f_i^0 + \int_0^t e^{-(t-s)} \tilde{Q}_e^+(f_i(s), f_i(s)) ds, \quad i = 1, 2.$$

As before, the convexity of the squared Wasserstein distance in Proposition 3.1 and the contraction of the gain operator in Theorem 5.1 imply

$$\begin{aligned} W_2^2(f_1(t), f_2(t)) &\leq e^{-t} W_2^2(f_1^0, f_2^0) \\ &\quad + \int_0^t e^{-(t-s)} W_2^2(\tilde{Q}_e^+(f_1(s), f_1(s)), \tilde{Q}_e^+(f_2(s), f_2(s))) ds \\ &\leq e^{-t} W_2^2(f_1^0, f_2^0) \\ &\quad + \int_0^t e^{-(t-s)} W_2^2(f_1(s), f_2(s)) ds. \end{aligned}$$

Therefore, the function  $y(t) = e^t W_2^2(f_1(t), f_2(t))$  satisfies the inequality

$$y(t) \leq y(0) + \int_0^t y(s) ds$$

and thus  $y(t) \leq y(0) e^t$  by Gronwall's lemma, concluding the argument.  $\square$

**5.2. Evolution of Fourier metrics.** We start by writing a closed form of the Boltzmann equation in Fourier variables. In fact, it is not difficult using Bobylev's identity in [3], [4], [5], [6] to get

$$\widehat{\tilde{Q}_e^+(f, f)} = \frac{1}{4\pi} \left\langle \int_{S^2} \hat{f}(t, k_-) \hat{f}(t, k_+) d\sigma \right\rangle$$

where

$$k_- = \frac{1+\tilde{e}}{4}k - \frac{1+\tilde{e}}{4}|k|\sigma \quad \text{and} \quad k_+ = \frac{3-\tilde{e}}{4}k + \frac{1+\tilde{e}}{4}|k|\sigma.$$

Let us start by analyzing the evolution of the distance  $d_2$  that in view of the properties in Propositions 3.1 and 3.2 should verify the same non-strict contraction as the transport distance  $W_2$ .

**Theorem 5.3.** *Given  $f$  and  $g$  in  $\mathcal{P}_2(\mathbb{R}^3)$  with equal mean velocity,*

$$d_2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) \leq \frac{3+e^2+\beta^2}{4} d_2(f, g).$$

*Proof.* Using the Fourier representation formula above, we deduce

$$\frac{\widehat{\tilde{Q}_e^+(f, f)}(k) - \widehat{\tilde{Q}_e^+(g, g)}(k)}{|k|^2} = \frac{1}{4\pi} \left\langle \int_{S^2} \left[ \frac{\hat{f}(k_-)\hat{f}(k_+) - \hat{g}(k_-)\hat{g}(k_+)}{|k|^2} \right] d\sigma \right\rangle$$

for all  $k \in \mathbb{R}_o^3$ . We now estimate the integrand as

$$\begin{aligned} \left| \frac{\hat{f}(k_-)\hat{f}(k_+) - \hat{g}(k_-)\hat{g}(k_+)}{|k|^2} \right| &\leq \sup_{k \in \mathbb{R}_o^3} \left\{ \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^2} \right\} \left( \frac{|k_-|^2 + |k_+|^2}{|k|^2} \right) \\ &= d_2(f, g) \left( \frac{|k_-|^2 + |k_+|^2}{|k|^2} \right), \end{aligned}$$

and thus

$$d_2(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) \leq \frac{1}{4\pi} \left\langle \int_{S^2} \left( \frac{|k_-|^2 + |k_+|^2}{|k|^2} \right) d\sigma \right\rangle d_2(f, g).$$

We observe that

$$\frac{|k_-|^2 + |k_+|^2}{|k|^2}$$

is a function of the angle between the unit vectors  $k/|k|$  and  $\sigma$  and the random variable  $\eta$ , and that

$$I := \frac{1}{4\pi} \left\langle \int_{S^2} \frac{|k_-|^2 + |k_+|^2}{|k|^2} d\sigma \right\rangle = \frac{3+e^2+\beta^2}{4}.$$

