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Kinetic models with randomly perturbed binary collisions

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Abstract We introduce a class of Kac-like kinetic equations on the real line, with general random collisional rules which, in some special cases, identify models for granular gases with a background heat bath [11], and models for wealth redistribution in an agent-based market [6]. Conditions on these collisional rules which guarantee both the existence and uniqueness of equilibrium profiles and their main properties are found. The characterization of these stationary states is of independent interest, since we show that they are stationary solutions of different evolution problems, both in the kinetic theory of rarefied gases [14,33] and in the econophysical context [6].

Keywords Boltzmann-like equations · econophysics · granular gases · long-time behavior

Mathematics Subject Classification (2000) MSC 82C40 · MSC 60F05

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1 Introduction

In this paper, we are concerned with the study of the time evolution and the asymptotic behavior of the spatially homogeneous kinetic equation

$$\begin{cases} \partial_t \mu_t + \mu_t = Q^+(\mu_t, \mu_t) \\ \mu_0 = \bar{\mu}_0 \end{cases} \tag{1}$$

which caricatures a Boltzmann–like equation in one spatial dimension. The solution $\mu_t = \mu_t(\cdot)$ is a time-dependent probability measure on $\mathbb{R}$, describing, in its most common physical applications, the distribution of particle velocity in a homogeneous gas, which is initially distributed according to the probability measure $\bar{\mu}_0$. The gain operator Q^+ models velocity changes due to binary particle collisions. Our fundamental assumption is that Q^+ is a generalized Wild convolution. More precisely, for all bounded and continuous test functions $g \in C_b(\mathbb{R})$, the probability measure $Q^+(\mu, \mu)$ is characterized by

$$\int g(v) Q^+(\mu, \mu)(dv) = \mathbb{E}\Big[\int_{\mathbb{R}} \int_{\mathbb{R}} g\Big(v_1 A_1 + v_2 A_2 + A_0 \Big) \mu(dv_1) \mu(dv_2) \Big], \tag{2}$$

where (A_0, A_1, A_2) is a random vector of $\mathbb{R}^3$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{E}$ denotes the expectation with respect to P.

The operator Q^+, characterized in (2), describes an interaction in which the post–collisional velocities are the result of a binary collision randomly perturbed by an external background. As we shall see, this modification induces an evolution process for the probability measure which stabilizes in time to a steady profile heavily dependent on this random background. The physical relevance of this generalized collision rule is mainly related to the dissipative Boltzmann equation. Indeed, we will show that a particular choice of this random term, in a dissipative binary collision process, produces the same steady state of the classical Boltzmann equation with standard dissipative binary collisions, in presence of a thermal bath [14].

A less standard application of interactions of type (2) is concerned with the construction of kinetic models for conservative economies. These models consider the evolution of wealth distribution in a market of agents which interact through binary trades, see for example [5, 28]. Recently, a kinetic model which includes the effects of taxation and subsequent redistribution has been introduced in [6]. This model is closely related to a dissipative Boltzmann equation in which the wealth lost in a single collision, the taxation, is given back by a suitable transport-drift operator. In our framework, the generalized collision represents a binary trade in which part of the money, exchanged by the agents, is taken and, then, redistributed by an external third subject, according to a certain random rule. Thus, unlike [6], the interaction we consider reproduces a market in which both taxation and redistribution are simultaneously present in each single trade. We will show that a particular choice of this random contribution produces the same steady states of the model considered in [6].

For $A_0 = 0$ and for suitable choices of (A_1, A_2), the one-dimensional kinetic equation (1) reduces to well-known simplified models for a spatially homogeneous gas, in which particles move only in one spatial direction. The basic assumption is that particles change their velocities only because of binary collisions. When two particles collide, then their velocities change from v and w, respectively, to

$$v' = p_1 v + q_1 w \qquad w' = p_2 v + q_2 w \tag{3}$$

where (p_1, q_1) and (q_2, p_2) are two identically distributed random vectors (not necessarily independent) with the same law of (A_1, A_2).

The first model of type (1)&(2) has been introduced by Kac [24], with collisional parameters $p_i = \sin\tilde{\theta}$ and $q_i = \cos\tilde{\theta}$, $i = 1, 2$, for a random angle $\tilde{\theta}$ uniformly distributed on $[0, 2\pi)$. The dynamics describes a gas in which the colliding molecules exchange a random fraction of their kinetic energies. This idea has been extended in [29] to gases with inelastically colliding molecules, which lose a random part of their energy in each interaction. The inelastic Kac equation corresponds to (1)&(2) with $A_0 = 0$, $A_1 = |\sin\tilde{\theta}|^d \sin\tilde{\theta}$ and $A_2 = |\cos\tilde{\theta}|^d \cos\tilde{\theta}$, where $d > 0$ is the parameter of inelasticity. Recently, more general versions of (1)&(2), with $A_0 = 0$, have been considered: their applications range from gases under the influence of a background heat bath [11] to models for the redistribution of wealth in simple market economies [18, 25]. In most of the above mentioned cases, A_1 and A_2 are positive random variables such that $\mathbb{E}[A_1^2 + A_2^2] = 1$ (conservation of energy) [24, 18], or $\mathbb{E}[A_1 + A_2] = 1$ (conservation of momentum) [18, 25].

In the classical Boltzmann equation [12, 13] relaxation to Maxwellian equilibrium (Gaussian density) is shown to be a universal behavior of the solution. On the contrary, the equilibria of model (1) depend on the precise form of the microscopic interactions (3) and, in most cases, they are not explicitly known.

The study of the steady states is crucial for models of wealth distribution. Indeed, the quality of a model is typically assessed by comparing the steady states to the empirical distribution of real data. For instance, it is usually accepted that the wealth distribution should approach, for large times, a stationary (or, in general, a self-similar) profile which should exhibit a *Pareto tail* [16, 17].

The case $A_0 = 0$ has been extensively studied in many aspects. In particular, the asymptotic behavior of the solutions of (1) has been satisfactory treated in [1, 2, 9], while the problem of propagation of smoothness has been addressed in [19, 25, 26].

As far as we know, the case $A_0 \neq 0$, while relevant in various applications which will be dealt with in this paper, has never been studied before. This case corresponds to binary interactions in which the particle velocities change from v and w, respectively, to

$$v' = p_1 v + q_1 w + r_1 \qquad w' = p_2 v + q_2 w + r_2, \tag{4}$$

where (p_1, q_1, r_1) and (q_2, p_2, r_2) are two identically distributed random vectors with the same law of (A_1, A_2, A_0). We will now describe the specific examples.

Inelastic Kac models with background

The interactions of type (4) can be related to the study of a dissipative gas in a thermal bath presented in [11, 14]. In one space-dimension, a dissipative Kac-like model has been introduced and discussed in [29]. As already mentioned, this model corresponds to the choice

$$A_1 = |\sin(\tilde{\theta})|^d \sin(\tilde{\theta}), \quad A_2 = |\cos(\tilde{\theta})|^d \cos(\tilde{\theta}) \tag{5}$$

where $\tilde{\theta}$ is uniformly distributed on $[0, 2\pi)$. As shown in [3, 29], a consequence of the dissipation is that a solution to the Kac equation, corresponding to an initial value with finite second moment, converges in time toward the probability mass located in zero. In addition to the physical dissipative interaction (5), let us now assume that particles velocities are subject to random fluctuations r_i, induced by an external background, whose distribution is the same of A_0. Assume further that A_0 and (A_1, A_2) are stochastically independent with $A_0 \neq 0$ and $\mathbb{E}[A_0] = 0$.

As observed in [11], a consequence of adding this random fluctuation is that the post-collision velocities have total energy bounded away from zero. Indeed, since A_0 and (A_1, A_2) are stochastically independent and $\mathbb{E}[A_0] = 0$, (4) implies

$$\begin{aligned}\mathbb{E}\left[(v')^2 + (w')^2\right] &= (v^2 + w^2)\mathbb{E}[A_1^2 + A_2^2] + 4vw\mathbb{E}[A_1 A_2] + 2\mathbb{E}[A_0^2] \\ &= \mathbb{E}[(A_1 v + A_2 w)^2] + \mathbb{E}[(A_1 w + A_2 v)^2] + 2\mathbb{E}[A_0^2] \geq 2\mathbb{E}[A_0^2],\end{aligned}$$

with $\mathbb{E}(A_0^2) > 0$. The main consequence of this fact is a different behavior of the solution to equation (2) with respect to the usual dissipative Kac-like model [29]: if the initial distribution has finite second moment, then the solution does not converge in time toward a degenerate probability distribution concentrated in zero. The same phenomenon happens if one adds to the dissipative Boltzmann equation a thermal bath [14].

It is possible to establish a direct link between the steady states of the present dissipative collisional models with random fluctuations and the steady states of the dissipative Boltzmann equation in presence of both a diffusion [14] and of friction and/or drift [33]. Indeed, the steady states of the various dissipative Boltzmann equations, studied in the above mentioned papers, are steady states of the Boltzmann problem (1), under suitable choices of the random variables (A_1, A_2, A_0). We will detail these correspondences in Section 3.

Kinetic models of a simple market economy with redistribution

As already mentioned, Boltzmann–type kinetic models have been recently used to study the evolution of wealth redistribution in a simple market economy. The largest part of these models refers to conservative economies, characterized by the fact that the mean wealth of the market does not vary in time. The binary interactions of type (3) considered in this context are such that the total money involved into the trade is conserved either pointwise or in the mean. These models, however, do not take into account additional effects which influence the evolution of wealth distribution in real market economies. For this reason, in [6] the authors focused their attention on a kinetic model which includes both taxes in each trading process and a form of redistribution. The latter arises by adding to the kinetic equation a linear transport-drift operator.

