Manuscript submitted to
AIMS' Journals
Volume , Number **0**,

Website: http://AIMsciences.org

pp. **X–XX**

ON THE MINIMIZATION PROBLEM OF SUB-LINEAR CONVEX FUNCTIONALS

Naoufel Ben Abdallah

Laboratoire MIP, University Paul Sabatier Toulouse
118 route de Narbonne F-31062 Toulouse Cedex 9, France

Irene M. Gamba

ICES & Department of Mathematics and ICES, University of Texas
Austin, Texas 78712 USA

Giuseppe Toscani

University of Pavia, Department of Mathematics
Via Ferrata 1, 27100 Pavia, Italy

(Communicated by)

Dedicated to Naoufel, who will remain with us forever

Abstract. The study of the convergence to equilibrium of solutions to Fokker-Planck type equations with linear diffusion and super-linear drift leads in a natural way to a minimization problem for an energy functional (entropy) which relies on a sub-linear convex function. In many cases, conditions linked both to the non-linearity of the drift and to the space dimension allow the equilibrium to have a singular part. We present here a simple proof of existence and uniqueness of the minimizer in the two physically interesting cases in which there is the constraint of mass, and the constraints of both mass and energy. The proof includes the localization in space of the (eventual) singular part. The major example is related to the Fokker-Planck equation introduced in [6, 7] to describe the evolution of both Bose-Einstein and Fermi-Dirac particles.

1. Introduction.

The quantum dynamics of many body systems is often modeled by a nonlinear Boltzmann equation which exhibits a gas-particle-like collision behavior. The application of quantum assumptions to molecular dynamics encounters leads to some divergences from the classical kinetic theory [2] and despite their formal analogies the Boltzmann equation for classical and quantum kinetic theory

2000 *Mathematics Subject Classification.* Primary: 35B40; Secondary: 35K55, 39B62.

Key words and phrases. Fokker-Planck equation, Bose-Einstein distribution, large time behaviour, sub-linear convex functionals.

The second author acknowledges partial support from NSF grant DMS-0807712. The third author is supported by MIUR project "Optimal mass transportation, geometrical and functional inequalities with applications". The warm hospitality of the Laboratoire MIP, of the University Paul Sabatier in Toulouse, where this work have been partially done, is kindly acknowledged. Support from the Institute from Computational Engineering and Sciences at the University of Texas at Austin is also gratefully acknowledged. The authors thank the second reviewer for the very useful comments and corrections as well as generous suggestions for improvement of this work presentation.

present very different features. Although the quantum Boltzmann equation, for a single species of particles, is valid for a gas of fermions as well as for a gas of bosons, blow up of the solution in finite time may occur only in the latter case. As a consequence the quantum Boltzmann equation for a gas of bosons represents the more challenging case both mathematically and numerically. In particular this equation has been successfully used for computing non-equilibrium situations where Bose-Einstein condensate occurs. From Chapman and Cowling [2] one can learn that the Boltzmann Bose-Einstein equation is established by imposing that, when the mean distance between neighboring molecules is comparable to the size of the quantum wave fields in which molecules are imbedded, a state of congestion results. For a gas composed of Bose-Einstein identical particles, according to quantum theory, the presence of a like particle in the velocity-range dv increases the probability that a particle will enter that range; the presence of $f(v)dv$ particles per unit volume increases this probability in the ratio $1+\epsilon f(v)$.

The fundamental assumption which leads to the correction in the Boltzmann collision operator, namely the fact that the presence of $f(v)dv$ particles per unit volume increases the probability that a particle will enter the velocity range dv in the ratio $1+\epsilon f(v)$, has been recently used by Kaniadakis and Quarati [6, 7] to propose a correction to the drift term of the Fokker-Planck equation in presence of quantum indistinguishable particles, bosons or fermions. For Bose-Einstein particles, their model reads

$$\frac{\partial f}{\partial t} = \nabla \cdot \left[\nabla f + vf(1+\epsilon f)\right]; \qquad v \in \mathbb{R}^d, d \geq 1. \tag{1}$$

By a direct inspection, one can easily verify that equation (1) then admits the Bose-Einstein distribution as stationary state. Indeed, the Bose-Einstein distribution [2]

$$f_\lambda(v) = \frac{1}{\epsilon}\left[e^{|v|^2/2+\lambda} - 1\right]^{-1}, \tag{2}$$

satisfies the equation

$$\nabla f_\lambda(v) + vf_\lambda(v)(1+\epsilon f_\lambda(v)) = 0$$

for any fixed positive constant λ. The constant λ is related to the mass of Bose-Einstein distribution

$$m_\lambda = \int_{\mathbb{R}^d} \frac{1}{\epsilon}\left[e^{|v|^2/2+\lambda} - 1\right]^{-1} dv,$$

and, since the mass is decreasing as soon as λ increases, the maximum value of m_λ is attained at $\lambda = 0$. If $d = 3$, the value

$$m_c = m_0 = \int_{\mathbb{R}^3} \frac{1}{\epsilon}\left[e^{|v|^2/2} - 1\right]^{-1} dv < +\infty \tag{3}$$

defines the *critical mass*. The Fokker-Planck equation (1) considered by Kaniadakis and Quarati in [6, 7] is a particular case of the general one obtained by varying the diffusion constant in front of the diffusion operator. In this case, the equation reads

$$\frac{\partial f}{\partial t} = \nabla \cdot \left[\theta\nabla f + vf(1+\epsilon f)\right]; \qquad v \in \mathbb{R}^d, d \geq 1, \tag{4}$$

where θ is a positive constant, and the steady state, which satisfies the equation

$$\theta\nabla f(v) + vf(v)(1+\epsilon f(v)) = 0 \tag{5}$$

is now given by the Bose-Einstein distribution

$$f_{\lambda,\theta}(v) = \frac{1}{\epsilon}\left[e^{|v|^2/(2\theta)+\lambda} - 1\right]^{-1}. \tag{6}$$