In fact, we can compute

$$\begin{aligned} |k_-|^2 &= |k|^2 \left( \frac{1+\tilde{e}}{4} \right)^2 2(1 - \cos \vartheta) \\ |k_+|^2 &= |k|^2 \left[ \left( \frac{3-\tilde{e}}{4} \right)^2 + \left( \frac{1+\tilde{e}}{4} \right)^2 + 2 \left( \frac{3-\tilde{e}}{4} \right) \left( \frac{1+\tilde{e}}{4} \right) \cos \vartheta \right] \end{aligned} \tag{35}$$

where  $\vartheta$  is the angle between the unit vectors  $k/|k|$  and  $\sigma$  from which the value of  $I$  is obtained. Putting together previous estimates we get the contraction in  $d_2$  with the same constant as  $W_2^2$  as desired.  $\square$

Now, let us see that we can also control Fourier-based distances with exponent  $2 + \alpha$ ,  $\alpha \in [0, \infty)$ . Let us set

$$\begin{aligned} \mathfrak{A}(\alpha, e, \eta) &:= \frac{1}{2} \left\langle \int_0^\pi \left\{ \left[ \left( \frac{1+\tilde{e}}{4} \right)^2 2(1-\cos\vartheta) \right]^{\frac{2+\alpha}{2}} \right. \right. \\ &\quad \left. \left. + \left[ \left( \frac{3-\tilde{e}}{4} \right)^2 + \left( \frac{1+\tilde{e}}{4} \right)^2 + 2 \left( \frac{3-\tilde{e}}{4} \right) \left( \frac{1+\tilde{e}}{4} \right) \cos\vartheta \right]^{\frac{2+\alpha}{2}} \right\} \sin\vartheta \, d\vartheta \right\rangle \\ &= \frac{2}{4+\alpha} \left\langle \left( \frac{1+\tilde{e}}{2} \right)^{2+\alpha} + \frac{1-|\frac{1-\tilde{e}}{2}|^{4+\alpha}}{1-|\frac{1-\tilde{e}}{2}|^2} \right\rangle. \end{aligned} \quad (36)$$

Whenever there is no confusion, i.e. for  $e$  and  $\eta$  fixed, we will denote just by  $\mathfrak{A}(\alpha)$  the above constant.

**Theorem 5.4.** *Given  $f, g \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$  with equal moments up to order  $2 + [\alpha]$ , there exists an explicit constant  $\mathfrak{A}(\alpha, e, \eta) > 0$  given by (36) such that*

$$d_{2+\alpha}(\tilde{Q}_e^+(f, f), \tilde{Q}_e^+(g, g)) \leq \mathfrak{A}(\alpha, e, \eta) d_{2+\alpha}(f, g).$$

*Proof.* As in the proof of the previous theorem, we compute

$$\begin{aligned} \left| \frac{\widehat{\tilde{Q}_e^+(f, f)}(k) - \widehat{\tilde{Q}_e^+(g, g)}(k)}{|k|^{2+\alpha}} \right| &= \frac{1}{4\pi} \left| \left\langle \int_{S^2} \frac{\hat{f}(k^+) \hat{f}(k^-) - \hat{g}(k^+) \hat{g}(k^-)}{|k|^{2+\alpha}} \, d\sigma \right\rangle \right| \\ &\leq A \sup_{k \in \mathbb{R}^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^{2+\alpha}} \end{aligned}$$

where  $A$  is given by

$$A := \frac{1}{4\pi} \left\langle \int_{S^2} \frac{|k_+|^{2+\alpha} + |k_-|^{2+\alpha}}{|k|^{2+\alpha}} \, d\sigma \right\rangle. \quad (37)$$

By inserting the expressions of  $k_-$  and  $k_+$  into (37) and computing the integral we conclude  $A = \mathfrak{A}(\alpha, e, \eta)$  and the proof follows.  $\square$

As a consequence, we obtain an estimate on contraction/expansion of the Fourier distances  $d_{2+\alpha}$  between solutions.