Here we obtain a similar model by assuming that the economic trades between agents are described by an interaction of type (4). In particular, for a given positive constant $0 < \epsilon < 1$, let us consider the post-interaction wealths including redistribution given by

$$A_1 = (1-\epsilon)\tilde{A}_1 \qquad A_2 = (1-\epsilon)\tilde{A}_2 \qquad A_0 = \epsilon\tilde{A}_0 \tag{6}$$

where $\mathbb{E}[\tilde{A}_1 + \tilde{A}_2] = 1$ and $\mathbb{E}[\tilde{A}_0] = m_0 = \int v\bar{\mu}_0(dv)$. Note that this assumption implies that the total mean wealth is left unchanged. Interaction (6) includes taxation operated by a third subject at the level of the single trade and a subsequent random redistribution. The mixing parameters $(\tilde{A}_1, \tilde{A}_2)$ can be chosen among the variety of models present in the literature, see e.g. [25]. The classical model introduced in [15] corresponds to

$$\tilde{A}_1 = \lambda + \tilde{\eta}(1-\lambda), \quad \tilde{A}_2 = \tilde{\eta}(1-\lambda)$$

where: $\tilde{\eta}$ is a random variable taking values in $[0,1]$ symmetrically distributed around $1/2$, and $\lambda \in [0,1]$ is a parameter, the so called saving propensity. The pure gambling [4] is obtained for $\tilde{A}_1 = \tilde{A}_2 = \tilde{\eta}$.

A further variant of the previous model is obtained by setting

$$A_1 = (1-\epsilon\Delta)\tilde{A}_1 \qquad A_2 = (1-\epsilon\Delta)\tilde{A}_2 \qquad A_0 = \epsilon\Delta\tilde{A}_0 \tag{7}$$

where $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_0)$ and Δ are stochastically independent with $P\{\Delta = 1\} = 1 - P\{\Delta = 0\} = \delta$. The presence of Δ in (7) simulates a market with tax evasion. In such a situation, common to many western countries, it is realistic to assume that taxation does not act on the totality of trades, but it occurs only with a probability δ.

As we shall see in Section 3.2, one can fix the values of (A_0, A_1, A_2) in such a way that the steady state of the model (1) produces the same steady states as the model considered in [6].

2 Main results

We start by writing the Boltzmann equation (1) in Fourier variables. By setting $\phi(t,\xi) = \int e^{i\xi v}\mu_t(dv)$, and using Bobylev's identity [7], one obtains that $\phi(t,\xi)$ obeys to the equation

$$\begin{cases} \partial_t\phi(t,\xi) + \phi(t,\xi) = \widehat{Q}^+\Big[\phi(t,\cdot),\phi(t,\cdot)\Big](\xi) \qquad (t>0, \xi\in\mathbb{R}) \\ \phi(0,\xi) = \phi_0(\xi) \end{cases} \tag{8}$$

where

$$\widehat{Q}^+\Big[\phi(t,\cdot),\phi(t,\cdot)\Big](\xi) := \mathbb{E}[\phi(t,A_1\xi)\phi(t,A_2\xi)e^{i\xi A_0}] \tag{9}$$

and $\phi_0(\xi) = \int e^{i\xi v}\bar{\mu}_0(dv)$.

As in the case of the Kac equation, it is easy to see that (8) admits a unique solution ϕ which can be written as a Wild series [35]

$$\phi(t,\xi) = \sum_{n\geq 0} e^{-t}(1-e^{-t})^n q_n(\xi), \tag{10}$$

where $q_0(\xi) = \phi_0(\xi)$ and, for $n \geq 1$,

$$q_n(\xi) = \frac{1}{n}\sum_{j=0}^{n-1} \hat{Q}^+(q_j, q_{n-1-j})(\xi). \tag{11}$$

Hence, if μ_t is the unique solution of (1) with initial condition $\bar{\mu}_0$, then its Fourier-Stieltjes transform is given by (10).

2.1 Steady states

The stationary equation associated to (8) is

$$\phi_\infty(\xi) = \hat{Q}^+[\phi_\infty(\cdot), \phi_\infty(\cdot)](\xi) \qquad (\xi \in \mathbb{R}). \tag{12}$$

It can be proved that, under suitable hypotheses, a solution to (12) exists. To show that steady states exist it is enough to recast the problem as a problem of fixed point equation for distributions. In terms of probability distributions, (12) reads

$$Q^+(\mu,\mu) = \mu. \tag{13}$$

Note that, given any probability distribution μ, by virtue of (2) the probability distribution $Q^+(\mu,\mu)$ is the law of the random variable

$$A_0 + Y_1 A_1 + Y_2 A_2,$$

where Y_1, Y_2 and (A_0, A_1, A_2) are stochastically independent and Y_i's have law μ.

In what follows, let us set

$$\mathcal{M}_\gamma := \left\{ \mu \text{ probability measure on } \mathcal{B}(\mathbb{R}) : \int_{\mathbb{R}} |v|^\gamma \mu(dv) < +\infty \right\},$$

and, for every m in $\mathbb{R}$ and $\gamma \geq 1$,

$$\mathcal{M}_{\gamma,m} := \left\{ \mu \in \mathcal{M}_\gamma \, : \int_{\mathbb{R}} v\mu(dv) = m \right\}.$$

In the rest of the paper we set

$$m_0 := \int v\bar{\mu}_0(dv),$$

whenever $\int |v|\bar{\mu}_0(dv) < +\infty$, and

$$\bar{m} := \frac{\mathbb{E}[A_0]}{1 - \mathbb{E}[A_1 + A_2]},$$

when $\mathbb{E}[|A_i|] < +\infty$ $(i = 0, 1, 2)$ and $\mathbb{E}[A_1 + A_2] \neq 1$.

The convex function $\Theta : [0, \infty) \to [0, \infty]$, defined by

$$\Theta(\gamma) := \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma], \tag{14}$$

where $0^0 := 0$, will play a very important role in what follows. In point of fact, the main results of the paper are stated under one of the following set of assumptions:

(H_1) There is γ in $(0, 1]$ such that

$$\Theta(\gamma) < 1, \ \mathbb{E}[|A_0|]^\gamma < +\infty \text{ and } \int |v|^\gamma \bar{\mu}_0(dv) < +\infty. \tag{15}$$

(H_2) There is γ in $(1, 2]$ such that (15) holds true. In addition

$$\mathbb{E}[A_1 + A_2] \neq 1 \text{ and } m_0 = \bar{m}.$$

(H_3) There is γ in $(1, 2]$ such that (15) holds true. In addition

$$\mathbb{E}[A_1 + A_2] = 1, \ \mathbb{E}[A_0] = 0 \text{ and } m_0 \text{ is arbitrary.}$$

In the next proposition we collect some known results on the existence of solutions of equation (13).

Proposition 1 ([31],[32]) *If either (H_1) or (H_2) holds, then there is a unique solution μ_∞ of (13) in $\mathcal{M}_\gamma$. In addition, if $1 \leq \gamma \leq 2$, then μ_∞ belongs to $\mathcal{M}_{\gamma,\bar{m}}$.*

If (H_3) holds, then there is a unique solution μ_∞ of (13) in $\mathcal{M}_{\gamma,m_0}$.

It is worth mentioning that if (H_1) holds, it is possible to describe μ_∞ in terms of a suitable series of random variables, see Lemma 2 in Section 5.

While it is easy to check when μ_∞ is a point mass probability, it is more difficult to obtain necessary and sufficient conditions for boundedness of moments up to a certain order. Some results in this direction are given in the next proposition.

Proposition 2 *Let μ_∞ be the probability distribution characterized in* Proposition 1.

(i) *If (H_1) or (H_2) holds, then μ_∞ is a point mass probability if and only if $m(1 - (A_1 + A_2)) = A_0$ almost surely (a.s.) for some real number m. If (H_3) holds, then μ_∞ is a point mass probability if and only if $m_0(1 - (A_1 + A_2)) = A_0$ a.s..*

(ii) *If $\Theta(\beta) < 1$ and $\mathbb{E}[|A_0|^\beta] < +\infty$ for some $\beta > 2$, then $\Theta(s) < 1$ for every $\gamma \leq s \leq \beta$ and $\int |v|^\beta \mu_\infty(dv) < +\infty$.*

(iii) *Let A_0, A_1 and A_2 be positive random variables with $P\{A_0 \neq 0\} > 0$. If, for some $\beta \geq \max\{1, \gamma\}$, $\int |v|^\beta \mu_\infty(dv) < +\infty$ and $\int v \mu_\infty(dv) > 0$, then $\mu_\infty\{[0, +\infty)\} = 1$ and $\Theta(\beta) < 1$.*

2.2 Trend to equilibrium

We recall that the Kantorovich-Wasserstein distance of order $\gamma > 0$ between two probability measures μ and ν is defined by

$$\mathcal{W}_\gamma(\mu,\nu) := \inf_{(X',Y')} (\mathbb{E}|X' - Y'|^\gamma)^{\frac{1}{1\vee\gamma}} \tag{16}$$

where the infimum is taken over all pairs (X', Y') of real random variables whose marginal probability distributions are μ and ν, respectively, and $1 \vee \gamma := \max\{1,\gamma\}$.

If $(\nu_n)_n$ is a sequence of probability measures belonging to $\mathcal{M}_\gamma$ and ν_∞ belongs to $\mathcal{M}_\gamma$, then $\mathcal{W}_\gamma(\nu_n,\nu_\infty) \to 0$ as $n \to +\infty$ if and only if ν_n converges weakly to ν_∞ and

$$\int |x|^\gamma \nu_n(dx) \to \int |x|^\gamma \nu_\infty(dx).$$

See, e.g., [30]. Recall that ν_n converges weakly to ν_∞ when $\int g(x)\nu_n(dx) \to \int g(x)\nu_\infty(dx)$ for every g in $C_b(\mathbb{R})$.

We are now ready to state our main results concerning the long time behavior of the solutions.

Proposition 3 *Assume that, for a suitable γ in $(0,2]$, one of $(H_1)-(H_3)$ is valid and let μ_∞ be the probability distribution characterized in* Proposition 1*. Then, for every $t > 0$,*

$$\mathcal{W}_\gamma(\mu_t,\mu_\infty) \le \bar{k}_\gamma e^{-t\frac{(1-\Theta(\gamma))}{(1\vee\gamma)}} \mathcal{W}_\gamma(\bar{\mu}_0,\mu_\infty).$$

where $\bar{k}_\gamma = 1$ if $\gamma \le 1$ and $\bar{k}_\gamma = 2^{1/\gamma}$ if $1 < \gamma \le 2$. Furthermore:

(i) *if either (H_1) holds with $\gamma = 1$ and $m_0 = \bar{m}$ or (H_2) holds, then $\int v\mu_t(dv) = \bar{m}$ for every $t > 0$;*
(ii) *if (H_3) holds, then $\int v\mu_t(dv) = m_0$ for every $t > 0$.*

3 Examples

In what follows, we will see some applications of the previous results.

