Explicit equilibria in a kinetic model of gambling

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We introduce and discuss a nonlinear kinetic equation of Boltzmann type which describes the evolution of wealth in a pure gambling process, where the entire sum of wealths of two agents is up for gambling, and randomly shared between the agents. For this equation the analytical form of the steady states is found for various realizations of the random fraction of the sum which is shared to the agents. Among others, the exponential distribution appears as steady state in case of a uniformly distributed random fraction, while Gamma distribution appears for a random fraction which is Beta distributed. The case in which the gambling game is only *conservative-in-the-mean* is shown to lead to an explicit heavy tailed distribution.

I. INTRODUCTION

Various concepts and techniques of statistical mechanics have been fruitfully applied for years to a wide variety of complex extended systems, physical and non-physical, in an effort to understand the emergent properties appearing in them. Economics is, by far, one of the complex extended systems to which methods borrowed from statistical mechanics for particle systems have been applied [1–8]. Starting from the original idea developed by Angle [9, 10], a variety of both discrete and continuous models has been proposed and studied in view of the relation between parameters in the microscopic roles and the resulting macroscopic statistics [2, 4, 5, 11, 12]. A typical ingredient on the microscopic level is a mechanism for saving, first introduced in [2]. It ensures that agents exchange at most a certain fraction of their wealth in each trade event; this is in contrast to the original molecular dynamics for gases. Moreover, randomness plays a rôle in virtually all available models, taking into account that many trades are risky, so that the exact amount of money changing hands is not known a priori. In most of the models introduced so far, the trading mechanism leaves the total mean wealth unchanged. Then, depending on the specific choice of the saving mechanism and the stochastic nature of the trades, the studied systems produce (or not) wealth curves with the desired Pareto tail [13, 14].

Other kinetic models have been recently considered, which, while maintaining the kinetic description, introduce more sophisticated rules for trading. For example, a description of the behavior of a stock price has been developed by Cordier, Pareschi and Piatecki in [15]. Further, there have been efforts to include non-microscopic effects, like global taxation (and subsequent redistribution), in recent works of Guala [16], Pianegonda, Iglesias, Abramson and Vega [17], Garibaldi, Scalas and Viarengo [11] and Bisi, Spiga and Toscani [18, 19].

Despite the high number of studies devoted to the subject, well documented by various recent review papers [20–24], analytical solutions or explicit steady states for wealth distribution densities are rarely present in the literature. For continuous models, the few exceptions are the self-similar solution with Pareto tails found by Slanina [8] for a kinetic model of a non-conservative economy, and the steady state solution (the exponential distribution) which appears both in the Chakraborti and Chakrabarti model [2] when the saving propensity is assumed to be zero [25, 26], and in a linear kinetic equation modeling taxation and uniform redistribution [19]. For discrete models, explicit expressions for steady states have been found in [12]. In a suitable scaling limit, some of these results can be used to re-obtain the exponential distribution as equilibrium for the underlying continuous model.

The goal of this paper is to study a Boltzmann–like kinetic equation for the evolution of wealth which is based on binary interactions obtained from model [2] with zero saving propensity (pure gambling trade interactions). With respect to previous studies on the same model [25, 26], we will allow the random fraction of wealth which governs the microscopic interaction to be different from the uniform distribution.

Two different situations, which correspond to require that the binary trade is pointwise conservative or not [13, 14], will be analyzed in details, by taking advantage from the the possibility to make use of the Laplace transform version of the kinetic equation [27].

Due to this representation, we will prove that various cases are explicitly solvable, and lead to an analytic expression of the steady profile. In particular, choosing the random fraction to be a symmetric Beta distribution, one obtains as equilibrium a Gamma distribution with a parameter which depends on the parameter of the Beta distribution. A special case emerges here, since, as previously observed [25, 26], the exponential distribution emerges for a random fraction uniformly distributed. Also, the case of the *winner takes all* game can be studied in details as limit of Beta ε, with parameter tending to zero. In all these cases, however, the equilibria possess moments of any order.

Interestingly enough, the case of the *conservative in the mean* gambling trade [4, 14] can be treated likewise, by choosing the random fractions to be inverse Beta. In this case, however, the steady state distribution is shown to be a inverse Gamma distribution with fat tails, which contains as a particular case the distribution found by Slanina [8] in a different context. Via a simple example, this founding clarifies the role

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of the *social* use of the wealth present in the market. The possibility to access to the wealth available in the common *reservoir* allows the formation of the rich class.