**Theorem 5.5.** *Let  $\alpha > 0$  be such that  $\mathfrak{A}(\alpha, e, \eta) < \infty$ . Let  $f_1$  and  $f_2$  be two solutions to (31) corresponding to initial values  $f_1^0, f_2^0$  with equal moments up to  $2 + [\alpha]$ . Then, for all  $t \geq 0$ ,*

$$d_{2+\alpha}(f_1(t), f_2(t)) \leq d_{2+\alpha}(f_1^0, f_2^0) e^{-C(\alpha, e, \eta)t}, \quad (38)$$

with  $C(\alpha, e, \eta) = 1 - \mathfrak{A}(\alpha, e, \eta)$ .

*Proof.* The Fourier expression of equation (31) is given by

$$\frac{\partial \hat{f}}{\partial t} = \frac{1}{4\pi} \int_{S^2} \hat{f}(k_+) \hat{f}(k_-) \, d\sigma - \hat{f} = \widehat{\tilde{Q}_e^+(f, f)} - \hat{f},$$

whose solution satisfies

$$\hat{f}(t, k) = e^{-t} \hat{f}(0, k) + \int_0^t e^{-(t-s)} \widehat{\tilde{Q}_e^+(f, f)}(s, k) \, ds. \quad (39)$$

Taking the expressions of the two solutions  $\hat{f}_1(t)$  and  $\hat{f}_2(t)$  in (39), subtracting them and dividing by  $|k|^{2+\alpha}$  with  $k \in \mathbb{R}_o^3$ , we get

$$e^t \frac{(\hat{f}_1 - \hat{f}_2)(t, k)}{|k|^{2+\alpha}} = \frac{\hat{f}_1(0, k) - \hat{f}_2(0, k)}{|k|^{2+\alpha}} + \int_0^t e^s \frac{(\widehat{Q_e^+}(f_1, f_1) - \widehat{Q_e^+}(f_2, f_2))(s, k)}{|k|^{2+\alpha}} ds.$$

Using Theorem 5.4 and taking the supremum in  $k \in \mathbb{R}_o^3$ , we obtain

$$e^t d_{2+\alpha}(\hat{f}_1, \hat{f}_2)(t) \leq d_{2+\alpha}(\hat{f}_1(0), \hat{f}_2(0)) + \mathfrak{A}(\alpha, e, \eta) \int_0^t e^s d_{2+\alpha}(\hat{f}_1, \hat{f}_2)(s) ds.$$

Let us set  $w(\tau) = e^t d_{2+\alpha}(\hat{f}_1, \hat{f}_2)(t)$ . Then

$$w(t) \leq w(0) + \mathfrak{A}(\alpha, e, \eta) \int_0^t w(s) ds,$$

which by Gronwall's inequality implies  $w(t) \leq w(0) e^{\mathfrak{A}(\alpha, e, \eta)t}$ , concluding the proof.  $\square$

The function  $\mathfrak{A}(\alpha) : [0, \infty) \rightarrow \mathbb{R}^+$  is convex by direct inspection. Taking into account that  $\mathfrak{A}(0) = 1$ , there are only three possible scenarios for the qualitative behavior of  $\mathfrak{A}$ . These are characterized by the sign of  $\mathfrak{A}'(0)$ . In case  $\mathfrak{A}'(0) \geq 0$ , the function  $\mathfrak{A}(\alpha)$  has a minimum at  $\alpha = 0$  due to convexity, and thus  $\mathfrak{A}(\alpha) > 1$  for all  $\alpha > 0$ . In this case, there does not exist any  $\bar{\alpha} \in \mathbb{R}_+$  such that  $\mathfrak{A}(\bar{\alpha}) < 1$  and there are no contraction, only expansion, estimates of  $d_s$  for  $s > 2$ .