3.1 Kinetic models of a simple market economy with redistribution

Let us first consider the kinetic model for wealth with redistribution, described in the Introduction. In this leading example, the random variables A_0, A_1, A_2 are given by (7). It follows that

$$\mathbb{E}[A_1 + A_2] = 1 - \epsilon\delta < 1.$$

Hence, since $\mathbb{E}[\tilde{A}_0] = m_0 < +\infty$, one can invoke Propositions 1 and 3 to prove both the existence and uniqueness in $\mathcal{M}_1$ of a steady state and the (exponential) convergence to this steady state of any solution with finite initial moment of order one equal to m_0.

It turns out that the steady states of a model with redistribution may have finite moments up to an order higher than that resulting from the case without redistribution. This can be easily verified by comparing the steady states corresponding to $\epsilon > 0$, say $\mu_\infty^{(\delta,\epsilon)}$, with the steady states without redistribution, we will denote by $\mu_\infty^{(0,0)}$, obtained by setting $\delta = \epsilon = 0$. Resorting to Theorem 5.3 in [22], one knows that, given $\beta > 1$, $\int v^\beta \mu_\infty^{(0,0)}(dv) < +\infty$ if and only if $\tilde{\Theta}(\beta) := \mathbb{E}[\tilde{A}_1^\beta + \tilde{A}_2^\beta] < 1$. On the other hand, if $\epsilon > 0$ and $\mathbb{E}[A_0^\beta] < +\infty$, by Proposition 2 (ii)-(iii), $\int v^\beta \mu_\infty^{(\delta,\epsilon)}(dv) < +\infty$ if and only if $\Theta(\beta) < 1$. Since $\Theta(\beta) = [1 + \delta[(1-\epsilon)^\beta - 1]]\tilde{\Theta}(\beta)$ and $[1 + \delta[(1-\epsilon)^\beta - 1]] < 1$, one can easily give examples in which $\int v^\beta \mu_\infty^{(0,0)}(dv) = +\infty$ while $\int v^\beta \mu_\infty^{(\delta,\epsilon)}(dv) < +\infty$. For instance if $\tilde{A}_0 = m_0$ and $\tilde{A}_1$ and $\tilde{A}_2$ have common density

$$f(x) = \begin{cases} 0 & \text{for } x \leq 1/3 \\ \frac{1}{9x^4} & \text{for } x > 1/3 \end{cases}$$

then

$$\tilde{\Theta}(\beta) = \begin{cases} \frac{2}{3^{\beta-1}(3-\beta)} & \text{for } 0 < \beta < 3 \\ +\infty & \text{for } \beta \geq 3. \end{cases}$$

Hence, $\tilde{\Theta}(1) = 1$, $\tilde{\Theta}(2) = 2/3 < 1$ and there exists $\beta_0 \in (2,3)$ such that $\tilde{\Theta}(\beta_0) = 1$. This implies that $\Theta(\beta_0) < 1$ for every ϵ and δ in $(0,1)$ and hence $\int v^{\beta_0} \mu_\infty^{(0,0)}(dv) = +\infty$ while $\int v^{\beta_0} \mu_\infty^{(\delta,\epsilon)}(dv) < +\infty$.

The previous discussion does not solve another interesting problem connected with wealth taxation and redistribution: the existence of an amount of taxation which is *optimal* in a suitable sense. For example one can look for redistributions yielding the steady states that have minimal variance, that is the minimal dispersion around the mean wealth. We leave this point to a further research.

3.2 Connections with other form of redistribution

We show here that the law of (A_0, A_1, A_2) can be fixed in such a way that the steady states of the redistribution model proposed in [6] are the same of the steady states of model (1).

Let us start by recalling that the model introduced in [6] can be formulated, in Fourier variables, as

$$\frac{\partial}{\partial t}\phi(t,\xi) + \phi(t,\xi) = \hat{Q}_\epsilon(\phi,\phi)(t,\xi) + \hat{R}_\chi^\epsilon(\phi)(t,\xi) \tag{17}$$

with

$$\hat{Q}_\epsilon(\phi,\phi) := \mathbb{E}\big[\phi\big(A_1^*\xi\big)\phi\big(A_2^*\xi\big)\big], \tag{18}$$

and

$$\hat{R}_\chi^\epsilon(\phi)(\xi) := -\epsilon\chi\xi\frac{\partial}{\partial\xi}\phi(\xi) + i\epsilon(\chi+1)m_0\xi\phi(\xi) \qquad (\chi \geq -1). \tag{19}$$

In (18) the random variables A_1^* and A_2^* are positive and such that $\mathbb{E}[A_1^* + A_2^*] = 1 - \epsilon$. Furthermore $\frac{\partial}{\partial \xi}\phi(0,0) = i \int v\bar{\mu}_0(dv) = im_0$. Note that the interaction operator in (17) consists of a *dissipative* collision operator $\hat{Q}_\epsilon(\phi,\phi)$ and a redistribution (differential) operator $\hat{R}^\epsilon_\chi(\phi)(\xi)$. It is worth recalling that, if ϕ is the Fourier-Stieltjes transform of a (regular) density f, then $\hat{R}^\epsilon_\chi(\phi)$ is the Fourier-Stieltjes transform of

$$R^\epsilon_\chi(f)(v) = \epsilon \frac{\partial}{\partial v}\Big[(\chi v - (\chi+1)m_0)\, f(v)\Big].$$

The possible steady states of (17) must satisfy

$$\phi(\xi) = \hat{Q}_\epsilon(\phi,\phi)(\xi) + \hat{R}^\epsilon_\chi(\phi)(\xi). \tag{20}$$

Existence of a global solution $\phi(\xi,t)$ to (17) has been proved in [6] when $\int v\bar{\mu}_0(dv) = m_0$. Nothing was proved about the existence of a steady state. This problem can be solved in a surprisingly easy way by establishing a connection between the steady states of the model [6] and special cases of our model.

First of all let us fix $\chi = -1$ in (19). In this case equation (20) reduces to

$$\phi(\xi) = \hat{Q}_\epsilon(\phi,\phi)(\xi) + \epsilon\xi\frac{\partial}{\partial \xi}\phi(\xi). \tag{21}$$

With an analogous computation as in [8,9], equality (21) is equivalent to

$$\phi(\xi) = \int_0^1 \hat{Q}_\epsilon(\phi,\phi)(\xi u^{-\epsilon})du. \tag{22}$$

It is immediate to see that equation (22) can be rephrased as

$$\phi(\xi) = \mathbb{E}\big[\phi\big(U^{-\epsilon}A_1^*\xi\big)\phi\big(U^{-\epsilon}A_2^*\xi\big)\big], \tag{23}$$

where U and (A_1^*, A_2^*) are stochastically independent and U is uniformly distributed on $[0,1]$. Hence, the steady state (21) coincides with the steady state (12) corresponding to $(A_0, A_1, A_2) := (0, U^{-\epsilon}A_1^*, U^{-\epsilon}A_2^*)$. Since in this case $\Theta(1) = \mathbb{E}[A_1 + A_2] = 1$, in order to apply Proposition 1 it is necessary that A_1^* and A_2^* satisfy

$$\Theta(\gamma) = \frac{1}{1-\gamma\epsilon}\mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma] < 1, \tag{24}$$

for some $1 < \gamma \leq \min\{2, 1/\epsilon\}$. If $\mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma] < 1$, then (24) holds true for every $\epsilon < (1 - \mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma])/\gamma$.

Let us now consider the case $\chi = 0$, in which (19) corresponds to a pure transport operator, thus producing a uniform redistribution. In this case, equation (20) becomes

$$\phi(\xi) = \hat{Q}_\epsilon(\phi,\phi)(\xi) + i\epsilon m_0\xi\phi(\xi), \tag{25}$$

or, equivalently,

$$\phi(\xi) = \frac{1}{1 - i\epsilon m_0 \xi} \hat{Q}_\epsilon(\phi, \phi)(\xi). \tag{26}$$

Let us observe that if A_0 is an exponential random variable of mean ϵm_0, that is with density $h_0(v) = \exp\{-v/(\epsilon m_0)\}/(\epsilon m_0)$ $(v > 0)$, then

$$\mathbb{E}[e^{i\xi A_0}] = \int_0^{+\infty} e^{i\xi v} h_0(v) dv = \frac{1}{1 - i\epsilon m_0 \xi}.$$

Under the additional assumption that A_0 and (A_1^*, A_2^*) are stochastically independent, (26) can be equivalently written as

$$\phi(\xi) = \mathbb{E}\big[e^{i\xi A_0} \phi\big(A_1^* \xi\big) \phi\big(A_2^* \xi\big)\big].$$

Hence, it is enough to choose $A_1 = A_1^*, A_2 = A_2^*$ and A_0 as above to identify the steady state (25) with the steady state (12). Note that, since in this case $\mathbb{E}[A_1 + A_2] = 1 - \epsilon < 1$, the assumption H_1 (for $\gamma = 1$) in Proposition 1 is satisfied.