The two parameters (λ, θ) are associated to the mass and energy of (6). One can easily deduce from equation (5) that the steady state (6) is the most general one can obtain from the Fokker-Planck equation (4). In fact, even in presence of a multiplicative constant γ in front of the drift term, one is led to a steady state which satisfies (5), with the constant θ substituted by the new constant θ/γ.

In particular, the mass given by

$$m_{\lambda,\theta} = \int_{\mathbb{R}^3} f_{\lambda,\theta}(v)dv\,, \tag{7}$$

still has its maximum attained at $\lambda = 0$ for all values of $\theta > 0$, while the kinetic energy is given by

$$\mathcal{E}_{\lambda,\theta} = \int_{\mathbb{R}^3} |v|^2 f_{\lambda,\theta}(v)dv\,, \tag{8}$$

It is then easy to observe that, due to the scaling properties of the distribution (6),

$$m_{\lambda,\theta} \;= \theta^{3/2}\, m_{\lambda,1}; \qquad \mathcal{E}_{\lambda,\theta} \;= \theta^{5/2}\, \mathcal{E}_{\lambda,1}\,. \tag{9}$$

The relations (9) imply

$$\left(\frac{m_{\lambda,\theta}}{m_{\lambda,1}}\right)^5 \left(\frac{\mathcal{E}_{\lambda,1}}{\mathcal{E}_{\lambda,\theta}}\right)^3 \;=\; 1\,. \tag{10}$$

Hence at the critical mass value $m_c = m_{0,1}$, attained at $\lambda = 0$, the corresponding energy $\mathcal{E}_c := \mathcal{E}_{0,1}$ satisfies the compatibility condition

$$\left(\frac{m_{\lambda,\theta}}{m_c}\right)^5 \left(\frac{\mathcal{E}_c}{\mathcal{E}_{\lambda,\theta}}\right)^3 \;=\; 1\,. \tag{11}$$

One of the fundamental problems related to evolution equations that relax towards a stationary state characterized by the existence of a critical mass, is to show how, starting from an initial distribution with a super-critical mass $m > m_c$, the solution eventually develops a singular part (the condensate), and, as soon as the singular part is present, to be able to follow its evolution.

We remark that in general the condensation phenomenon is heavily dependent of the dimension of the physical space. In dimension $d \leq 2$, in fact, the maximal mass m_0 of the Bose-Einstein distribution (2) is unbounded, and the eventual formation of a condensate is lost.

In the linear Fokker-Planck equation, corresponding to the choice $\epsilon = 0$, convergence to the equilibrium Maxwell-Boltzmann (Gaussian) distribution is achieved by showing convergence of the Boltzmann H-functional

$$H(f) = \int \left(f(v)\log f(v) + \frac{|v|^2}{2} f(v) \right) dv$$

towards its minimum. An analogous proof for equation (4) would require the knowledge of the convergence of the Bose-Einstein entropy

$$H_B(f) = \int \left(f(v)\log f(v) - (1+\epsilon f(v))\log(1+\epsilon f(v)) + \frac{|v|^2}{2\theta} f(v) \right) dv$$

towards its minimizer (formally given by the Bose-Einstein distribution (6)). While the fact that the Bose-Einstein distribution (6) is a minimizer of $H_B(f)$ is commonly accepted in the physics literature, its rigorous justification needs some care. Some results in this direction have been shown by Glassey and Strauss [5], Lu [8] and maybe others. In a recent paper [4], Escobedo Mischler and Valle give almost general and optimal statements and proofs, with a unified treatment of the minimization problem in various cases, including the standard Boltzmann functional, as well as Bose-Einstein and Fermi-Dirac entropies. However the problem of finding a well posed optimization problem for unique stationary solutions to problem (4) that admit a singular part has not been addressed so far.

In the present manuscript we solved, by constrained minimization, the problem of finding stationary solutions to the Fokker-Planck equation (1), where the non-linear drift term is extended to a general super-linear class defined below. We deal first with constraining only the mass, and later with constraining both mass and energy. In both constraining sets, it is possible to show both existence and uniqueness of solutions that can exhibit a singular part (condensation) depending on the constrained mass for the first case, and on a relation between the constrained mass and energy for the latter one.