Suppose that  $\mathfrak{A}'(0) < 0$ . In this case, the contraction properties of  $d_s$  depend on whether

$$\lim_{\alpha \rightarrow \infty} \mathfrak{A}(\alpha) < 1$$

or

$$\lim_{\alpha \rightarrow \infty} \mathfrak{A}(\alpha) > 1.$$

In the former case,  $\mathfrak{A}(\alpha) < 1$  for  $\alpha > 0$ . Theorem 5.4 then implies that the  $d_s$ -metric is contractive for all values of the parameter  $s > 2$ . In the latter, since  $\mathfrak{A}(0) = 0$ , the convex function  $\mathfrak{A}(\alpha)$  has a minimum attained at some point  $\bar{\alpha} > 0$ , and at the same time there exists  $\tilde{\alpha} > \bar{\alpha}$  for which  $\mathfrak{A}(\tilde{\alpha}) = 1$ . Thus,  $\mathfrak{A}(\alpha) < 1$  in the interval  $0 < \alpha < \tilde{\alpha}$ , and at the same time  $\mathfrak{A}(\alpha) > 1$  for  $\alpha > \tilde{\alpha}$ . In this case Theorem 5.4 implies that the Boltzmann equation is contractive up to but not including order  $\tilde{\alpha}$ .

**Remark 5.6.** In order to clarify the behavior of  $\mathfrak{A}(\alpha, e, \eta)$ , we can fix the random variable  $\eta$  to assume only two values, while respecting conditions (5). This can be done by assuming that  $\eta$  only takes the value  $\sqrt{1 - e^2}/\varrho$  with probability  $\varrho^2/(1 + \varrho^2)$  and the value  $\sqrt{1 - e^2}\varrho$  with probability  $1/(1 + \varrho^2)$ . By varying the parameters  $\varrho$  and  $e$  one encounters the whole variety of possible behaviors of the function  $\mathfrak{A}(\alpha, e, \eta)$ . Since

$$\mathfrak{A}(\alpha, e, \eta) = \frac{2}{4 + \alpha} \left\langle \left( \frac{1 + \tilde{e}}{2} \right)^{2+\alpha} + \frac{1 - |\frac{1-\tilde{e}}{2}|^{4+\alpha}}{1 - |\frac{1-\tilde{e}}{2}|^2} \right\rangle,$$

$\mathfrak{A}(\alpha, e, \eta)$  results in the sum of four contributions, one of which is

$$C(\alpha, e, \eta) = \frac{1}{1 + \varrho^2} \frac{2}{4 + \alpha} \left( \frac{1 + e + \sqrt{1 - e^2}\varrho}{2} \right)^{2+\alpha}.$$

For any fixed values of  $\bar{\alpha} > 0$  and  $e$ , since the numerator grows like  $\varrho^{2+\alpha}$ , we can choose  $\varrho \gg 1$  in such a way that  $C(\alpha, e, \eta) > 1$ , and Theorem 5.4 implies that the Boltzmann equation is contractive up to but not including order  $\bar{\alpha}$ .

On the other hand, choosing for example  $\alpha = 2$  to simplify computations, one obtains easily

$$\mathfrak{A}(2, e, \eta) = \frac{1}{3} \left\langle \left( \frac{1+\tilde{e}}{2} \right)^4 + 1 + \left( \frac{1-\tilde{e}}{2} \right)^2 + \left( \frac{1-\tilde{e}}{2} \right)^4 \right\rangle = \frac{23 - e + \langle \tilde{e}^4 \rangle}{24}. \quad (40)$$

Choosing now  $1 - e \ll 1$ , and  $\varrho = \sqrt{1 - e^2}/e$ , one obtains that  $\tilde{e}$  assumes the value 0 with probability  $1 - e^2$  and the value  $1/e$  with probability  $e^2$ . Therefore  $\langle \tilde{e}^4 \rangle = 1/e^2$ , which implies  $\mathfrak{A}(2, e, \eta) < 1$  as long as  $1/e^2 - e < 2$ . In this second case Theorem 5.4 implies that the Boltzmann equation is contractive at least up to order 4.

**5.3. Existence and uniqueness of regular isotropic steady states.** Existence and uniqueness of steady states, as well as the size of their overpopulated tails, can be derived in full generality (that is, without imposing restrictive conditions on the random coefficient of restitution) by adapting to the present situation the methodology of [2], which refers to the inelastic Boltzmann equation for Maxwell molecules. This methodology, in fact, is based only on the contractivity properties of the  $d_s$ -metric, which are analogous to Theorems 5.4 and 5.5.