Finally, let us examine the physically relevant case in which $\chi > -1$ and $\chi \neq 0$. For any given $\epsilon \in (0, 1]$ set $\delta := \epsilon\chi$. With this choice, (20) becomes

$$\hat{Q}_\epsilon(\phi, \phi)(\xi) = \phi(\xi) + \delta\xi \frac{\partial}{\partial\xi} \phi(\xi) - i(\delta + \epsilon) m_0 \xi \phi(\xi).$$

Multiplying both sides by $e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta} - 1}$ we get

$$\begin{aligned}\hat{Q}_\epsilon(\phi, \phi)(\xi) e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}-1} &= e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}-1} \Big(\phi(\xi) + \delta\xi \frac{\partial}{\partial\xi}\phi(\xi) - i(\delta+\epsilon) m_0 \xi \phi(\xi)\Big) \\ &= \frac{\partial}{\partial\xi}\Big(\delta e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}} \phi(\xi)\Big),\end{aligned}$$

which, after integrating over $[0, \xi]$, gives

$$\delta e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}} \phi(\xi) = \int_0^\xi \hat{Q}_\epsilon(\phi, \phi)(\tau) e^{-i\tau m_0 \frac{\delta+\epsilon}{\delta}} \tau^{\frac{1}{\delta}-1} d\tau. \tag{27}$$

By the change of variable $\xi u^\delta = \tau$, we can write the previous equation in the equivalent form

$$\phi(\xi) = \int_0^1 \hat{Q}_\epsilon(\phi, \phi)(\xi u^\delta) e^{i(1-u^\delta)\frac{\delta+\epsilon}{\delta} m_0 \xi} du,$$

which can be rephrased as

$$\phi(\xi) = \mathbb{E}[\phi(U^\delta A_1^* \xi) \phi(U^\delta A_2^* \xi) e^{i(1-U^\delta)\frac{\delta+\epsilon}{\delta} m_0 \xi}],$$

with (A_1^*, A_2^*) and U stochastically independent and U uniformly distributed on $[0, 1]$. Hence (20) is equivalent to

$$\phi(\xi) = \mathbb{E}[\phi(A_1\xi)\phi(A_2\xi) e^{iA_0\xi}]$$

for

$$A_1 = U^\delta A_1^* \qquad A_2 = U^\delta A_2^* \qquad A_0 = (1 - U^\delta)\frac{\delta+\epsilon}{\delta} m_0. \tag{28}$$

Since $\delta = \epsilon\chi > -\epsilon$, it is immediate to find that $\mathbb{E}[A_1 + A_2] = (1-\epsilon)/(1+\delta) < 1$. Hence Proposition 1 applies. In particular, $\mathbb{E}[A_0]/(1 - \mathbb{E}[A_1 + A_2]) = m_0$ and hence the solution μ_∞ given in Proposition 1 satisfies $\int v\mu_\infty(dv) = m_0$.

3.3 Inelastic Kac models with background and connection with dissipative models with diffusion

Let us consider now the case in which

$$A_1 = |\sin(\tilde\theta)|^d \sin(\tilde\theta), \quad A_2 = |\cos(\tilde\theta)|^d \cos(\tilde\theta) \tag{29}$$

with $\tilde\theta$ uniformly distributed on $[0, 2\pi)$. When $A_0 = 0$, this assumption leads to the inelastic Kac model [29], which describes the cooling of a one-dimensional spatially homogeneous Maxwell–like gas. In fact, if $A_0 = 0$ and $\int |v|^{2/(1+d)}\bar\mu_0(dv) < +\infty$, μ_t converges weakly to the probability mass concentrated at 0 (cf. [3]). As we shall see, by adding a random fluctuation, described by a random variable $A_0 \neq 0$, one can obtain non-degenerate steady states. If A_1 and A_2 are given by (29), then

$$\mathbb{E}[A_1 + A_2] = 0,$$

and, whenever $\gamma > 2/(1+d)$,

$$\Theta(\gamma) = \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma] = \frac{1}{2\pi}\int_0^{2\pi} [|\sin(\theta)|^{(d+1)\gamma} + |\cos(\theta)|^{(d+1)\gamma}]d\theta < 1.$$

Let us assume that for some positive ϵ, with $\epsilon + 2/(1+d) < 2$, $\mathbb{E}|A_0|^{2/(1+d)+\epsilon} < +\infty$ (with $\mathbb{E}[A_0] = m_0 := \int v\bar\mu_0(dv)$ if $d < 1$) and $\int |v|^{2/(1+d)+\epsilon}\bar\mu_0(dv) < +\infty$, then Propositions 1–3 apply. In particular, if $\mathbb{E}[A_0] = 0$ and $P\{A_0 \neq 0\} > 0$, the steady state is a non-degenerate probability distribution with finite moments of any order.

As a special case let us choose $A_0 = A_{0,a} - A_{0,b}$ with $A_{0,a}$ and $A_{0,b}$ exponentially distributed with density $v \mapsto \exp\{-v/a\}/a$ and $v \mapsto \exp\{-v/b\}/b$ $(v > 0)$. Moreover, assume that $A_{0,a}, A_{0,b}, A_1, A_2$ are stochastically independent. Since

$$\mathbb{E}[e^{i\xi A_0}] = \mathbb{E}[e^{i\xi A_{0,a}}]\mathbb{E}[e^{-i\xi A_{0,b}}] = \frac{1}{(1 - ia\xi)(1 + ib\xi)} = \frac{1}{1 - i(a-b)\xi + ab\xi^2},$$

if $a := (m_0 + \sqrt{m_0^2 + 4\sigma^2})/2$ and $b := (-m_0 + \sqrt{m_0^2 + 4\sigma^2})/2$ the stationary equation (12) becomes

$$\mathbb{E}[\phi(A_1\xi)\phi(A_2\xi)] = \phi(\xi)(1 - im_0\xi + \sigma^2\xi^2). \tag{30}$$

If we set $\hat Q_d^+(\phi, \phi) := \mathbb{E}[\phi(A_1\xi)\phi(A_2\xi)]$, then (30) can be equivalently written as

$$\phi(\xi) = \hat Q_d^+(\phi, \phi)(\xi) - \sigma^2\xi^2\phi(\xi) + im_0\xi\phi(\xi). \tag{31}$$

It is interesting to note that equation (31) describes the steady states of the inelastic Kac equation in presence of a thermal bath and a transport term. Indeed, if ϕ is the Fourier-Stieltjes transform of a density f, then $-\sigma^2\xi^2\phi(\xi) + im_0\xi\phi(\xi)$ is the Fourier-Stieltjes transform of

$$\sigma^2 \frac{\partial^2}{\partial v^2} f(v) - m_0 \frac{\partial}{\partial v} f(v).$$

In particular, the analysis of Section 2 allows to prove existence of a steady state for the dissipative Kac equation with diffusion. The problem of the solvability of equations of type

$$Q(f,f) + \sigma^2 \Delta f = 0, \tag{32}$$

in terms of nonnegative integrable densities $f \in L^1_+(\mathbb{R}^3)$, and where Q is the Boltzmann collision operator, is a well-known problem in kinetic theory of rarefied gases. When Q is the dissipative collision operator for Maxwellian molecules, existence of non trivial weak solutions has been proved by Cercignani, Illner and Stoica [14].

Also, as clearly discussed by Villani in [33], apart from collisions, other physically relevant problems in kinetic theory of granular gases lead to the addition of various terms. These either model external physical forces or arise from particular situations. One of these situations is described by equation (32). A second one is obtained by subtracting a drift term to the Boltzmann collision operator. This leads to the problem of finding steady states of the equation

$$Q(f,f) - \sigma^2 \nabla \cdot (vf) = 0. \tag{33}$$

Let us remark that in one dimension of the velocity space, equation (33) is a particular case of equation (20) with $\chi = -1$, which has been solved in the previous Sub-section.

4 Probabilistic representation of the solutions

The core of the proofs of our results is a suitable probabilistic representation of the solution μ_t. The idea to represent the solutions of the Kac equation in a probabilistic way dates back, at least, to the work of McKean [27]. However, it has been fully formalized and employed in the derivation of analytic results for the Kac equation only in the last decade, starting from [10] and [23].

Our approach here follows the same steps used in [1] and [2] and it is based on the concept of random recursive binary trees. It is worth recalling that a binary tree is a (planar and rooted) tree where each node is either a leaf (that is, it has no successor) or it has 2 successors. We define the size of a binary tree τ, in symbol $|\tau|$, by the number of internal nodes. Hence, any binary tree with $2k+1$ nodes has size k and possesses $k+1$ leaves. Given a binary tree, let us denote by $\emptyset$ its root. For every node σ, different from the root, let us denote by $|\sigma|$ the size of σ, which is the number nodes (including σ) in the path connecting the root to σ. Hence, if $|\sigma| = k$, we can identify σ with a vector with k components, i.e. $\sigma = (\sigma_1, \dots, \sigma_k)$, where $\sigma_i = 1$ or 2

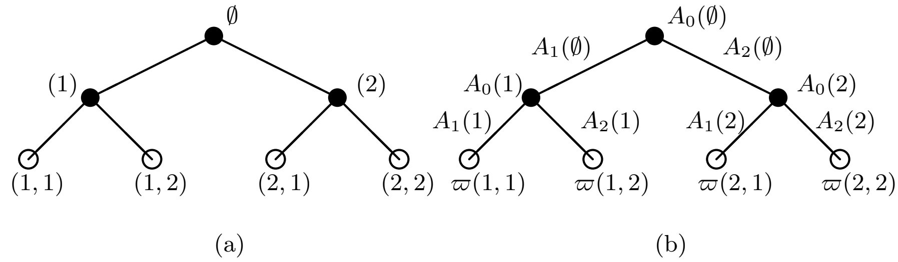

Fig. 1 (a) Representation of the nodes of a 4-leafed binary tree. (b) Random variables indexed by the nodes of a 4-leafed binary tree

according as to whether the i–th node of the descending path is a right or left node. See Figure 1 (a) for an example.

Hence any binary tree can be identified by a subset of

$$\mathbb{U} = \{\emptyset\} \cup [\cup_{k \geq 1}\{1,2\}^k].$$

Moreover let us set $(\sigma, \sigma_{k+1}) := (\sigma_1, \ldots, \sigma_k, \sigma_{k+1})$ and for every $1 \leq i \leq k$, $\sigma|i := (\sigma_1, \ldots, \sigma_i)$ and $\sigma|0 = \emptyset$.

In the rest of the paper, given a binary tree τ, we shall denote by $\mathcal{L}(\tau)$ the set of the leaves of τ and by $\mathcal{I}(\tau)$ the set of the internal nodes of τ.

We now describe a tree evolution process which gives rise to the so called "random binary recursive tree". The evolution process starts with $T_0 = \{\emptyset\}$, an empty tree, with just an external node (the root). The first step in the growth process is to replace this external node by an internal one with 2 successors that are leaves. In this way one obtains $T_1 = \{\emptyset, (1), (2)\}$. Then, with probability 1/2 (i.e. one over the number of leaves), one of these 2 leaves is selected and again replaced by an internal node with 2 successors. One continues along the same rules. If one iterates this process, at time $k \geq 2$, a binary tree T_k, with k internal nodes, is obtained. For more details on binary recursive trees see, for instance, [20].