The paper is organized as follows. In the next section we introduce the the discrete model as a simple closed economy in which agents trade money. Then, the continuous model which is described by a nonlinear kinetic equation of Boltzmann type, and its main features are discussed in some detail in Section III. The analytical solutions in the case of the pure gambling trade (1) are described in Section IV. Section V deals with the conservative in the mean only gambling rule.

II. A DISCRETE MODEL ECONOMY

Wealth exchange processes in a discrete market are characterized by binary trades. A fixed number of N agents in a system are allowed to interact (trade) stochastically and thus wealth is exchanged between them. In each time step τ a pair of agents (i, j) is chosen randomly and interact in essentially instantaneous "collisions". In the collision wealth changes as follows

$$w_i(\tau+1) = w_i(\tau) + \Delta w, \qquad w_j(\tau+1) = w_j(\tau) - \Delta w,$$

where $w_i(\tau)$ and $w_j(\tau)$ are wealths of i-th and j-th agents at time τ and $w_i(\tau+1)$ and $w_j(\tau+1)$ are that at the next time step $\tau+1$. The amount Δw (to be won or to be lost by an agent) is determined by the nature of interaction. If the agents are allowed to interact for a long enough time, a steady state equilibrium distribution for individual wealth is achieved. The equilibrium distribution does not depend on the initial configuration (initial distribution of wealth among the agents).

In a pure gambling process [5], the entire sum of wealths of two agents is up for gambling. Some random fraction of this sum is shared by one agent and the rest goes to the other. The randomness is introduced into the model through a parameter ε which is a random number drawn from a probability distribution in $[0,1]$. In general it is assumed that ε is independent of a pair of agents, so that a pair of agents is not likely to share the same fraction of aggregate wealth the same way when they interact repeatedly. The interaction rule reads

$$\begin{aligned} w_i(\tau+1) &= \varepsilon[w_i(\tau) + w_j(\tau)], \\ w_j(\tau+1) &= (1-\varepsilon)[w_i(\tau) + w_j(\tau)], \end{aligned} \tag{1}$$

where the pair of agents (indicated by i and j) are chosen randomly. The amount of wealth that is exchanged is now $\Delta w = \varepsilon[w_i(\tau) + w_j(\tau)] - w_i(\tau)$.

The interaction rule (1) is a particular case of the model introduced by Chakraborti and Chakrabarti [2], given by

$$\begin{aligned} w_i(\tau+1) &= \lambda w_i(\tau) + \varepsilon(1-\lambda)[w_i(\tau) + w_j(\tau)], \\ w_j(\tau+1) &= \lambda w_j(\tau) + (1-\varepsilon)(1-\lambda)[w_i(\tau) + w_j(\tau)], \end{aligned}$$

where the constant parameter $0 \le \lambda < 1$ denotes the saving propensity. Both numerical experiments [21], and theoretical studies [25, 26] show that, if ε is a random number drawn from a uniform distribution in $[0,1]$, and $\lambda = 0$, the individual wealth distribution at equilibrium emerges out to be the exponential distribution. The same studies (cfr. [28] for an exhaustive review) however show that, in case $\lambda > 0$, the previous model produces at equilibrium only Gamma-like distributions, while the form of the exact solution is still an open question.

The main feature of trade (1) is such the amount of wealth which is restituted in a single trade coincides with the amount of wealth the two agents use for gambling (*pointwise conservative trade*). In this gambling market there is no rule of the (conserved) total amount of wealth initially in the hands of the agents. In other words, agents do not take advantage from the amount of wealth available in the market.