Our proof relies on existence and uniqueness of minimizers for appropriate energy (entropy) functionals which are uniquely defined in terms of the super-linear drift term. The constrained mass-energy problem satisfies an d-dimensional extension of the compatibility condition (11) that characterizes criticality for the minimization of mass-energy $(m, \mathcal{E})$ pairs. The the d-dimensional scaling invariant ratio is given by

$$\mathcal{R}_{m,\mathcal{E}} = \left(\frac{m}{m_c}\right)^{d+2} \left(\frac{\mathcal{E}_c}{\mathcal{E}}\right)^{d}. \tag{12}$$

The criticality condition for a Dirac singular mass formation in the solution of the mass and energy constrained minimization problem is given by the value of $\mathcal{R}_{m,\mathcal{E}}$ compared to unity, as shown in section 3.2 of this manuscript.

This introduction will be completed after the presentation of some preliminaries concerned with the Fokker-Plank equation with a super-linear drift term. In section 2 we will discuss the corresponding steady states we will introduce the natural entropy functional. In section 3 we present the minimization problem, both for the mass constraint case and later for mass and energy constraints. The solutions are fully characterized by their criticality constraints.

1.1. **Preliminaries.** We start by noticing that the Fokker-Planck type equation for Bose-Einstein particles (4) belongs to the more general class of Fokker-Planck type equations

$$\frac{\partial f}{\partial t} = \nabla \cdot \left[\theta \nabla f + v D(f)\right], \tag{13}$$

where θ is a positive constant, and the *drift* function $D(f)$ is *super-linear* and *strictly* increasing from $D(0) = 0$. We associate this definition to any function $D(\cdot)$ such that

$$\int_1^{+\infty} \frac{1}{D(\rho)}\, d\rho \leq K < +\infty. \tag{14}$$

In analogy with the Bose-Einstein case (1) considered by Kaniadakis and Quarati, where $D(\rho) = \rho(1+\rho)$, we will assume further that $D(\cdot)$ is linear close to zero

$$\lim_{\rho \to 0^+} \frac{D(\rho)}{\rho} = 1. \tag{15}$$

Note that the linear drift $D(\rho) = \rho$ which leads to the standard Fokker-Planck equation with a Maxwellian equilibrium is the border-line case. Condition (14) does not hold, since the integral

$$\int_1^R \frac{1}{\rho}\, d\rho$$

has a logarithmic growth as R converges to infinity.

In the remainder of the paper, we will be interested in the equilibrium states of the Fokker-Planck equation (13) and in the possibility that these equilibria admit a singular part (condensation). As known in the situation in which the drift term in the Fokker-Planck equation is linear, while the diffusion operator is nonlinear [1], both equilibria and entropy functionals are uniquely characterized in a systematic way.

An interesting problem related to these equilibria is the minimization problem of the associated entropy functional, under the constraint of fixed mass and energy. As we will see, this problem can be solved in a clear and elegant way, by resorting only to the properties (14) and (15) of the drift function.

One of the results of our analysis will be that, if the super-linearity of the drift is sufficiently strong, formation of singularities can happen also in dimension one of the velocity space. This is the case, among others, of the drift functions $D_N(\rho) = \rho(1+\rho^N)$, when $N > 2$. This introduces the possibility to use some of the results in [1], which are confined to dimension one, to study the relaxation to equilibrium even in the situation in which the initial datum has a mass which is above the critical one, and one can have formation of a condensed part in finite time. This problem will be object of a separate research.

Our minimization proof is alternative to the one recently presented in [4], from which differs in many aspects. In particular, it is stated here that the (eventual) singular part is localized in the origin.

2. **Steady states and entropy.**

The steady states of equation (13), of given mass $m > 0$, solve

$$\theta\nabla f + vD(f) = 0. \tag{16}$$

Since $D(f) > 0$ when $f > 0$, the solution to (16) satisfies

$$\frac{1}{D(f)}\nabla f + \nabla\frac{|v|^2}{2\theta} = 0. \tag{17}$$

Let us set

$$\Phi'(f) = -\int_f^{+\infty} \frac{1}{D(\rho)}\, d\rho. \tag{18}$$

Thanks to condition (14), $\Phi'(\rho)$ in $(-\infty, 0)$ is strictly increasing for $\rho \in (0,\infty)$, with

$$\lim_{\rho\to\infty} \Phi'(\rho) = 0 \quad \text{and.} \quad \lim_{\rho\to 0^+} \Phi'(\rho) = -\infty$$

Substituting $\Phi(\cdot)$ into (17), we obtain that the steady state is a solution to

$$\nabla\left\{\Phi'(f) + \frac{|v|^2}{2\theta}\right\} = 0, \tag{19}$$

or, what is the same,

$$\Phi'(f) = -\frac{|v|^2}{2\theta} - \lambda, \tag{20}$$

λ being an integration constant. We remark that, by the definition of Φ', the constant λ has to be nonnegative. If not, choosing $v = 0$ into (20) we obtain a contradiction. Since $\Phi'(\cdot)$ is strictly increasing, denoting by $[\Phi']^{-1}$ its inverse, we can invert (20) to find the steady states of equation (13)

$$f_{\lambda,\theta}(v) = [\Phi']^{-1}\left(-\frac{|v|^2}{2\theta} - \lambda\right). \tag{21}$$

Remark 1. The value of the constant λ is related to the mass of the steady state. Using the fact that $\Phi'(\rho)$ is non-decreasing with respect to ρ, so that the same monotonicity property holds for its inverse function, we can conclude that the value of $f_{\lambda,\theta}(v)$ at a fixed argument $\bar{v}$ is a decreasing function of the parameter λ. In consequence, the maximum value of the masses of the family of steady states $f_{\lambda,\theta}$ is achieved in correspondence to $\lambda = 0$.