The Wild series expansion (10)-(11) can be translated in a probabilistic representation of the solutions as sums of random variables indexed by the nodes of a binary recursive random tree. On a sufficiently large probability space $(\Omega, \mathcal{F}, P)$ let the following be given:

- a family $(X_\sigma)_{\sigma \in \mathbb{U}}$ of independent random variables with common probability distribution $\bar{\mu}_0$;
- a family $\big(A_0(\sigma), A_1(\sigma), A_2(\sigma)\big)_{\sigma \in \mathbb{U}}$ of independent positive random vectors with the same distribution of (A_0, A_1, A_2);
- a sequence of binary recursive random trees $(T_n)_{n \in \mathbb{N}}$;
- a stochastic process $(\nu_t)_{t \geq 0}$ with values in $\mathbb{N}_0$ such that $P\{\nu_t = k\} = e^{-t}(1 - e^{-t})^k$ for every integer $k \geq 0$.

Write $A(\sigma) = (A_0(\sigma), A_1(\sigma), A_2(\sigma))$ and assume further that

$$(A(\sigma))_{\sigma \in \mathbb{U}}, \quad (T_n)_{n \geq 1}, \quad (X_\sigma)_{\sigma \in \mathbb{U}} \quad \text{and} \quad (\nu_t)_{t > 0}$$

are stochastically independent.

For each node $\sigma = (\sigma_1, \dots, \sigma_k)$ in $\mathbb{U}$ set

$$\begin{aligned}\varpi(\sigma) := &\prod_{i=0}^{|\sigma|-1} A_{\sigma_{i+1}}(\sigma|i)\\ &= A_{\sigma_1}(\emptyset)A_{\sigma_2}(\sigma_1)A_{\sigma_3}((\sigma_1,\sigma_2))\cdots A_{\sigma_k}((\sigma_1,\dots,\sigma_{k-1}))\end{aligned}$$

and $\varpi(\emptyset) = 1$. In order to understand better the definition of $\varpi(\sigma)$, imagine to attach to each node σ the random vector $A(\sigma) = (A_0(\sigma), A_1(\sigma), A_2(\sigma))$. Given a node $\sigma = (\sigma_1, \dots, \sigma_k)$, consider all the internal nodes of the descending path connecting the root to σ, i.e. $\emptyset, \sigma_1, (\sigma_1, \sigma_2), \dots (\sigma_1, \dots, \sigma_{k-1})$. Then $\varpi(\sigma)$ is obtained by multiplying, for every such internal node, say σ', either the component $A_1(\sigma')$ or $A_2(\sigma')$, according that the successive node of σ' in the path is a right or a left node. For example, if $\sigma = (1, 2)$ then $\varpi((1, 2)) = A_1(\emptyset)A_2(1)$, see Figure 1 (b). Now define

$$W_0 := X_\emptyset \text{ and } G_0 := 0$$

and, for any $n \geq 1$,

$$W_n := \sum_{\sigma\in\mathcal{L}(T_n)} \varpi(\sigma)X_\sigma, \qquad G_n := \sum_{\sigma\in\mathcal{I}(T_n)} \varpi(\sigma)A_0(\sigma), \qquad Z_n := W_n + G_n.$$

Proposition 4 *Equation* (8) *has a unique solution* ϕ*, which coincides with the characteristic function of* Z_{ν_t}*, i.e.*

$$\phi(t,\xi) = \mathbb{E}[e^{i\xi Z_{\nu_t}}] = \sum_{n=0}^{\infty} e^{-t}(1-e^{-t})^n \mathbb{E}[e^{i\xi Z_n}] \qquad (t > 0,\ \xi \in \mathbb{R}).$$

Proof . We need some preliminary results on recursive binary trees. Any binary tree has a recursive structure: a binary tree τ is either just an external node or an internal node with 2 subtrees, $\tau^{(1)}, \tau^{(2)}$, that are again binary trees. For every $k \geq 0$ let $\mathbb{T}_k$ denote the set of all binary trees with size k. By Proposition 3.1 in [1], we know that if $(T_k)_{k\geq 0}$ is a sequence of random binary recursive trees, then for every $k \geq 1$, $j = 0, \dots, k-1$ and every τ in $\mathbb{T}_k$,

$$\begin{aligned}P\Big\{T_k^{(1)} = \tau^{(1)},\ T_k^{(2)} = \tau^{(2)}\Big||T_k^{(1)}| = j\Big\}\\ = P\{T_j = \tau^{(1)}\}P\{T_{k-j-1} = \tau^{(2)}\}\mathbb{I}\{|\tau^{(1)}| = j\}\end{aligned} \tag{34}$$

and for $k \geq 1$

$$P\{|T_k^{(1)}| = j\} = \frac{1}{k} \tag{35}$$

for every $j = 0, \dots, k-1$. Now observe that, in order to prove the proposition we need only to prove that $q_n(\xi) = \mathbb{E}[e^{i\xi Z_n}]$, for every $n \geq 0$. This is clearly true for $n = 0$. For $n \geq 1$, write

$$Z_n = A_0(\emptyset) + \sum_{j=1}^{2} A_j(\emptyset)\Big\{\Big[\sum_{\sigma\in\mathcal{L}(T_n^{(j)})} \prod_{i=0}^{|\sigma|-1} A^{(j)}_{\sigma_{i+1}}(\sigma|i) X^{(j)}_\sigma\Big] + \Big[\sum_{\sigma\in\mathcal{I}(T_n^{(j)})} \prod_{i=0}^{|\sigma|-1} A^{(j)}_{\sigma_{i+1}}(\sigma|i) A_0^{(j)}(\sigma)\Big]\Big\}$$

where $A^{(j)}(\sigma) = A((j,\sigma))$, $X^{(j)}_\sigma = X_{(j,\sigma)}$ and, by convention, if $\mathcal{L}(T_n^j) = \emptyset$ the first terms in square brackets is equal to $X^j_\emptyset = X_j$. Since $(A^{(j)}(\sigma), X^{(j)}_\sigma)_{\sigma\in\mathbb{U}}$, $j = 1, 2$, are independent, with the same distribution of $(A(\sigma), X_\sigma)_{\sigma\in\mathbb{U}}$, using (34) and the induction hypothesis one proves that

$$\mathbb{E}\Big[e^{i\xi Z_n}\Big|A(\emptyset), |T_n^{(1)}|, |T_n^{(2)}|\Big] = \prod_{j=1}^{2} q_{|T_n^{(j)}|}(\xi A_j(\emptyset)) e^{i\xi A_0(\emptyset)}. \tag{36}$$

At this stage the conclusion follows easily by using (35); indeed:

$$\begin{aligned}\mathbb{E}[e^{i\xi Z_n}] &= \mathbb{E}\Big[\prod_{j=1}^{2} q_{|T_n^{(j)}|}(\xi A_j(\emptyset)) e^{i\xi A_0(\emptyset)}\Big] \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\Big[q_j(\xi A_1) q_{n-j-1}(\xi A_2) e^{i\xi A_0}\Big] = q_n(\xi).\end{aligned}$$

5 Proofs of Section 2

Proposition 1, for $\gamma = 2$, is proved in [31], while for general $\gamma \in (0, 2)$ it can be seen as a special case of a (more general) result contained in [32]. The proof of Proposition 1 given in [32] is based on some contraction properties of the Wasserstein metrics. Here we provide a proof for $\gamma \in (0, 2]$ based on a martingale method inspired by [31]. In this way we obtain some additional information on the solution, that will be useful to prove Proposition 2.

We will repeatedly use the von Bahr-Esseen inequality (see [34]), which states the following: *Let $Y_1, \dots, Y_n$ be a sequence of random variables such that $\mathbb{E}|Y_i|^r < +\infty$ $(1 \leq i \leq n)$ for some r in $[1, 2]$ and let $S_m := \sum_{i=1}^m Y_i$ for $1 \leq m \leq n$. If*

$$\mathbb{E}[Y_{m+1}|S_m] = 0 \ a.s. \qquad m = 1, \dots, n-1 \tag{37}$$

then

$$\mathbb{E}|S_n|^r \leq 2\sum_{i=1}^{n} \mathbb{E}|Y_i|^r. \tag{38}$$

In the following we need to consider a sequence $(T_n^*)_{n\geq 0}$ of (deterministic) balanced binary trees. That is, starting from $T_0^* = \{\emptyset\}$, for each n, T_n^* is the binary tree obtained from T^*_{n-1} replacing each leaf by an internal node with 2 successors. In particular, for every n, T^*_n has 2^n leaves. For instance, the tree in Figure 1 (a) is T^*_2. Recall that $\mathcal{L}(T^*_n)$ ($\mathcal{I}(T^*_n)$, respectively) denotes the set of the leaves (the internal nodes, respectively) of T^*_n.

Define $\mathtt{K}(\mu) := Q^+(\mu,\mu)$ and denote by $\mathtt{K}^n(\delta_m)$ the n-iterate of the transformation $\mathtt{K}$ applied to the mass probability concentrated on the real value m. Finally, for $n \geq 0$, set

$$M^*_n := m \sum_{v \in \mathcal{L}(T^*_n)} \varpi(v) + \sum_{v \in \mathcal{I}(T^*_n)} \varpi(v) A_0(v). \tag{39}$$

In the rest of the paper L^γ will stand for $L^\gamma(\Omega, \mathcal{F}, P)$.

Lemma 1 *If (H_2) or (H_3) holds for some $\gamma \in (1,2]$ and (M^*_n) is defined as in (39) with $m = m_0$, then*

(*a*) *$\mathtt{K}^n(\delta_{m_0})$ is the law of M^*_n;*
(*b*) *$(M^*_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{G}^*_n)_{n\geq 1}$, where, for every n, $\mathcal{G}^*_n$ is the σ-field generated by $(A(\sigma) : \sigma \in T^*_{n-1})$. Furthermore, $\mathbb{E}(M^*_n) = m_0$ for every n;*
(*c*) *$\sup_n \mathbb{E}|M^*_n|^\gamma < +\infty$, hence $(M^*_n)_{n\geq 0}$ converges a.s. and in L^1 to a random variable M^*_∞ such that $\mathbb{E}(M^*_\infty) = m_0$ and $\mathbb{E}|M^*_\infty|^\gamma < +\infty$;*
(*d*) *The law μ_∞ of M^*_∞ is a solution of (13) in $\mathcal{M}_\gamma$;*
(*e*) *If (H_3) holds true, then*

$$M^*_\infty = m_0 \Xi_\infty + \sum_{n\geq 0} \sum_{\sigma \in \mathcal{L}(T^*_n)} \varpi(\sigma) A_0(\sigma)$$

where Ξ_∞ is the almost sure limit of $\sum_{\sigma\in\mathcal{L}(T^*_n)} \varpi(\sigma)$ for $n \to +\infty$.