The idea of using this wealth as a *reservoir* for trades can be easily implemented allowing agents to trade with random profit

$$\begin{aligned} w_i(\tau+1) &= \varepsilon_1[w_i(\tau) + w_j(\tau)], \\ w_j(\tau+1) &= \varepsilon_2[w_i(\tau) + w_j(\tau)], \end{aligned} \tag{2}$$

where the parameters $(\varepsilon_1, \varepsilon_2)$ are now random numbers drawn from a joint probability distribution such that

$$\langle \varepsilon_1 + \varepsilon_2 \rangle = 1, \tag{3}$$

where $\langle \cdot \rangle$ denotes as usual the mathematical expectation. Within this picture, $w_i(\tau+1) + w_j(\tau+1)$ can be strictly less than $w_i(\tau) + w_j(\tau)$, and in this case the lost wealth is achieved by the market, or the reverse situation is verified, and the additional wealth is taken from the market. Condition (3), however, guarantees that in the mean the wealth present in the market is left unchanged. Note that if one assumes that $\varepsilon_i \ge \gamma > 0$, $i = 1, 2$ then agents are prevented from loosing all their wealth in a single trade. Note also that, choosing $(\varepsilon_1, \varepsilon_2) = (\varepsilon, 1-\varepsilon)$ the collision rule (2) becomes (1).

In order to produce a fair game, it will be assumed that the random numbers ε_1 and ε_2 are identically distributed. For trades of type (1), this assumption simply forces the random fraction ε to be symmetric with respect to the value $1/2$.

We remark that binary trades of type (2), which have never been considered before, can be the basis for a generalization of the (pointwise conservative) trades with saving propensity introduced in [2].

III. A CONTINUOUS KINETIC MODEL

As already explained, given a fixed number N of agents in a system, the interaction rules (2) give rise to a stochastic process $(w_1(\tau), \dots, w_N(\tau))$ which evolves as follows: at each time τ a pair of agents (i, j) is chosen randomly and interact according to (2). The resulting process is clearly a discrete time Markov process with state space $(\mathbb{R}^+)^N$. Processes of this type are thoroughly studied e. g. in the context of kinetic theory of ideal gases. Indeed, if the variables w_i are interpreted as energies corresponding to i-th particle, one can map the process to the mean-field limit of the Maxwell model of elastic particles [29–31]. The full information about the process in time τ is contained in the N-particle joint probability distribution $P_N(\tau, w_1, w_2, \dots, w_N)$. However, one can write a kinetic equation for one-marginal distribution function

$$P_1(\tau, w) = \int P_N(\tau, w, w_2, \dots, w_N) dw_2 \cdots dw_N,$$

involving only one- and two-particle distribution functions [29, 30]

$$P_1(\tau+1, w) - P_1(\tau, w) = \Big\langle \frac{1}{N} \Big[\int P_2(\tau, w_i, w_j) \Big(\delta(w - \varepsilon_1(w_i + w_j)) + \delta(w - \varepsilon_2(w_i + w_j)) \Big) dw_i\, dw_j - 2P_1(\tau, w) \Big] \Big\rangle,$$

which may be continued to give eventually an infinite hierarchy of equations of BBGKY type [29, 31]. The standard approximation, which neglects the correlations between the wealth of the agents induced by the trade, gives the factorization

$$P_2(\tau, w_i, w_j) = P_1(\tau, w_i) P_1(\tau, w_j),$$

which implies a closure of the hierarchy at the lowest level. Rescaling the time as $t = 2\tau/N$ in the thermodynamic limit $N \to \infty$, one obtains for the one-particle distribution function $f(t, w)$ the Boltzmann-like kinetic equation

$$\frac{\partial f(t,w)}{\partial t} = \frac{1}{2} \Big\langle \int f(t, w_1) f(t, w_2) \Big(\delta(w - \varepsilon_1(w_1 + w_2)) + \delta(w - \varepsilon_2(w_1 + w_2)) \Big) dw_1\, dw_2 \Big\rangle - f(t, w), \tag{1}$$

which describes the process (2) in the limit $N \to \infty$. A rigorous derivation of equation (1), based on the so-called propagation of chaos, can be derived following the classical Kac's argument [32, 33]. In this case one need to assume that in the discrete model the interactions take place at random times which correspond to the jumps of a Poisson process of rate N. See, e.g, [34].