It has to be remarked that the approach in [2] is not suitable to recover the (eventual) regularity of the steady profile. A regularity result for the steady state of the inelastic Boltzmann equation for Maxwell molecules has been obtained in a recent paper by Bobylev and Cercignani [8]. In this paper they were concerned with properties of the self-similar profiles of the Boltzmann equation for both elastic and inelastic collisions, and, in addition to the existence, they obtained results on the regularity of the steady profiles by showing that the Fourier transform of the steady profile satisfies a suitable upper bound. Their method takes advantage of the existence of a super-solution to the rescaled equation in Fourier variables (BKW-mode). In our collisional setting, the situation is more involved, and it requires a precise analysis.

In Fourier variables, the steady state of (9) is a solution of the integral equation

$$\frac{1}{4\pi} \left\langle \int_{S^2} \hat{f}(k_-) \hat{f}(k_+) d\sigma \right\rangle = \hat{f}(k), \quad (41)$$

where  $k_+$  and  $k_-$  are given by the relations

$$k_- = \frac{1+\tilde{e}}{4}k - \frac{1+\tilde{e}}{4}|k|\sigma \quad \text{and} \quad k_+ = \frac{3-\tilde{e}}{4}k + \frac{1+\tilde{e}}{4}|k|\sigma.$$

Since isotropy is not destroyed by the collision operator, by choosing isotropic initial values, one concludes with the isotropy of the (eventual) steady state. Taking this property into account, the following result can be obtained as a consequence of Theorem 5.5 (see [2] for details).

**Corollary 5.7.** *Equation (31) has a unique isotropic steady state  $f_\infty$  in the set of isotropic probability measures with unit mass, zero mean velocity and unit temperature. Moreover, given any solution  $f$  to (31) for the initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$  with zero mean velocity and unit pressure tensor,*

$$d_{2+\alpha}(f(t), f_\infty) \leq d_{2+\alpha}(f_0, f_\infty) e^{-C(\alpha, e, \eta)t}$$

for all  $t \geq 0$ ,  $0 < \alpha < 1$ . Thus, if  $\mathfrak{A}(\alpha, e, \eta) < 1$ ,  $f(t)$  converges to the stationary state as  $t \rightarrow \infty$  in the  $d_{2+\alpha}$  sense.

**Remark 5.8.** The previous result shows that the stationary states attract all solutions with initial data having zero mean velocity and unit pressure tensor. The assumption of having unit pressure tensor can be weakened to having initial unit temperature by proceeding similarly to the homogeneous cooling state analysis in [11], [2].

Let us define

$$\begin{aligned} a^2(e, \eta, \theta) &= \frac{|k_-|^2}{|k|^2} = \left(\frac{1+\tilde{e}}{4}\right)^2 2(1 - \cos \vartheta) \\ b^2(e, \eta, \theta) &= \frac{|k_+|^2}{|k|^2} = \left[\left(\frac{3-\tilde{e}}{4}\right)^2 + \left(\frac{1+\tilde{e}}{4}\right)^2 + 2\left(\frac{3-\tilde{e}}{4}\right)\left(\frac{1+\tilde{e}}{4}\right)\cos \vartheta\right]. \end{aligned} \quad (42)$$

Recalling the definition of  $k_+$  and  $k_-$  given in (35), it is immediate to show that

$$a + b \geq 1; \quad \frac{1}{2} \left\langle \int_0^\pi (a^2 + b^2) \sin \theta d\theta \right\rangle = 1. \quad (43)$$

The first property in (43) is a direct consequence of the equality  $k_+ + k_- = k$ , while the second is the equality  $\mathfrak{A}(0) = 1$  in (37). Let us set  $x = |k|$ . Then, for any function  $\psi(x)$ , the left-hand side of (41) can be rewritten in the form

$$R[\psi(x)] = \frac{1}{2} \left\langle \int_0^\pi \psi(ax)\psi(bx) \sin \theta d\theta \right\rangle. \quad (44)$$

Under the conditions of Corollary 5.7, the Boltzmann equation has a unique steady state  $\hat{f}_\infty(x)$ , of unit mass, zero mean velocity and unit second moment.