Following the general framework of statistical equilibrium theory, the steady states of the Fokker-Planck equation (13) can be alternatively seen as the extremal points of a suitable entropy functional. To identify such entropy functional, consider that defining

$$\Phi(f) = \int_0^f \Phi'(\rho)\, d\rho, \tag{22}$$

the function $\Phi(\rho)$, which has a derivative that is non-decreasing, is convex for $\rho \geq 0$.

Hence, as usual when dealing with Fokker-Planck type equations [1], let us consider the *natural* entropy functional

$$H(f) = \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2\theta} f + \Phi(f)\right) dv. \tag{23}$$

If we now look for extremals of the entropy functional (23) on the set of functions f that belong to $\mathcal{F}_m$, where, for a given positive constant m

$$\mathcal{F}_m = \left\{ f \geq 0 : \int_{\mathbb{R}^d} f\, dv = m \right\}, \tag{24}$$

it is standard to conclude that the (possible) extremals solve

$$\int_{\mathbb{R}^d} \left\{ \Phi'(f) + \frac{|v|^2}{2\theta} + \lambda \right\} \delta f\, dv = 0. \tag{25}$$

Hence the steady state (21) with λ chosen to satisfy the constraint on the mass, is the (unique) extremal of the entropy functional (22). Note that, at least from a formal point of view, the entropy functional $H(f)$ decreases in time along the solution to the Fokker-Planck equation (13). In fact

$$\frac{dH(f)}{dt} = \int_{\mathbb{R}^d} \left[\Phi'(f) + \frac{|v|^2}{2\theta}\right] \frac{\partial f}{\partial t}\, dv = \int_{\mathbb{R}^d} \left[\Phi'(f) + \frac{|v|^2}{2\theta}\right] \nabla \cdot [\theta \nabla f + v D(f)]\, dv =$$
$$- \int_{\mathbb{R}^d} \theta D(f) \left| \nabla \left[\Phi'(f) + \frac{|v|^2}{2\theta}\right]\right|^2 dv \leq 0. \tag{26}$$

Thus, one can expect that, starting from an initial density of given mass, the solution to the Fokker-Planck equation (13) converges towards the steady state (21) with the same mass, and the convergence in relative entropy can be stated to show exponential convergence to equilibrium. In practice, however, various problems arise if the drift function $D(\rho)$ is growing too fast as ρ goes to infinity. To clarify this point, let us point out that, as shown before, the maximal value of the masses of the steady states (21) is reached for $\lambda = 0$. In this case, choosing for simplicity $\theta = 1$, we obtain

$$\int_{\mathbb{R}^d} f_{0,1}(v)\,dv = \int_{\mathbb{R}^d} [\Phi']^{-1}\left(-\frac{|v|^2}{2}\right) = B_d \int_0^{+\infty} [\Phi']^{-1}\left(-\frac{\rho^2}{2}\right)\rho^{d-1}\,d\rho \tag{27}$$

where B_d denotes the measure of the surface of the unit ball in $\mathbb{R}^d$. Since Φ', as defined in (18) is non-positive, the choice of coordinates s given by

$$\Phi'(s) = -\frac{\rho^2}{2}$$

allows to compute the last integral on (27) in terms of the function Φ, and consequently, of the drift function D. Taking into account that, thanks to properties (14) and (15) of the drift function one obtains

$$\lim_{s\to+\infty} \Phi'(s) = 0, \qquad \lim_{s\to 0^+} \Phi'(s) = -\infty,$$

the maximal (critical) mass $m_c = m_c(D)$ associated to the drift function D, is furnished by the integral

$$m_c(D) = \int_{\mathbb{R}^d} f_{0,1}(v)\,dv = B_d \int_0^{+\infty} s\Phi''(s)\left[-2\Phi'(s)\right]^{(d-2)/2}\,ds, \tag{28}$$

or, what is the same

$$m_c = B_d \int_0^{+\infty} \frac{s}{D(s)}\left[2\int_s^{+\infty} \frac{1}{D(\rho)}\,d\rho\right]^{(d-2)/2} ds. \tag{29}$$

The main example of *super-linear* drift is represented by the Bose-Einstein-Fokker-Planck equation (1) introduced by Kaniadakis and Quarati. In this relevant case, $D(\rho) = \rho(1+\rho)$. Then,

$$\Phi'(f) = -\int_f^{+\infty} \frac{1}{D(\rho)}\,d\rho = \log\frac{f}{1+f}.$$

The maximal mass of the steady state in $\mathbb{R}^d$ is given by

$$m_c = B_d \int_0^{+\infty} \frac{1}{1+s}\left[2\log\frac{1+s}{s}\right]^{(d-2)/2} ds. \tag{30}$$

It is easy to show that the integral is unbounded if $d = 1, 2$, while the same is bounded if the dimension $d \ge 3$. In other words, when $d = 1, 2$ by varying the value of the parameter λ from $+\infty$ to 0, the family (21) of steady states ranges its mass from 0 to $+\infty$. In the remaining case $d \ge 3$, since the integral in (30) is bounded, this is no longer possible, and the possible masses for the family (21) vary from 0 to m_c, which represents a critical mass for the problem.