Owing to equations of type (1), the study of the time-evolution of the wealth distribution among individuals in a simple economy, together with a reasonable explanation of the formation of tails in this distribution has been recently achieved in [14] (see also [13, 35]). The Boltzmann-like equation (1) can be fruitfully written in weak form. It corresponds to say that the solution $f(t, w)$ satisfies, for all smooth functions $\phi(v)$

$$\frac{d}{dt} \int_{\mathbb{R}_+} f(t, w) \phi(w)\, dw = \frac{1}{2} \Big\langle \int_{\mathbb{R}_+^2} \big(\phi(w') + \phi(w'_*) - \phi(w) - \phi(w_*) \big) f(t, w) f(t, w_*)\, dw\, dw_* \Big\rangle, \tag{2}$$

where the post-trade wealths (w', w'_*) obey to the rule (2)

$$w' = \varepsilon_1(w + w_*), \qquad w'_* = \varepsilon_2(w + w_*).$$

Note that (2) implies that $f(t, w)$ remains a probability density if it so initially

$$\int_{\mathbb{R}_+} f(t, w)\, dw = \int_{\mathbb{R}_+} f_0(w)\, dw = 1.$$

Moreover, on the basis of (3), the choice $\phi(w) = w$ shows that also the total mean wealth is preserved in time

$$m(t) = \int_{\mathbb{R}_+} w f(t, w)\, dw = \int_{\mathbb{R}_+} w f_0(w)\, dw = m(0). \tag{3}$$

Consequently, without loss of generality, in what follows, we assign to the initial density a unit mean

$$\int_{\mathbb{R}_+} w f_0(w)\, dw = 1. \tag{4}$$

Setting $\phi(w) = \exp\{-\xi w\}$ in (2), [27], one gets the Boltzmann equation for the Laplace transform $\widetilde{f}$ of f, where

$$\widetilde{f}(t, \xi) = \int_{\mathbb{R}^+} e^{-\xi w} f(t, w)\, dw.$$

Direct computations [14] show that $\widetilde{f}(t, \xi)$ satisfies the equation

$$\frac{\partial \widetilde{f}(t,\xi)}{\partial t} + \widetilde{f}(t, \xi) = \frac{1}{2} \left\langle \widetilde{f}(t, \xi\varepsilon_1)^2 + \widetilde{f}(t, \xi\varepsilon_2)^2 \right\rangle.$$

As proven in [14] by resorting to the Fourier transform version of the Boltzmann equation (1), explicitly computable conditions on the gambling variables ε_i, $i = 1, 2$, guarantee that the distribution $f(t, w)\, dw$ weakly converges to a *universal* probability distribution $f_\infty(w)\, dw$. This fact follows from Thm. 3.3 in [14] and Thm. 2 in [36], which links both the convergence and the boundedness of moments of the equilibrium solution to the sign of the key function $G(s)$, defined as

$$G(s) := \langle \varepsilon_1^s + \varepsilon_2^s \rangle - 1.$$

In point of fact, the results of [14, 36] can be easily adapted to show that if the condition

$$G'(1) < 0 \tag{5}$$

holds, the Laplace transform of $f_\infty(w)$ is a solution of

$$\widetilde{f}_\infty(\xi) = \frac{1}{2} \left\langle \widetilde{f}_\infty(\xi\varepsilon_1)^2 + \widetilde{f}_\infty(\xi\varepsilon_2)^2 \right\rangle.$$

Moreover this solution is unique in the class of all probability functions with unit mean, as given by (4), property that can be equivalently written as

$$\widetilde{f}'_\infty(\xi)|_{\xi=0} = -1.$$

By further requiring that the random variables ε_1 and ε_2 are distributed with the same law, the equilibrium solution $\widetilde{f}_\infty$ is found to be the unique solution to

$$\widetilde{f}_\infty(\xi) = \left\langle \widetilde{f}_\infty(\xi\varepsilon)^2 \right\rangle, \tag{6}$$

where the random variable ε is distributed according to the common law of ε_1 and ε_2. Equation (6) can be better understood by saying that, if Z is a random variable with law f_∞, the law of Z, defined in (6), is a distributional fixed point of the equation

$$Z \stackrel{d}{=} \varepsilon(Z_1 + Z_2), \tag{7}$$

where $\stackrel{d}{=}$ means identity in distribution (in law) and one assumes that the random variables Z_1, Z_2 and Z have the same probability law, while the variables Z_1, Z_2 and ε are assumed to be stochastically independent. Equations of type (7) are well-known and extensively studied, see e.g. [37, 38].