Let us remark that  $0 \leq R[\psi] \leq 1$  if  $0 \leq \psi \leq 1$ , and  $R[\psi] \leq R[\phi]$  if  $0 \leq \psi \leq \phi$ . Hence the iteration is monotone increasing and converges point-wise if we choose the initial approximation  $0 \leq \phi_0 \leq 1$  in such a way that  $\phi_0 \leq R[\phi_0]$ . As observed in [8],  $\phi_0(x) = \exp\{-x^2/2\}$  allows us to obtain a monotone increasing sequence. In fact, since the function  $e^{-r}$ ,  $r \geq 0$  is convex, by Jensen's inequality we obtain

$$\left\langle e^{-\frac{1}{2}(a^2+b^2)x^2} \right\rangle \geq e^{-\langle \frac{1}{2}(a^2+b^2)x^2 \rangle} = e^{-x^2/2}. \quad (45)$$

This implies that the limit  $\hat{f}_\infty(x) \geq 0$ . The trivial limit  $\hat{f}_\infty(x) = 1$  will be excluded if there exists a non-zero function  $\psi_0(x)$  such that

$$\phi_0(x) \leq \psi_0(x), \quad (46)$$

and at the same time  $\psi_0(x)$  generates a monotone decreasing sequence.

Inspired by the ideas of Desvillettes et al in [19], given a fixed positive constant  $\rho$ , we introduce the fixed point operator

$$\mathbf{R}[\psi](x) := \begin{cases} \hat{f}_\infty(x) & \text{if } x < \rho \\ R[\psi(x)] & \text{if } x \geq \rho \end{cases}$$

on bounded complex functions  $\psi : \mathbb{R} \rightarrow \mathbb{C}$ . Notice that  $\mathbf{R}$  is closely related to the Fourier transform of the collision kernel.

**Lemma 5.9.** *Let  $0 \leq \hat{f}_\infty(x) \leq 1$  be the steady state of the Boltzmann equation, and let us define*

$$\psi_0(x) := \begin{cases} \hat{f}_\infty(x) & \text{if } x < \rho \\ \exp(-\mu x) & \text{if } x \geq \rho. \end{cases}$$

*Then, if the random variables  $a(e, \eta, \theta)$  and  $b(e, \eta, \theta)$  are such that*

$$P(a < \delta) + P(b < \delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \quad (47)$$

*there exist positive constants  $\rho$  and  $\mu$  such that*

$$R[\psi_0](x) \leq \psi_0(x).$$

*Proof.* Since the steady state is an isotropic probability density function of unit mass, zero mean velocity and unit second moment, there exist positive constants  $M < 1/2$  and  $\rho$  such that (cfr. [19])

$$0 \leq \hat{f}_\infty(x) \leq e^{-Mx^2} \quad \text{if } x \leq \rho. \quad (48)$$

Hence, we can fix  $\rho$  and  $M$  to obtain

$$\psi_0(x) \leq e^{-Mx^2} \quad \text{if } x \leq \rho. \quad (49)$$

Clearly, thanks to the definition of  $\psi_0$ , if  $x \leq \rho$ , there is nothing to prove. Therefore, let us consider the possible cases corresponding to  $x > \rho$ . Since  $a + b \geq 1$ , if both  $ax \geq \rho$ ,  $bx \geq \rho$ ,

$$\langle \psi_0(ax)\psi_0(bx)e^{\mu x} \rangle \leq 1.$$

If now both  $ax < \rho$  and  $bx < \rho$ , using bound (49), we obtain

$$\langle \psi_0(ax)\psi_0(bx)e^{\mu x} \rangle \leq \langle e^{g(x)} \rangle,$$

where

$$g(x) = \mu x - M(a^2 + b^2)x^2.$$

Since  $a + b \geq 1$ , it follows that  $a^2 + b^2 \geq 1/2$ . Thus

$$g(x) \leq \mu x - \frac{1}{2}(a^2 + b^2)x^2 \leq 0 \quad \text{if } \mu \leq \frac{M}{2\rho}. \quad (50)$$