Consider now a general drift function of the form $D(\rho) = \rho(1+\rho^N)$, where $N > 0$ is a given constant. In this case,

$$\Phi'(f) = \frac{1}{N}\log\left(\frac{f^N}{1+f^N}\right), \tag{31}$$

and the maximal mass in dimension $d = 1$ is given by

$$m_c = \int_0^{+\infty} \frac{1}{1+s^N}\left[\frac{2}{N}\log\left(1+\frac{1}{s^N}\right)\right]^{-1/2} ds. \tag{32}$$

Since the integrand behaves like $1/x^{N/2}$ as x goes to $+\infty$, the integral in (32) is bounded as soon as $N > 2$. Hence, we have the same phenomenon of existence of a bounded critical mass in dimension 1 of the velocity variable.

The relationship between the Fokker-Planck equation (13), its steady states and the related entropy functional induces in a natural way the minimization problem under mass constraint. The minimization problem associated to an entropy functional where both mass and energy are constrained, is easily obtained from the H-functional (23), by studying minimizers of

$$H(f) = \int_{\mathbb{R}^d} \left(\left(\frac{|v|^2}{2\theta} + \lambda \right) f + \Phi(f) \right) dv, \tag{33}$$

for a non-negative pair of Lagrange multipliers (λ, θ).

3. **Sub-linear entropy functionals and their minimizers.**

3.1. **Minimization under mass constraint.** We first present the minimization problem, under mass constraint only, related to sub-linear entropies. We exhibit necessary and sufficient conditions both for the existence of condensates, and for the localization in velocity space of these condensate solutions. In the remainder of this section, without loss of generality, we will assume $\theta = 1$. Identical results, however, still hold for a general θ. More precisely, we consider the entropy function $H(f)$ defined in (23) which we recall here

$$H(f) = \int (\frac{|v|^2}{2} f + \Phi(f)) \, dv.$$

The function Φ was assumed to be strictly convex and decreasing on $[0, \infty)$, satisfying

$$\Phi(0) = 0 \quad , \quad \lim_{s \to +\infty} \Phi'(s) = 0 \quad ; \quad \lim_{s \to 0^+} \Phi'(s) = -\infty.$$

Since Φ' vanishes at $+\infty$, then $\lim_{t \to +\infty} \frac{\Phi(t)}{t} = 0$.

We will now minimize the functional H on the set

$$\mathcal{F}_m = \left\{ f \in L^1_+(\mathbb{R}^d); \quad \int_{\mathbb{R}^d} f \, dv = m \right\}, \tag{34}$$

Since the function Φ is sub-linear, the Dunford-Pettis criterion for compactness of minimizing sequences cannot be satisfied. Therefore sequences of functions with bounded entropies will eventually weak* converge in the sense of measures. This issue has been extensively studied by Demengel and Temam [3] where one has: Let f_n be a sequence in $\mathcal{F}_m$ weak star converging in measures to $f \in \mathcal{M}_b(\mathbb{R}^d)$. In this context we consider the Lebesgue decomposition $f = f_r + f_s$ of a measure f, where f_r is the absolutely continuous part of f and f_s its singular part. Such measures should be understood as $f(dv) = f_r(v)dv + f_s(dv)$, and its corresponding Lebesgue integral for the singular part is given with the standard notation

$$\int \varphi \, f_s dv \ := \ \int \varphi(v) \, f_s(dv) \,. \tag{35}$$

Then

$$\liminf H(f_n) \geq H(f) = \int (\frac{|v|^2}{2} f + \Phi(f_r)) \, dv.$$

In other words, the sub-linear part of the entropy does not see the singular part of the measure f. Consequently the entropy H can be extended to the set

$$\overline{\mathcal{F}_m} = \left\{ f \in \mathcal{M}_b^+(\mathbb{R}^d) \quad ; \quad \int_{\mathbb{R}^d} f\, dv = m \right\}, \tag{36}$$

by setting

$$H(f) = \int (\frac{|v|^2}{2} f + \Phi(f_r))\, dv, \quad \text{where } f = f_r + f_s.$$

Note that the functional H is convex and lower semi-continuous in the weak star topology of measures. Hence, as seen from the previous section, the candidates for the minimizers are given by

$$f_\lambda(v) = [\Phi']^{-1}(-\frac{|v|^2}{2} - \lambda)$$

where $\lambda \in \mathbb{R}^+ = [0, \infty)$.

Next, we define the following mass function depending on the parameter λ

$$M(\lambda) = \int f_\lambda(v)\, dv\,.$$

This is a decreasing function of λ that satisfies

$$\lim_{\lambda \to 0^+} M(\lambda) = M(0) \in (0, \infty] \quad ; \quad \lim_{\lambda \to +\infty} M(\lambda) = 0\,. \tag{37}$$

The first result of this section is the following

Theorem 3.1. *For any positive number m, the minimization problem on $\overline{\mathcal{F}_m}$ has a unique minimizer f.*

- *In the sub-critical case, $m \leq M(0)$, there exists a unique $\lambda_m \geq 0$ such that $m = M(\lambda_m)$. The unique minimizer is given in this case by*

$$f = f_{\lambda_m}.$$

- *In the super-critical case, $m > M(0)$, (and in this case $M(0)$ is finite), the unique minimizer is given by*

$$f = f_0 + (m - M(0))\delta(v),$$

where δ is the Dirac measure at the velocity $v = 0$.