IV. THE PURE GAMBLING TRADE MODEL

Let us start from the pure gambling case in which $(\varepsilon_1, \varepsilon_2) = (\varepsilon, 1-\varepsilon)$. Having in mind that numerical experiments are usually done with a random number drawn from a uniform distribution in $[0,1]$, which leads the discrete market to an exponential distribution at equilibrium [21], we fix the random number ε to be a symmetric $Beta(a,a)$ random variable, with $(a>0)$. We recall that a random variable is $Beta(a,b)$ distributed if its density is

$$\beta_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \qquad x \in (0,1),$$

see, for instance, [39].

The case $a=1$, where ε is a random number uniformly distributed on $(0,1)$, confirms the numerical outcome. In this case, in fact, (6) becomes

$$\widetilde{f}_\infty(\xi) = \int_0^1 \widetilde{f}^2_\infty(\xi x)\, dx.$$

It is easy to see that

$$\widetilde{f}_\infty(\xi) = (1+\xi)^{-1}$$

is a solution of (6), such that $\widetilde{f}'_\infty(\xi)_{|\xi=0} = -1$. Since $(1+\xi)^{-1}$ is the Laplace transform of the exponential distribution of unit mean

$$f_\infty(w) = e^{-w} \qquad (w \ge 0),$$

the exponential distribution is an analytical steady solution to the pure gambling trade market, in case ε is a uniform random number in $(0,1)$. In this case, since $0<\varepsilon<1$ with probability one,

$$G'(1) = \langle \varepsilon \log \varepsilon \rangle < 0$$

and condition (5) is verified. Consequently, on the basis of the results of [14, 36], the exponential distribution is the unique solution to (6) with unit mean.

To treat the more general case it suffices to recall that, if $\alpha_1, \alpha_2, \xi > 0$, then formula 3.197.4 in [40] gives

$$\begin{aligned} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{(1+\xi x)^{\alpha_1+\alpha_2}} dx \\ = (1+\xi)^{-\alpha_1}. \end{aligned} \tag{8}$$

Using identity (8) with $\alpha_1 = \alpha_2 = a$, one obtains that for a general $a>0$ the function

$$\widetilde{f}_\infty(\xi) = \left(1 + \xi \frac{1}{a}\right)^{-a}$$

solves equation (6). Hence, the equilibrium solution results in a Gamma distribution of unit mean, with shape parameter a and scale parameter $1/a$, $Gamma(a, 1/a)$ distribution for short,

$$f_\infty(w) = \frac{a^a w^{a-1} e^{-aw}}{\Gamma(a)} \qquad (w>0). \tag{9}$$

Again, since ε is a symmetric $Beta(a,a)$ random variable, so that $0<\varepsilon<1$ with probability one, the function

$$G(s) := 2\langle \varepsilon^s \rangle - 1,$$

satisfies condition (5), and (9) is the unique equilibrium solution with unit mean.

The uniform distribution ($a=1$) appears like a natural separation between two different behaviors of the equilibrium solutions. In case $a<1$, $Gamma(a,1/a)$ density (9) is monotonically decreasing, starting from $f_\infty(0) = +\infty$. In the opposite case $a>1$, there is appearance of a peak *around* the unit mean value. The average wealth is unchanged for every a but, when $a>1$, the number of agents with a wealth closer to the average value increases or, in other words, the wealth distribution becomes more fair for larger a. Analogously, for $a<1$ the distribution gives more weight to agents with a wealth close to zero.

As a consequence, measures of the inequality of the wealth distribution, such as the Gini coefficient, increase for decreasing a and tend to zero for $a \to +\infty$. Some insight can be gained by the simple computation of the variance of a random variable X_∞ distributed according to f_∞. It holds

$$Var(X_\infty) = \int_{\mathbb{R}^+} w^2 f_\infty(w)\, dw - 1 = \frac{1}{a}.$$

Hence the variance (the spreading) decreases as a increases.

There are two interesting limiting cases. The first one is obtained by letting $a \to +\infty$. In this case ε converges in distribution to the constant value $1/2$ and one can immediately see that the steady state is a degenerate distribution concentrated on the value 1 of the mean wealth. This corresponds to a perfectly fair distribution, in which all agents end up with the same wealth. The other limit case is the *winner takes all* game, which can be obtained letting $a \to 0$. In this case the steady state is concentrating on 0, while its variance is blowing up. This corresponds to the discrete situation in which a finite number of agents end up with no wealth, except one which *takes all*. In the continuous case, the value $a=0$ cannot be assumed directly, since the exchange of the limits $a \to 0$ and $t \to \infty$ is not allowed by the lack of regularity of the equilibrium solution for $a=0$.