Proof . (a) is immediate for $n = 1$. In fact

$$\begin{aligned}\mathtt{K}^1(\delta_{m_0}) &= \mathcal{D}(A_1(\emptyset)m_0 + A_2(\emptyset)m_0 + A_0(\emptyset)) \\ &= \mathcal{D}\Big(\sum_{\sigma\in\mathcal{L}(T^*_1)} \varpi(\sigma)m_0 + \sum_{\sigma\in\mathcal{I}(T^*_1)} \varpi(\sigma)A_0(\sigma)\Big),\end{aligned}$$

where, for every random variable X, $\mathcal{D}(X)$ denotes the law of X. Now, by induction, we obtain

$$\begin{aligned}\mathtt{K}^n(\delta_{m_0}) &= \mathtt{K}(\mathtt{K}^{n-1}(\delta_{m_0})) \\ &= \mathtt{K}\Big(\mathcal{D}\Big(\sum_{\sigma\in\mathcal{L}(T^*_{n-1})} \varpi(\sigma)m_0 + \sum_{\sigma\in\mathcal{I}(T^*_{n-1})} \varpi(\sigma)A_0(\sigma)\Big)\Big).\end{aligned}$$

At this stage, denote by T^{*1}_n (T^{*2}_n, respectively) the left (right, respectively) binary subtree of T^*_n. For every σ in $\mathbb{U}$ and $i = 1, 2$ set $\varpi^i(\sigma) = \varpi((i,\sigma))/A_i(\emptyset)$ if $A_i(\emptyset) \neq 0$ and $\varpi^i(\sigma) = 0$ if $A_i(\emptyset) = 0$. It is plain to check that

$$\begin{aligned}&\sum_{\sigma\in\mathcal{L}(T^{*i}_n)} \varpi^i(\sigma)m_0 + \sum_{\sigma\in\mathcal{I}(T^{*i}_n)} \varpi^i(\sigma)A_0((i,\sigma)) \\ &i = 1, 2 \text{ are independent random variables} \\ &\qquad \text{with the same law of } M^*_{n-1}.\end{aligned} \tag{40}$$

Hence,

$$
\begin{aligned}
\mathtt{K}^n(\delta_{m_0}) = \mathcal{D}\Big(& A_1(\emptyset)\big(\sum_{\sigma\in\mathcal{L}(T_n^{*1})}\varpi^1(\sigma)m_0 + \sum_{\sigma\in\mathcal{I}(T_n^{*1})}\varpi^1(\sigma)A_0((1,\sigma))\big)\\
&+ A_2(\emptyset)\big(\sum_{\sigma\in\mathcal{L}(T_n^{*2})}\varpi^2(\sigma)m_0 + \sum_{\sigma\in\mathcal{I}(T_n^{*2})}\varpi^2(\sigma)A_0((2,\sigma))\big) + A_0(\emptyset))\Big)\\
&= \mathcal{D}(M_n^*).
\end{aligned}
\tag{41}
$$

As far as (b) is concerned, note that M_n^* is integrable and $\mathcal{G}_n^*$–measurable. Moreover,

$$
M_n^* = M_{n-1}^* + \sum_{\sigma\in\mathcal{L}(T_{n-1}^*)}\varpi(\sigma)[(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)]
$$

and hence

$$
\mathbb{E}[M_n^*|\mathcal{G}_{n-1}^*] = M_{n-1}^*
$$

under (H_2) and (H_3). Furthermore, $\mathbb{E}[M_n^*] = m_0$ for every n. Since M_n^* is a martingale then (37) is satisfied for $Y_j = M_{j-1}^* - M_{j-2}^*$ with the convention $M_{-1}^* := 0$. Hence we can apply the von Bahr-Esseen inequality (38) to get

$$
\begin{aligned}
\mathbb{E}[|M_n^*|^\gamma] &\le 2\mathbb{E}[|M_0^*|^\gamma] + 2\sum_{j=1}^n \mathbb{E}[|M_j^* - M_{j-1}^*|^\gamma]\\
&= 2|m_0|^\gamma + 2\sum_{j=1}^n \mathbb{E}\Big[\Big|\sum_{\sigma\in\mathcal{L}(T_{j-1}^*)}\varpi(\sigma)[(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)]\Big|^\gamma\Big].
\end{aligned}
$$

Now, since $\mathbb{E}[(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)|\mathcal{G}_{j-1}^*] = \mathbb{E}[(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)] = 0$ for every σ in $\mathcal{L}(T_{j-1}^*)$, using the von Bahr-Esseen inequality (38) once again we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{\sigma\in\mathcal{L}(T_{j-1}^*)}\varpi(\sigma)[(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)]\Big|^\gamma\Big|\mathcal{G}_{j-1}^*\Big]\\
&\qquad\le 2\sum_{\sigma\in\mathcal{L}(T_{j-1}^*)}|\varpi(\sigma)|^\gamma\mathbb{E}\Big[|(A_1(\sigma)+A_2(\sigma)-1)m_0 + A_0(\sigma)|^\gamma\Big|\mathcal{G}_{j-1}^*\Big]
\end{aligned}
$$

almost surely. Hence,

$$
\mathbb{E}[|M_n^*|^\gamma] \le 2|m_0|^\gamma + K\sum_{j=1}^n \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{j-1}^*)}|\varpi(\sigma)|^\gamma\Big]
$$

where $K = 4\mathbb{E}[|(A_1 + A_2 - 1)m_0 + A_0|^\gamma] < +\infty$ by assumption. Now it is easy to see that, for every $k \geq 1$,

$$\begin{aligned}\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_k^*)} |\varpi(\sigma)|^\gamma\Big] &= \mathbb{E}\Big[\sum_{\sigma'\in\mathcal{L}(T_{k-1}^*)} |\varpi(\sigma')|^\gamma(|A_1(\sigma')|^\gamma + |A_2(\sigma')|^\gamma)\Big] \\ &= \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma]\mathbb{E}\Big[\sum_{\sigma'\in\mathcal{L}(T_{k-1}^*)} |\varpi(\sigma')|^\gamma\Big] \\ &= \Theta(\gamma)\mathbb{E}\Big[\sum_{\sigma'\in\mathcal{L}(T_{k-1}^*)} |\varpi(\sigma')|^\gamma\Big]\end{aligned}$$

and then, for every $k \geq 0$,

$$\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_k^*)} |\varpi(\sigma)|^\gamma\Big] = \Theta(\gamma)^k \tag{42}$$

with $\Theta(\gamma) < 1$. Hence, $\sup_n \mathbb{E}[|M_n^*|^\gamma] < +\infty$ and, from the elementary martingale theory, it follows that $(M_n^*)_{n\geq 0}$ converges a.s. and in L^1 to a random variable M_∞^* such that $\mathbb{E}[M_\infty^*] = m_0$ and $\mathbb{E}[|M_\infty^*|^\gamma] < +\infty$. The proof of (c) is completed. In order to prove (d) set $\phi_n(\xi) = \mathbb{E}[\exp(i\xi M_n^*)]$. By (40), it is clear that (41) is equivalent to

$$\phi_n(\xi) = \mathbb{E}[\phi_{n-1}(A_1\xi)\phi_{n-1}(\xi A_2)e^{i\xi A_0}].$$

From (c) we know that $\phi_n(\xi)$ converges to $\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi M_\infty^*)]$ as $n \to +\infty$. Hence, by dominated convergence theorem, we get

$$\phi_\infty(\xi) = \mathbb{E}[\phi_\infty(A_1\xi)\phi_\infty(\xi A_2)e^{i\xi A_0}]$$

and the proof of (d) is completed. Arguing as in the proof of (c) it is easy to see that under (H_3), the terms $\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)m_0$ and $\sum_{\sigma\in\mathcal{I}(T_n^*)} \varpi(\sigma)A_0(\sigma)$, which form M_n^*, are both uniformly integrable martingales. Hence (e) easily follows.

Lemma 2 *If (H_1) holds for some γ in $(0,1]$, then for every real number m,*

- (a) *$\mathcal{K}^n(\delta_m)$ is the law of M_n^* defined in (39);*
- (b) *$\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)$ converges to 0 in L^γ;*
- (c) *$G_n^* = \sum_{\sigma\in\mathcal{I}(T_n^*)} \varpi(\sigma)A_0(\sigma)$ is a Cauchy sequence in L^γ and hence it converges in L^γ to the random variable*

$$G_\infty^* = \sum_{n\geq 0}\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)A_0(\sigma) = \sum_{\sigma\in\mathbb{U}} \varpi(\sigma)A_0(\sigma);$$

- (d) *M_n^* converges to G_∞^* in L^γ and the law μ_∞ of G_∞^* is a solution of (13) in $\mathcal{M}_\gamma$;*

Proof . The proof of (a) is the same of the proof of (a) in Proposition 1. Furthermore, since $\gamma \in (0,1]$, from (42) one deduces

$$\mathbb{E}\left[\Big|\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)\Big|^\gamma\right] \leq \mathbb{E}\left[\sum_{\sigma\in\mathcal{L}(T_n^*)} |\varpi(\sigma)|^\gamma\right] = \Theta(\gamma)^n.$$

Using the fact that $\Theta(\gamma) < 1$, (b) follows. In order to prove (c) observe that, if $n > m$,

$$\mathbb{E}[|G_n^* - G_m^*|^\gamma] = \mathbb{E}\left[\Big|\sum_{j=m}^{n-1}\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)A_0(\sigma)\Big|^\gamma\right] \leq \mathbb{E}[|A_0|^\gamma]\sum_{j=m+1}^{n} \Theta(\gamma)^j \to 0$$

for $n, m \to +\infty$, i.e. $(G_n^*)_n$ is a Cauchy sequence in L^γ. As for assertion (d), combining (b) and (c) we obtain that $(M_n^*)_n$ converges in L^γ to G_∞^*. Finally, arguing as in the proof of (d) of Lemma 1, we obtain that G_∞^* is a solution of (13) in $\mathcal{M}_\gamma$.