Moreover, any minimizing sequence in $\overline{\mathcal{F}_m}$ weak star converges in $\mathcal{M}_b$ towards the unique minimizer.

Proof. The convergence of the minimizing sequences is insured by the lower weak semicontinuity of the functional H and from the uniqueness of the minimizer. Since we have an explicit formula for the minimizers, we shall directly prove that $H(g) > H(f)$ for any element of $\overline{\mathcal{F}_m}$ different from f.

(i) The sub-critical case. We assume here that $m \leq M(0)$. Let $g = g_s + g_r$ be a measure in $\overline{\mathcal{F}_m}$. Then

$$H(g) - H(f) = \int \frac{|v|^2}{2} g_s + \int \frac{|v|^2}{2}(g_r - f) + \int (\Phi(g_r) - \Phi(f))\,.$$

Taylor formula gives

$$\Phi(g_r) - \Phi(f) = \Phi'(f)(g_r - f) + \int_0^1 (1-t)\Phi''(tg_r + (1-t)f)(g_r - f)^2 dt$$

On the other hand $\Phi'(f) = -(|v|^2/2 + \lambda)$. Therefore

$$H(g) - H(f) = \int \frac{|v|^2}{2} g_s + \lambda \int (f - g_r) + \int \int_0^1 (1-t)\Phi''(tg_r + (1-t)f)(g_r - f)^2 dt\, dv. \tag{38}$$

Since $\lambda \geq 0$, $\int g_r \leq m$ and Φ is strictly convex, the three integrals on the right-hand side of (38) are nonnegative. Therefore f is a minimizer.

Let us prove that it is the unique one. For $H(g) = H(f)$ it is necessary that the three integrals vanish. But since f is locally bounded and Φ is strictly convex, the last integral vanishes if and only if $g_r = f$ almost everywhere. The mass constraint then implies $g_s = 0$.

(ii) The super-critical case. Assume that $m > M(0)$. Since the δ function is not seen neither by the linear part of the entropy nor by the kinetic part (because $|v|^2$ vanishes on its support), then formula (38) with $\lambda = 0$ gives

$$H(g) - H(f_0 + (m - M(0))\delta(v)) = \int \frac{|v|^2}{2} g_s + \int \int_0^1 (1-t)\Phi''(tg_r + (1-t)f_0)(g_r - f_0)^2 dt\, dv.$$

Again, the right hand side is non-negative and vanishes if and only if

$$g_r = f_0\,, \quad |v|^2 g_s = 0.$$

Therefore $g_s = \alpha\delta(v)$ and the mass constraint gives the value $\alpha = m - M(0)$. □

3.2. **Minimization under mass and energy constraint.** The problem of minimization of a suitable entropy functional constraining mass and energy is surprisingly different in some ways and nevertheless coherent with the previous minimization under mass constrain only. The new minimization problem reads

$$S(f) = \inf_{g \in \overline{\mathcal{F}_{m,\mathcal{E}}}} S(g) \tag{39}$$

where

$$\overline{\mathcal{F}_{m,\mathcal{E}}} = \left\{ f \in \mathcal{M}_b^+(\mathbb{R}^d) \quad ; \quad \int_{\mathbb{R}^d} f\, dv = m; \quad \int_{\mathbb{R}^d} \frac{|v|^2}{2} f\, dv = \mathcal{E} \right\}, \tag{40}$$

and

$$S(g) = \int \Phi(g_r)\, dv. \tag{41}$$

The double parameter minimization problem is now endowed simply with the entropy functional $H(g) = S(g)$, in which the term corresponding to free energy does not appear in the functional to be minimized. (Indeed the energy is not free any longer since we are constraining it with a Lagrange multiplier θ.)

The corresponding Euler Lagrange equations formulation for this minimization problem yields

$$\Phi'(f) = -\left(\frac{|v|^2}{2\theta} + \lambda\right),$$

where the Lagrange multipliers θ and λ are nonnegative.

Following from (21), let us define a steady state by

$$\mathcal{M}_{\lambda,\theta} = [\Phi']^{-1}\left[-\left(\frac{|v|^2}{2\theta} + \lambda\right)\right]. \tag{42}$$

It is clear that the total mass of $\mathcal{M}_{\lambda,\theta}$ is maximal for $\lambda = 0$ (as well as its energy). Let $(M(\lambda,\theta), \mathcal{E}(\lambda,\theta))$, be the mass and energy corresponding to $\mathcal{M}_{\lambda,\theta}$. Then, rescaling the integration variable yields

$$M(\lambda,\theta) = \theta^{d/2} B_d \int_0^{+\infty} |u|^{d-1} [\Phi']^{-1}(-(\frac{|u|^2}{2} + \lambda))\, du, \tag{43}$$

$$\mathcal{E}(\lambda,\theta) = \theta^{d/2+1} B_d \int_0^{+\infty} \frac{|u|^{d+1}}{2} [\Phi']^{-1}(-(\frac{|u|^2}{2} + \lambda))\, du. \tag{44}$$