V. ANALYTIC EQUILIBRIA WITH HEAVY TAILS

In agreement with the previous section, we assume that

$$\varepsilon_i = \frac{1}{4\theta_i}, \qquad i = 1,2, \tag{10}$$

where, for a given $a > 1$, θ_i $(i = 1,2)$ is a $Beta(a+1/2, a-1/2)$ random variable. It is immediate to reckon that the random variables ε_i are such that $\langle \varepsilon_i \rangle = 1/2$ $i = 1,2$. Consequently (3) holds and the model is conservative in the mean. We can invoke again [14, 36] to prove that there is convergence to a unique equilibrium solution such that its Laplace transform satisfies

$$\widetilde{f}_\infty(\xi) = \int_0^1 \widetilde{f}_\infty(\xi/(4x))^2 \beta_{a+1/2,a-1/2}(x)dx, \tag{11}$$

with $\widetilde{f}'_\infty(\xi)|_{\xi=0} = -1$. In this case one needs to show that condition (5) holds true for

$$\begin{aligned} G(s) &:= \langle \varepsilon_1^s + \varepsilon_2^s \rangle - 1 \\ &= 2\int_0^1 \frac{1}{(4x)^s} \beta_{a+1/2,a-1/2}(x)dx - 1 \\ &= \frac{2^{1-2s}\Gamma(2a)\Gamma(a-s+\frac{1}{2})}{\Gamma(2a-s)\Gamma(a+\frac{1}{2})} - 1. \end{aligned} \tag{12}$$

The proof of this condition is not direct. For the sake of brevity, we postpone the computations to the Appendix.

Now, we shall prove that the solution of (11) is obtained by taking the Laplace transform of the so called Inverse-Gamma distribution [41] of shape parameter a and scale parameter $a-1$, that is

$$f_\infty(w) = \frac{(a-1)^a}{\Gamma(a)} \frac{e^{-\frac{(a-1)}{w}}}{w^{a+1}} \qquad (w \geq 0), \tag{13}$$

which is peaked around the mean value 1 and has heavy tails, in that it decays at infinity like $w^{-(a+1)}$. An analytical solution of type (13), with a polynomial decay corresponding to $a = 3/2$ has been discovered in [8] as self-similar solution of a kinetic model of a non-conservative economy. Motivated by the analogy with a dissipative Maxwell gas, Slanina [8] considered a collisional kinetic model of Boltzmann type, based on dissipative binary collisions, given by

$$\begin{aligned} & w' = pw + qw_*, \quad w'_* = qw + pw_*; \\ & p \geq q > 0, \quad \sqrt{p} + \sqrt{q} = 1. \end{aligned} \tag{14}$$

Note that, within condition (14) on the mixing parameters p and q,

$$w' + w'_* = (1 - 2\sqrt{pq})(w + w_*) < w + w_*,$$

which implies that the mean value $m(t)$ at time t decays exponentially to zero at the rate $2\sqrt{pq}$. The standard way to look for self–similarity is to scale the solution. More precisely, define the rescaled solution g by

$$g(t,w) = m(t) f\big(t, m(t)w\big),$$

which implies that $\int wg(t,w)\,dw = 1$ for all $t \geq 0$.

In terms of the Laplace transform $\widetilde{g}$ of g, it is found that the equation satisfied by $\widetilde{g}$ reads [35]

$$\frac{\partial \widetilde{g}}{\partial t} + \xi(p+q-1)\frac{\partial \widetilde{g}}{\partial \xi} = \widetilde{g}(p\xi)\widetilde{g}(q\xi) - \widetilde{g}(\xi). \tag{15}$$

Steady solutions to equation (15) satisfy

$$\xi(p+q-1)\frac{\partial \widetilde{g}}{\partial \xi} = \widetilde{g}(p\xi)\widetilde{g}(q\xi) - \widetilde{g}(\xi). \tag{16}$$