Consider now the case in which  $ax \leq \rho$ , while  $bx > \rho$ . In this case

$$\langle \psi_0(ax)\psi_0(bx)e^{\mu x} \rangle \leq \langle e^{h(x)} \rangle,$$

where

$$h(x) = \mu(1 - a)x - Mb^2x^2.$$

Since  $a + b \geq 1$ , it follows that  $b \geq 1 - a$ , and

$$h(x) \leq z(bx) = \mu bx - M(bx)^2 \leq \frac{\mu^2}{4M^2}. \quad (51)$$

In fact, the function  $z(r)$  has a maximum at  $\bar{r} = \mu/(2M)$ . Moreover, since  $z(r)$  decreases for  $r > \bar{r}$ , if  $r \geq 3\bar{r}$ ,

$$z(r) \leq z(3\bar{r}) = -3\frac{\mu^2}{4M^2}. \quad (52)$$

Let us split the calculation of the mean value into the sets  $A = \{bx \geq 3\bar{r}\}$  and  $A^c = \{bx < 3\bar{r}\}$ . Thanks to conditions (51) and (52) one obtains

$$\langle e^{h(x)} \rangle \leq P(A) \exp\left\{-3\frac{\mu^2}{4M^2}\right\} + P(A^c) \exp\left\{\frac{\mu^2}{4M^2}\right\}. \quad (53)$$

Let us set  $\delta = 3\bar{r} = 3\mu/(2M)$ . By hypothesis, since  $x > \rho$ ,

$$P(A^c) = P(bx < \delta) \leq P(b\rho < \delta) \rightarrow 0 \quad \text{if } \delta \rightarrow 0.$$

Consider that we can rewrite (53) as

$$\begin{aligned} \langle e^{h(x)} \rangle &\leq (1 - P(A^c)) \exp\left\{-\frac{1}{3}M\delta^2\right\} + P(A^c) \exp\left\{\frac{1}{3}M\delta^2\right\} \\ &= 1 - \frac{1}{3}(1 - 4P(A^c))M\delta^2 + o(\delta^2) \leq 1 \end{aligned}$$

if  $\delta$  is sufficiently small. Now, this condition on  $\delta$  can be satisfied by choosing  $\mu$  sufficiently small. This is not in contrast with condition (50). Since the case in which  $ax > \rho$  while  $bx \leq \rho$  can be treated likewise, the lemma is proven.  $\square$

**Remark 5.10.** Condition (47) excludes some pathological situations related to the definition of the random variable  $\eta$  that describes the randomness of the coefficient of restitution  $e$ . For example, condition (47) is violated if  $\eta$  is concentrated on some particular point,

$$P(\eta = 1 - e) = p > 0.$$

In this case, in fact,  $P(b(e, \eta, 0) = 0) = p$ , and condition (47) is false.

Lemma 5.9 implies that, starting from  $\psi_0$ , the iteration process leads to a monotone decreasing sequence. On the other hand, it is clear that, for  $\mu$  sufficiently small,

$$0 \leq \phi_0(x) \leq \psi_0(x) \leq 1.$$

Given  $\mu > 0$ , define  $K_\mu$  as the set of functions  $\psi$  with  $\psi(0) = 1$ ,  $\psi'(0) = \hat{f}'_\infty(0)$ , and satisfying the estimates

$$|\psi(x)| \leq \exp(-\kappa x^2) \quad \text{for } x < \rho, \quad |\psi(x)| \leq \exp(-\mu x) \quad \text{for } x \geq \rho. \quad (54)$$

The previous inequalities prove the following

**Theorem 5.11.** *For any pair of functions  $a$  and  $b$  satisfying conditions (43) and (47), the integral equation (44) has a nontrivial solution  $\hat{f}_\infty(x)$  such that  $\hat{f}_\infty(x)$  belongs to the Gevrey class  $K_\mu$  defined by (54).*

**Remark 5.12.** An analogous regularity result can be proven for the steady state to the one-dimensional kinetic model (21) [24]. In this case, it is important to know that the mean wealth of the stationary state is equal to one.

**5.4. Fat tails of stationary states.** In this work, we will only examine the case of the fourth moment, postponing the complete analysis of moment evolution to future research. Here, we will show that under certain conditions on the random variable, the fourth moment diverges or is controlled uniformly.