In order to understand the thresholds for condensation, as in the case of minimization under mass constraint, it is interesting to consider the limiting case $\lambda = 0$. Let us define the critical masses and energies by

$$\begin{cases} M_c = M(0,1) = B_d \int_0^{+\infty} |u|^{d-1} [\Phi']^{-1}(-\frac{|u|^2}{2})\, du \\ \mathcal{E}_c = \mathcal{E}(0,1) = B_d \int_0^{+\infty} |u|^{d+1} [\Phi']^{-1}(-\frac{|u|^2}{2})\, du. \end{cases} \tag{45}$$

Clearly, formulas (43) and (44) imply the relation

$$\left(\frac{M(0,\theta)}{M_c}\right)^{d+2} \left(\frac{\mathcal{E}_c}{\mathcal{E}(0,\theta)}\right)^{d} = 1, \tag{46}$$

as shown in (11) in the classical case for Bose Einstein distributions in 3-dimensions.

In particular, it is natural to define the sub-critical ensemble as follows

$$SC_d = \{(m,\mathcal{E}) \in (0,\infty)^2, \quad \mathcal{R}_{m,\mathcal{E}} = \left(\frac{m}{M_c}\right)^{d+2} \left(\frac{\mathcal{E}_c}{\mathcal{E}}\right)^{d} \le 1\}. \tag{47}$$

As a consequence the following Theorem holds

Theorem 3.2. *For any given pair of positive numbers $(m,\mathcal{E})$, the minimization problem (39), (40) and (41) has a unique minimizer $\mathcal{M}$.*

- *In the sub-critical case, i.e. $(m,\mathcal{E}) \in SC_d$, there exists a unique pair $(\lambda,\theta) \in [0,\infty)\times(0,\infty)$ such that $m = M(\lambda,\theta)$ and $\mathcal{E} = \mathcal{E}(\lambda,\theta)$. The unique minimizer is given in this case by*
$$\mathcal{M} = \mathcal{M}_{\lambda,\theta}. \tag{48}$$
- *In the super-critical case, i.e. $(m,\mathcal{E}) \notin SC_d$, there exists a unique positive θ such that $\mathcal{E} = \mathcal{E}(0,\theta)$. The unique minimizer of (40) is given by*
$$\mathcal{M} = \mathcal{M}_{0,\theta} + (m - M(0,\theta))\delta(v), \tag{49}$$
where δ is the Dirac measure at the velocity $v = 0$.

Moreover, any minimizing sequence in $\overline{\mathcal{F}_{m,\mathcal{E}}}$ weak star converges in $\mathcal{M}_b$ towards the unique minimizer.

Proof. **Uniqueness:** For the proof of uniqueness we assume the existence of a *ground state* $\mathcal{M}$ for the minimization problem.

Since we have an explicit formula for the minimizer given by either by (48) or (49), we proceed like in the previous subsection and show that the functional reaches its minimum at that candidate state $\mathcal{M}$. Let g be another minimizer element of the ensemble $\overline{\mathcal{F}_{m,\mathcal{E}}}$. We denote both $g = g_r + g_s$ and $\mathcal{M} = \mathcal{M}_r + \mathcal{M}_s$ meaning their regular and singular parts respectively.

A key point for uniqueness it to ensure that the singular part of g is concentrated at $v = 0$.

First, since both g and $\mathcal{M}$ are in $\overline{\mathcal{F}_{m,\mathcal{E}}}$, they have the same mass m and energy $\mathcal{E}$. In particular

$$0 = \int \frac{|v|^2}{2\theta}(g - M) = \int \frac{|v|^2}{2\theta} g_s \, dv + \int \frac{|v|^2}{2\theta}(g_r - \mathcal{M}_r)\, dv \tag{50}$$

since the singular part of the *ground state* $\mathcal{M}_s$ is a delta measure at $v = 0$ by (49) .

On the other hand, since the minimizer $\mathcal{M}$ is given by (48) or (49), and in both cases its regular part can be written $\mathcal{M}_r = \mathcal{M}_{\lambda,\theta}$, this implies by (42) that $\Phi'(\mathcal{M}_r) = -(|v|^2/(2\theta) + \lambda)$ (recall that for the case (49) one has $\lambda = 0$).

Thus adding and subtracting $\int \lambda(g_r - \mathcal{M}_r)$ (50) becomes

$$0 = \int \frac{|v|^2}{2\theta} g_s \, dv + \int \lambda(\mathcal{M}_r - g_r) - \Phi'(\mathcal{M}_r)(g_r - \mathcal{M}_r)\, dv \tag{51}$$

Finally, using the entropy function, now defined in (41) for the mass and energy constrained problem, and the null form in (51) we can write the difference between the entropies of g and $\mathcal{M}$ as

$$S(g) - S(\mathcal{M}) = \int \left(\Phi(g_r) - \Phi(\mathcal{M}_r)\right)\, dv =$$

$$\int \left(\frac{|v|^2}{2\theta} g_s + \lambda(M_r - g_r) \right) dv + \int \left(\Phi(g_r) - \Phi(\mathcal{M}_r) - \Phi'(\mathcal{M}_r)(g_r - \mathcal{M}_r)\right) dv. \tag{52}$$

Identity (52) allows to prove the uniqueness both for sub-critical and super-critical cases.