Direct computations then show that the function

$$\widetilde{g}_\infty(\xi) = \left(1 + \sqrt{2\xi}\right) e^{-\sqrt{2\xi}} \tag{17}$$

solves (16) for all values of p and q satisfying the constraint $\sqrt{p} + \sqrt{q} = 1$. Note that (17) is the explicit Laplace transform of

$$g_\infty(w) = \frac{(1/2)^{3/2}}{\Gamma(3/2)} \frac{e^{-\frac{1}{2w}}}{w^{5/2}}. \tag{18}$$

Let us set $p = q = 1/4$ in (16). Then the steady solution (17) satisfies

$$-\frac{\xi}{2}\frac{\partial \widetilde{g}}{\partial \xi} + \widetilde{g}(\xi) = \widetilde{g}\left(\frac{\xi}{4}\right)^2. \tag{19}$$

Following [42], Sect. 6, (19) can be equivalently written in integral form as

$$\widetilde{g}(\xi) = \int_0^1 \widetilde{g}\left(\frac{\xi}{4}\rho^{-1/2}\right)^2 d\rho,$$

or, setting $\rho^{1/2} = x$

$$\widetilde{g}(\xi) = \int_0^1 2\widetilde{g}\left(\frac{\xi}{4x}\right)^2 x\,dx,$$

which is nothing but (11) with $a = 3/2$. Consequently the distribution (18) solves (11) with $a = 3/2$. This argument establishes a connection between the present problem and the non-conservative one introduced by Slanina [8], which leads to explicit computations.

In order to prove that the Laplace transform of (13) is the solution of (11), since a direct proof seems not straightforward like in the previous case, we recast the problem in a more probabilistic way. First of all, let us note that an Inverse–Gamma random variable Y of parameter $(a, a-1)$ can be obtained by taking $Y = 1/X$ where X is a $Gamma(a, 1/(a-1))$ random variable. Recall that X is a $Gamma(a, 1/(a-1))$ random variable if its density is

$$\frac{(a-1)^a w^{a-1} e^{(a-1)w}}{\Gamma(a)} \qquad (w \geq 0).$$

Recall also that its Laplace transform reads

$$(1 + \xi/(a-1))^{-a}. \tag{20}$$

As discussed in Section III, equation (11) can be rewritten in equivalent way as

$$Y \stackrel{d}{=} \frac{1}{4\theta}[Y_1 + Y_2] \tag{21}$$

where Y_1, Y_2, θ are independent random variables, Y, Y_1, Y_2 have density f_∞, while θ has density $\beta_{a+1/2,a-1/2}$. Recall that the symbol $\stackrel{d}{=}$ has to be meant as an identity in distribution. Hence, to prove (21) it suffices to show that $Y^{-1} \stackrel{d}{=} 4\theta[Y_1+Y_2]^{-1}$ or, equivalently,

$$X \stackrel{d}{=} \frac{4\theta X_1 X_2}{X_1 + X_2} \tag{22}$$

where X, X_1, X_2 are independent $Gamma(a, 1/(a-1))$ random variables. Summarizing, the result follows if one is able to show that

$$\frac{4\theta X_1 X_2}{X_1 + X_2} \tag{23}$$

has Laplace transform (20). Setting $G := X_1 + X_2$ and $B := X_1/(X_1+X_2)$ one rewrites (23) as

$$4\theta GB(1-B).$$

It is a classical result of probability theory that, given X_1 and X_2 which are independent $Gamma(a, 1/(a-1))$ distributed, the random variables G and B are stochastically independent and, moreover, that G has $Gamma(2a, 1/(a-1))$ distribution, while B has $Beta(a,a)$ distribution. For the proof of this property, we refer for instance to Chapter 10.4 in [39]. Moreover, using relation (8) one can reckon that θG has $Gamma(a+1/2, 1/(a-1))$ distribution simply by computing its Laplace transform. Using now the fact that θG and B are mutually independent, one can write the Laplace transform of $4\theta GB(1-B)$ in the point ξ as

$$\begin{aligned} L(\xi) &= \int_0^1 \frac{\beta_{a,a}(x)}{\left[1+4(a-1)^{-1}\xi x(1-x)\right]^{a+\frac{1}{2}}}\,dx \\ &= 2\int_0^{\frac{1}{2}} \frac{\beta_{a,a}(x)}{\left[1+4(a-1)^{-1}\xi x(1-x)\right]^{a+\frac{1}{2}}}\,dx. \end{aligned}$$