**Lemma 5.13.** *Let the restitution coefficient  $e$  and the random variable  $\eta$  be chosen so that  $\mathfrak{A}(2, e, \eta) < 1$ . If  $f^0$  is a Borel probability measure on  $\mathbb{R}^3$  such that*

$$\int_{\mathbb{R}^3} |v|^4 f^0(v) dv < \infty,$$

*then the solution  $f$  to (31) with initial datum  $f^0$  satisfies*

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^4 f(t, v) dv < \infty.$$

*Proof.* Without loss of generality we can assume that  $f^0$ , and hence  $f(t)$  for all  $\tau \geq 0$ , has zero mean velocity and unit temperature. We let

$$m_4(t) = \int_{\mathbb{R}^3} |v|^4 f(t, v) dv$$

denote the fourth order moment of  $f(t)$ . Then, using the weak formulation of the inelastic Boltzmann equation, we have:

$$\frac{dm_4(t)}{dt} = \int_{\mathbb{R}^3} |v|^4 \tilde{Q}_e(f(t), f(t))(v) dv \quad (55)$$

that can be computed as in [14] by

$$\begin{aligned} \int_{\mathbb{R}^3} |v|^4 \tilde{Q}_e(f, f)(v) dv &= -\langle \zeta \rangle \int_{\mathbb{R}^3} |v|^4 f(v) dv + \langle \mu_1 \rangle \left( \int_{\mathbb{R}^3} |v|^2 f(v) dv \right)^2 \\ &\quad + \langle \mu_2 \rangle \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w)^2 f(v) f(w) dv dw \end{aligned}$$

where

$$\mu_1 = \frac{1}{8}(\nu_1 + \nu_2 - \nu_3) \quad \text{and} \quad \mu_2 = \frac{1}{4}(\nu_1 - \nu_2)$$

with

$$\nu_1 = (\epsilon^2 + \epsilon'^2)^2 - 1 + \frac{4}{3}\epsilon^2\epsilon'^2, \quad \nu_2 = 2[\epsilon^2 + \epsilon'^2 - 1 + \frac{2}{3}\epsilon'^2], \quad \nu_3 = 4(\epsilon^2 - 1),$$

and

$$\zeta = \frac{1}{3}(1 + 4\epsilon - 7\epsilon^2 + 4\epsilon^3 - 2\epsilon^4) \quad \text{with} \quad \epsilon = \frac{1 - \tilde{e}}{2} \quad \text{and} \quad \epsilon' = 1 - \epsilon.$$

Now, (55) reads

$$\frac{dm_4(t)}{dt} = -\langle \zeta \rangle m_4(t) + m(t) \quad (56)$$

where  $m(t)$  is a combination of second order moments, which are bounded in time since the kinetic energy is preserved by equation (31). Moreover one can check from the expression of  $\zeta$  in terms of  $e$  that  $\langle \zeta \rangle = 1 - \mathfrak{A}(2, e, \eta) > 0$ . This ensures that  $m_4(t)$  is bounded uniformly in time if initially finite, and concludes the argument.  $\square$

The preceding result also shows the divergence of the fourth moment in case the random variable  $\eta$  and the restitution coefficient  $e$  are chosen to satisfy  $\mathfrak{A}(2, e, \eta) > 1$  but  $\mathfrak{A}(\alpha, e, \eta) < 1$  for some  $0 < \alpha < 2$ .

**Corollary 5.14.** *Let the restitution coefficient  $e$  and the random variable  $\eta$  be chosen so that  $\mathfrak{A}(2, e, \eta) > 1$  but  $\mathfrak{A}(\alpha, e, \eta) < 1$  for some  $0 < \alpha < 2$ . Then, the unique isotropic steady state  $f_\infty$  in  $\mathcal{P}_2(\mathbb{R}^3)$  of equation (31) with zero mean velocity and unit pressure tensor has unbounded fourth moment.*

*Proof.* With the notation of the previous subsection, the evolution of the fourth moment for isotropic densities given in Lemma 5.13 ensures that

$$\frac{dm_4(t)}{dt} = -\langle \zeta \rangle m_4(t) + m(t),$$

where  $m(t)$ , which is a combination of second order moments, is bounded from below. Recall that  $\langle \zeta \rangle = 1 - \mathfrak{A}(2, e, \eta) < 0$  to conclude.  $\square$

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