Proof of Proposition 1. The existence of a solution μ_∞ of (13) in $\mathcal{M}_\gamma$ is a consequence of Lemma 1 and Lemma 2. Let us prove the uniqueness. Let μ_1 and μ_2 be two solutions of (13) in $\mathcal{M}_\gamma$. Let $(Y_\sigma^{(1)})_\sigma$ and $(Y_\sigma^{(2)})_\sigma$ two sequences of independent random variables such that, for every $\sigma \in \mathbb{U}$, $Y_\sigma^{(1)}$ ($Y_\sigma^{(2)}$, respectively) has law μ_1 (μ_2, respectively) and, in addition,

$$(Y_\sigma^{(1)})_\sigma, \quad (Y_\sigma^{(2)})_\sigma, \quad (A(\sigma))_\sigma$$

are stochastically independent. Then, following the same lines of (a) in Lemma 1 and Lemma 2, it is easy to see that

$$\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)Y_\sigma^{(i)} + \sum_{\sigma\in\mathcal{I}(T_n^*)} \varpi(\sigma)A_0(\sigma)$$

has law μ_i $(i = 1,2)$. Let either (H_1) or (H_2) holds true and denote by $\mathcal{H}$ the σ–field generated by $(A(\sigma) : \sigma \in \mathbb{U})$, then

$$\begin{aligned}
\mathcal{W}_\gamma^{1\vee\gamma}(\mu_1,\mu_2) &\leq \mathbb{E}\Big[\Big|\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)(Y_\sigma^{(1)} - Y_\sigma^{(2)})\Big|^\gamma\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{\sigma\in\mathcal{L}(T_n^*)} \varpi(\sigma)(Y_\sigma^{(1)} - Y_\sigma^{(2)})\Big|^\gamma|\mathcal{H}\Big]\Big] \\
&\leq k_\gamma\mathbb{E}[\sum_{\sigma\in\mathcal{L}(T_n^*)} |\varpi(\sigma)|^\gamma\mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma|\mathcal{H}]] \qquad (43)\\
&= k_\gamma\mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma]\mathbb{E}[\sum_{\sigma\in\mathcal{L}(T_n^*)} |\varpi(\sigma)|^\gamma] \\
&= k_\gamma\mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma]\Theta(\gamma)^n.
\end{aligned}$$

The second inequality in (43) is immediate for $\gamma \leq 1$ with $k_\gamma = 1$. In order to prove it when $1 < \gamma \leq 2$, first note that every solution μ_i of equation

(13) has first moment equal to $\bar{m}$, which implies that $\mathbb{E}[Y_\sigma^{(1)} - Y_\sigma^{(2)}|\mathcal{H}] = 0$. Hence, when $1 < \gamma \le 2$, the second inequality in (43) follows with $k_\gamma = 2$ from the von Bahr-Esseen inequality (38). To conclude, note that in (43) we have $\Theta(\gamma)^n \to 0$ for $n \to +\infty$ and hence $\mu_1 = \mu_2$.

The proof of the uniqueness under (H_2) follows in a similar way, since the possible solutions μ_1 and μ_2 belong to $\mathcal{M}_{\gamma,m_0}$.

Proof of Proposition 2. The proof of (i) is straightforward.

The proofs of (ii) and (iii) are inspired by the proof of Theorem 5.3 in [22]. Let us first prove (ii). Note that, since Θ is a convex function, for every λ in $[0,1]$, $\Theta(\lambda\gamma + (1-\lambda)\beta) \le \lambda\Theta(\gamma) + (1-\lambda)\Theta(\beta)$, and hence $\Theta(s) < 1$ for every $\gamma \le s \le \beta$. In addition $\mathbb{E}|A_0|^s < +\infty$ since $\mathbb{E}|A_0|^\beta < +\infty$. Now fix $s \le \beta$, with $1 \le k < s \le k+1$, k integer. Then, for $x_i \ge 0$

$$\Big(\sum_{i=1}^3 x_i\Big)^s = \Big(\sum_{i=1}^3 x_i\Big)^{\frac{s}{k+1}(k+1)} \le \sum_{i=1}^3 x_i^s + \sum c_{j_1 j_2 j_3} (x_1^{j_1} x_2^{j_2} x_3^{j_3})^{\frac{s}{k+1}} \qquad (44)$$

for suitable constants $c_{j_1j_2j_3}$ and j_i are integers such that $j_i \le k$ and $j_1 + j_2 + j_3 = k+1$. Using (44) it is easy to see that

$$\mathbb{E}[|Y_1A_1 + Y_2A_2 + A_0|^s] \le \Theta(s)\mathbb{E}|Y|^s + c_1\mathbb{E}[|Y^k]^{\frac{s}{k}} + c_2 \qquad (45)$$

if Y, Y_1, Y_2 are independent random variables with the same law ν and (Y, Y_1, Y_2) is independent of (A_0, A_1, A_2). The constants c_1 and c_2 may depend on β but not on ν. Obviously (45) is equivalent to

$$\int |x|^s(\mathtt{K}\nu)(dx) \le \Theta(s)\int |x|^s\nu(dx) + c_1\Big[\int |x|^k\nu(dx)\Big]^{\frac{s}{k}} + c_2 \qquad (46)$$

Let either (H_2) or (H_3) hold true so that γ belongs to $(1,2]$. From Lemma 1 we know that $\mathtt{K}^n\delta_{m_0}$ converges weakly to μ_∞ and

$$\sup_n \int |x|^\gamma(\mathtt{K}^n\delta_{m_0})(dx) < +\infty.$$

Let us now prove that if, for $k \ge 1$ and $k < s \le k+1$

$$\sup_n \int |x|^k(\mathtt{K}^n\delta_{m_0})(dx) < +\infty \qquad \text{and} \qquad \Theta(s) < 1 \qquad (47)$$

is true, then

$$\sup_n \int |x|^s(\mathtt{K}^n\delta_{m_0})(dx) < +\infty \qquad \text{and} \qquad \int |x|^s\mu_\infty(dx) < +\infty. \qquad (48)$$

In fact, applying iteratively (46) starting from $\nu = \delta_{m_0}$, since $\int |x|^s\delta_{m_0}(dx) = |m_0|^s < +\infty$, one obtains

$$\int |x|^s(\mathtt{K}^n\delta_{m_0})(dx) \le |m_0|^s\Theta(s)^n + C\sum_{j=0}^{n-1}\Theta(s)^j$$

for a suitable constant C. Since $\Theta(s) < 1$, then one gets $\sup_n \int |x|^s (\mathtt{K}^n \delta_{m_0})(dx) < +\infty$ and

$$\int |x|^s \mu_\infty(dx) < +\infty.$$

Now, since $\gamma > 1$ and $\beta > 2$, then (47) holds for $k = 1$ and $s = 2$. As a consequence we obtain (48) for $s = 2$. Let us iterate this procedure for $k < \bar{k}$ with $\bar{k} < \beta \leq \bar{k} + 1$. The last step starts from the validity of (47) for $k = \bar{k}$ and $s = \beta$ which implies $\int |x|^\beta \mu_\infty(dx) < +\infty$.

If (H_1) holds and hence $0 < \gamma \leq 1$, then

$$\mathbb{E}[|A_0 + A_1 Y_1 + A_2 Y_2|] \leq \mathbb{E}[|Y|]\Theta(1) + \mathbb{E}[|A_0|].$$

Since $\Theta(1) < 1$ and $\mathbb{E}[|A_0|] < +\infty$, we get for every m

$$\int |x| (\mathtt{K}^n \delta_m)(dx) \leq |m| \Theta(1)^n + C \sum_{j=0}^{n-1} \Theta(1)^j$$

and hence, thanks to Lemma 2,

$$\sup_n \int |x| (\mathtt{K}^n \delta_m)(dx) < +\infty \qquad \text{and} \qquad \int |x| \mu_\infty(dx) < +\infty.$$

At this stage (47) is proved for $k = 1$ and $s = k + 1 = 2$ and we can go on as in the previous case. This proves (ii).

Let us prove (iii). If $\gamma \leq 1$, since μ_∞ is the law of G_n^* of Lemma 2 (c) one obtains that $\mu_\infty\{[0, +\infty)\} = 1$. Now assume that $\gamma \in (1, 2]$. Since $P\{A_0 \geq 0\} = 1$ and $P\{A_0 \neq 0\} > 0$, then $\mathbb{E}[A_0] \neq 0$ and only (H_2) has to be considered. By assumption $\bar{m} = \int x \mu_\infty(dx) > 0$ and, hence, from Lemma 1 $(c) - (d)$ we obtain that M_n^* is positive a.s. for every $n \geq 1$. This yields $\mu_\infty\{[0, +\infty)\}$.

As a consequence, using the fact that $\beta \geq 1$, we can write

$$X^\beta =^{\mathcal{D}} (A_0 + A_1 X_1 + A_2 X_2)^\beta \geq A_0^\beta + A_1^\beta X_1^\beta + A_2^\beta X_2^\beta$$

($=^{\mathcal{D}}$ denotes the identity in distribution) if X, X_1, X_2 are independent random variables with law μ_∞ and (X, X_1, X_2) and (A_0, A_1, A_2) are stochastically independent. Then

$$\mathbb{E}[X^\beta] \geq \Theta(\beta)\mathbb{E}[X_1^\beta] + \mathbb{E}[A_0^\beta] > \Theta(\beta)\mathbb{E}[X_1^\beta]$$

since we are assuming that $P\{A_0 \neq 0\} > 0$. Hence $\Theta(\beta) < 1$.

Let us state a useful result which is proved, with slightly different notation, in Lemma 2 of [2] (see also Proposition 4.1 in [1]).