At this stage, a simple change of variable shows that

$$\begin{aligned} L(\xi) &= \frac{\Gamma(2a)}{2^{2a-1}\Gamma(a)^2}\int_0^1 \frac{z^{a-1}(1-z)^{\frac{1}{2}-1}}{(1+\xi z/(a-1)))^{a+\frac{1}{2}}}\,dz \\ &= \int_0^1 \frac{1}{(1+\xi z/(a-1)))^{a+\frac{1}{2}}}\beta_{a,1/2}(z)\,dz. \end{aligned}$$

The last identity follows by the duplication formula $\Gamma(2a) = 2^{2a-1}\Gamma(a)\Gamma(a+1/2)/\Gamma(1/2)$ (see 8.335.1 in [40]). Using relation (8) once again we get $L(\xi) = (1+\xi/(a-1))^{-a}$ and (22) is proved.

Some remarks are in order. Within the choice (10), the conservative in the mean trade (2) is such that the two agents maintain at least $1/4$ of the total wealth used to trade. Thus, trade (2) is in a sense less risky than the conservative trade (1), where one of the two agents can exit from the trade with almost no wealth. In addition, it follows that the number of moments of the explicit equilibrium state which are finite increase with a. On the other hand, when a increases, the area described by the distribution of the random fraction ε on the interval $[1, +\infty)$ decreases, and the probability to use wealth of the society is also decreasing. Hence a fat Pareto tail is obtained through a big use of the common wealth.

VI. CONCLUSIONS

In this paper, we introduced and discussed the equilibrium solution of a nonlinear kinetic equation of Boltzmann type, modelling redistribution of wealth in a simple market economy in which trades are described by a standard gambling game. Due to the simplicity of the game trade, analytical solutions can be obtained in the case in which the post-trade wealths depend on the pre-trade ones through random variables which are Beta distributed. Previously known analytical solutions are here shown to appear for particular values of the underlying parameters of this model. The results enlighten the role of the interaction in producing Pareto tails. The lesson we can draw from this elementary market model is that Pareto tails cannot be produced through binary trade interactions in which agents only redistribute their wealths, while Pareto tails are produced through binary trades in which agents can take advantage from the amount of wealth available in the market.

APPENDIX A:

Let us prove that $G'(1) < 0$ when G is defined as in (12). Starting from (12), differentiation shows that

$$\begin{aligned} G'(s) = -2^{1-2s}&\frac{\Gamma(2a)\Gamma(a-s+\frac{1}{2})}{\Gamma(2a-s)\Gamma(a+\frac{1}{2})}\cdot \\ &\cdot\left\{2\log(2) + \psi(a-s+\frac{1}{2}) - \psi(2a-s)\right\} \end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Digamma function [40]. Using the duplication formula $2\psi(2x) = \psi(x) + \psi(x+1/2) + 2\log(2)$, see 8.365.6 [40], one obtains

$$G'(1) = -\frac{1}{4}\frac{\Gamma(2a)\Gamma(a-s+\frac{1}{2})}{\Gamma(2a-s)\Gamma(a+\frac{1}{2})}Q(a)$$

where $Q(a) := 2\log(2) + \psi(a-1/2) + \psi(a)$.

Now $\psi(1/2) = -\gamma - 2\log 2$ and $\psi(1) = -\gamma$ (γ being the Euler-Mascheroni constant), see 8.366.1/2 [40], and then $Q(1) = 0$.

The classical expansion formula (see 8.363.8 [40])

$$\psi'(x) = \sum_{k\geq 0}\frac{1}{(x+k)^2}$$

allows to conclude that, for every $a > 1$

$$Q'(a) = \sum_{k\geq 0} \frac{1}{(a-1/2+k)^2} - \sum_{k\geq 0} \frac{1}{(a+k)^2} > 0.$$

Hence $Q(a)$ is strictly monotone in $[1,+\infty)$ and since $Q(1) = 0$ it follows that $Q(a) > 0$ for every $a > 1$. This shows that $G'(1) < 0$.

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