Lemma 3 *Let $\gamma > 0$ such that $\Theta(\gamma) = \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma] < +\infty$. Then, for every $n \geq 0$,*

$$\mathbb{E}\Big[\sum_{\sigma \in \mathcal{L}(T_n)} |\varpi(\sigma)|^\gamma \Big] = \frac{\Gamma(\Theta(\gamma) + n)}{\Gamma(n+1)\Gamma(\Theta(\gamma))} =: c_n(\gamma). \tag{49}$$

Proof . Given the sequence $(T_n)_{n\geq 1}$ of random binary recursive trees, one can define a sequence $(V_n)_{n\geq 1}$ of $\mathbb{U}$-valued random variables such that

$$T_{n+1} = T_n \cup \{(V_n,1),(V_n,2)\}$$

for every $n \geq 0$, where $V_0 = \emptyset$ and $V_n \in \mathcal{L}(T_n)$. The random variable V_n corresponds to the random vertex chosen to generate T_{n+1} from T_n. Hence, by construction,

$$P\{V_n = \sigma | T_1,\dots,T_n\} = \mathbb{I}\{\sigma \in \mathcal{L}(T_n)\}\frac{1}{n+1}$$

for every $n \geq 1$. Since $T_0 = \emptyset$ and $\varpi(\emptyset) = 1$, $\mathbb{E}[\sum_{\sigma\in\mathcal{L}(T_0)} |\varpi(\sigma)|^\gamma] = 1$ and hence (49) is true for $n = 0$. For $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_n)} |\varpi(\sigma)|^\gamma\Big] \\
&\quad = \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{n-1})} |\varpi(\sigma)|^\gamma \Big[(|A_1(\sigma)|^\gamma + |A_2(\sigma)|^\gamma - 1)\mathbb{I}\{V_{n-1} = \sigma\} + 1\Big]\Big] \\
&\quad = \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{n-1})} |\varpi(\sigma)|^\gamma\Big]\Big(1 + \frac{\Theta(\gamma)-1}{n}\Big)
\end{aligned}$$

from the independence assumptions and since $(A_1(\sigma), A_2(\sigma))$ has the same law of (A_1, A_2) for every σ. Hence, by induction,

$$\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_n)} |\varpi(\sigma)|^\gamma\Big] = \prod_{j=1}^{n}\Big(1 + \frac{\Theta(\gamma)-1}{j}\Big) = \frac{\Gamma(\Theta(\gamma)+n)}{\Gamma(n+1)\Gamma(\Theta(\gamma))}$$

and (49) is proved.

In the following denote by ζ_n the law of Z_n.

Lemma 4 *Let* (H_1) *or* (H_2) *hold for some* γ *in* $(0,2]$, *then*

$$\mathcal{W}_\gamma^{1\vee\gamma}(\zeta_n,\mu_\infty) \leq k_\gamma c_n(\gamma)\mathcal{W}_\gamma^{1\vee\gamma}(\bar{\mu}_0,\mu_\infty) \tag{50}$$

for every $n \geq 0$, *where* μ_∞ *is the unique solution of* (13) *in* $\mathcal{M}_\gamma$ *and* $k_\gamma = 1$ *for* $0 < \gamma \leq 1$ *and* $k_\gamma = 2$ *for* $1 < \gamma \leq 2$. *Furthermore, if* $\gamma \geq 1$ *and* $m_0 = \bar{m}$ *(also for* $\gamma = 1$*) then* $\mathbb{E}(Z_n) = \int v\zeta_n(dv) = \bar{m}$ *for every* $n \geq 0$.

If (H_3) *holds for some* γ *in* $(1,2]$ *and* μ_∞ *is the unique solution of* (13) *in* $\mathcal{M}_{\gamma,m_0}$, *then* (50) *holds and* $\mathbb{E}(Z_n) = \int v\zeta_n(dv) = m_0$ *for every* $n \geq 0$.

Proof . The existence and uniqueness of μ_∞ is guaranteed by Proposition 1. On a sufficiently large probability space $(\Omega,\mathcal{F},P)$ consider a sequence $(Y_\sigma)_{\sigma\in\mathbb{U}}$, such that

- $(A(\sigma))_{\sigma\in\mathbb{U}}$,$(T_n)_{n\geq 0}$, and $(X_\sigma,Y_\sigma)_{\sigma\in\mathbb{U}}$ are independent;
- (X_σ,Y_σ) are independent and identically distributed for σ in $\mathbb{U}$, and each (X_σ,Y_σ) is an optimal transport plan for $\mathcal{W}_\gamma(\bar{\mu}_0,\mu_\infty)$, i.e. the law of X_σ is $\bar{\mu}_0$, the law of Y_σ is μ_∞ and $\mathbb{E}|X_\sigma - Y_\sigma|^\gamma = \mathcal{W}_\gamma^{1\vee\gamma}(\bar{\mu}_0,\mu_\infty)$.

Let us set $U_n = \sum_{\sigma\in\mathcal{L}(T_n)} \varpi(\sigma)Y_\sigma + G_n$. It is easy to see that the law of U_n is μ_∞ for every n. Now denote by $\mathcal{G}$ the σ–field generated by $(A(\sigma) : \sigma \in \mathbb{U}, (T_n)_{n\geq 1})$ and observe that

$$\begin{aligned}\mathcal{W}_\gamma^{1\vee\gamma}(\zeta_n,\mu_\infty) &\leq \mathbb{E}|Z_n - U_n|^\gamma = \mathbb{E}\Big[\mathbb{E}\Big[|\sum_{\sigma\in\mathcal{L}(T_n)} \varpi(\sigma)(X_\sigma - Y_\sigma)|^\gamma \Big|\mathcal{G}\Big]\Big] \\ &\leq k_\gamma \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_n)} |\varpi(\sigma)|^\gamma \mathbb{E}\Big[|X_\sigma - Y_\sigma|^\gamma\Big|\mathcal{G}\Big]\Big].\end{aligned}$$

Once again, the last inequality is immediate for $\gamma \leq 1$ with $k_\gamma = 1$. On the other hand, if $1 < \gamma \leq 2$, it follows with $k_\gamma = 2$ from (38), since $\mathbb{E}[X_\sigma] = \mathbb{E}[Y_\sigma]$ which implies $\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_n)} \varpi(\sigma)(X_\sigma - Y_\sigma)\Big|\mathcal{G}\Big] = 0$. Hence, using (49),

$$\begin{aligned}\mathcal{W}_\gamma^{1\vee\gamma}(\zeta_n,\mu_\infty) &\leq k_\gamma \mathbb{E}\Big[\sum_{s\in\mathcal{L}(T_n)} |\varpi(\sigma)|^\gamma\Big] \mathcal{W}_\gamma^{1\vee\gamma}(\bar\mu_0,\mu_\infty) \\ &= k_\gamma c_n(\gamma) \mathcal{W}_\gamma^{1\vee\gamma}(\bar\mu_0,\mu_\infty).\end{aligned}$$

In order to conclude the proof, let us study $\mathbb{E}[Z_n]$ when γ belongs to $[1,2]$. Observe that

$$\begin{aligned}\mathbb{E}[Z_n] &= \mathbb{E}[Z_{n-1}] \\ &\quad + \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{n-1})} \varpi(\sigma)[A_0(\sigma) + A_1(\sigma)X_{(\sigma,1)} + A_2(\sigma)X_{(\sigma,2)} - X_\sigma]\mathbb{I}\{V_n = \sigma\}\Big].\end{aligned}$$

If $\mathcal{G}_{n-1}$is the σ-field generated by $(T_1,\ldots,T_n; \varpi(\sigma) : \sigma \in \mathcal{L}(T_{n-1}))$, then

$$\begin{aligned}&\mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{n-1})} \varpi(\sigma)[A_0(\sigma) + A_1(\sigma)X_{(\sigma,1)} + A_2(\sigma)X_{(\sigma,2)} - X_\sigma]\mathbb{I}\{V_n = \sigma\}\Big] \\ &\quad = \mathbb{E}\Big[\sum_{\sigma\in\mathcal{L}(T_{n-1})} \varpi(\sigma)\mathbb{I}\{V_n = \sigma\}\mathbb{E}\big[A_0(\sigma) + A_1(\sigma)X_{(\sigma,1)} + A_2(\sigma)X_{(\sigma,2)} - X_\sigma\big|\mathcal{G}_{n-1}\big]\Big]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big[A_0(\sigma) + &A_1(\sigma)X_{(\sigma,1)} + A_2(\sigma)X_{(\sigma,2)} - X_\sigma\big|\mathcal{G}_{n-1}\big] \\ &= \mathbb{E}[A_1 + A_2 - 1]m_0 + \mathbb{E}[A_0] = 0\end{aligned}$$

under our assumptions. Hence, $\mathbb{E}[Z_n] = \mathbb{E}[Z_0] = m_0$, which completes the proof.

Proof of Proposition 3. Using Proposition 4, the convexity of $\mathcal{W}_\gamma^{\gamma\vee 1}$ and Lemma 4 one gets

$$\begin{aligned}&\mathcal{W}_\gamma^{\gamma\vee 1}(\mu_t,\mu_\infty) \leq \sum_{n\geq 0} e^{-t}(1-e^{-t})^n \mathcal{W}_\gamma^{\gamma\vee 1}(\zeta_n,\mu_\infty) \\ &\leq \sum_{n\geq 0} e^{-t}(1-e^{-t})^n k_\gamma c_n(\gamma) \mathcal{W}_\gamma^{\gamma\vee 1}(\bar\mu_0,\mu_\infty) = e^{-t(1-\Theta(\gamma))} k_\gamma \mathcal{W}_\gamma^{\gamma\vee 1}(\bar\mu_0,\mu_\infty)\end{aligned}$$

with $k_\gamma = 1$ if $\gamma \leq 1$ and $k_\gamma = 2$ if $1 < \gamma \leq 2$ and this proves the first part of the proposition. Now assume that (H_1) holds with $\gamma = 1$, if $m_0 = \bar{m}$, then

$$\int v\mu_t(dv) = \sum_{n\geq 0} e^{-t}(1-e^{-t})^n \mathbb{E}[Z_n] = \sum_{n\geq 0} e^{-t}(1-e^{-t})^n \bar{m} = \bar{m},$$

since $\mathbb{E}[Z_n] = \bar{m}$ as stated in of Lemma 4. The other cases can be treated in an analogous way.

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