If $(m, \mathcal{E}) \in SC_d$ (sub-critical case), with $\lambda \geq 0$, $\theta > 0$, then by (48) we have $\mathcal{M}_r = \mathcal{M}_{\lambda,\theta} = \mathcal{M}$ and thus $\int \mathcal{M}_r \, dv = \int \mathcal{M}\, dv = m = \int g \, dv \geq \int g_r \, dv$. Since g_s is nonnegative, invoking the strict convexity of Φ as in the previous subsection we clearly obtain from (52) that $S(g) - S(\mathcal{M}) \geq 0$ and the equality holds is and only if $g_r = \mathcal{M}_r = \mathcal{M}$ and $g_s = 0 = \mathcal{M}_s$, and consequently $g = \mathcal{M}$. Once more, we used the fact that $g = g_s + g_s$ and $\mathcal{M}$ have the same total mass, so that $g_r = \mathcal{M}$ implies $g_s = 0$.

If $(m, \mathcal{E}) \notin SC_d$ (super-critical case), the total mass and energy values are critical at $\lambda = 0$ as noted it in (46) and (47), and as it was originally observed in (11). In particular, by (49), the regular part of the minimizer is given by $\mathcal{M}_r = \mathcal{M}_{0,\theta}$, and the singular part $\mathcal{M}_s = (m - M(0,\theta)\delta(v)$ is also non-negative since $m \geq M(0,\theta)$.

Then, once more invoking that g_s is nonnegative and the strict convexity of Φ in (52), we conclude that $S(g) - S(\mathcal{M}) \geq 0$, and equality holds is and only if $g_r = \mathcal{M}_r$ and $\frac{|v|^2}{2\theta} g_s = 0$. Therefore g_s is a Dirac measure centered at zero and its mass is deduced from the total mass carried out by g. This immediately leads to $g = \mathcal{M}$.

In particular we obtained the minimality of $S(\mathcal{M})$ and the uniqueness of the minimizer $\mathcal{M}$.

Existence of a minimizer $\mathcal{M}$: We begin with the super-critical case. Let us consider a mass m and an energy $\mathcal{E}$ such that

$$\mathcal{R}_{m,\mathcal{E}} = \left(\frac{m}{M_c}\right)^{d+2} \left(\frac{\mathcal{E}_c}{\mathcal{E}}\right)^{d} \geq 1\,. \tag{53}$$

In addition, from (44, 45), the constant θ can be defined by

$$\mathcal{E} = \theta^{d/2+1}\mathcal{E}_c\,. \tag{54}$$

Therefore, combining (54) with (43), (45) and the ratio of super-criticality (53) yields

$$m^{d+2} \geq \left(M_c \theta^{d/2}\right)^{d+2} = M(0,\theta)^{d+2}. \tag{55}$$

In particular this last condition insures that $m \geq M(0,\theta)$ and therefore (by the minimization problem of constraining just masses) the minimizer must have a singular part, that is there is a state $\mathcal{M}$ such that $\mathcal{M} = \mathcal{M}_{0,\theta} + (m - M(0,\theta))\delta(v)$ is an element of $\overline{\mathcal{F}_{m,\mathcal{E}}}$ and the above computation of the entropy shows that $\mathcal{M}$ is the unique minimizer of S on $\overline{\mathcal{F}_{m,\mathcal{E}}}$.

We are left now with showing the existence of minimizers in the sub-critical case, i.e. we need to show that there exists a unique pair (λ,θ) such that $\mathcal{M}_{\lambda,\theta} \in \overline{\mathcal{F}_{m,\mathcal{E}}}$.

Using (43) allows us to recast the problem in terms of λ by taking

$$\theta^{d/2} = \frac{m}{B_d \int_0^{+\infty} |u|^{d-1} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du}.$$

Thus, equation (44) on the energy $\mathcal{E}$ now becomes

$$\mathcal{E}^d = \theta^{d(d+2)/2} {B_d}^d \left[\int_0^{+\infty} \frac{|u|^{d+1}}{2} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du\right]^d =$$

$$\frac{m^{d+2}\left[\int_0^{+\infty} \frac{|u|^{d+1}}{2} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du\right]^d}{{B_d}^2 \left[\int_0^{+\infty} |u|^{d-1} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du\right]^{d+2}}.$$

This is an identity that depends on $\lambda \geq 0$ which, using definition (21), can be reformulated as a functional identity $G(\lambda) = 1$, where G is defined by

$$G(\lambda) := \frac{m^{d+2}\left[\int_0^{+\infty} |u|^{d+1} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du\right]^d}{(2\mathcal{E})^d {B_d}^2 \left[\int_0^{+\infty} |u|^{d-1} [\Phi']^{-1}(-\frac{|u|^2}{2} - \lambda)\, du\right]^{d+2}}. \tag{56}$$

Hence, the sub-critical condition for $(m,\mathcal{E}) \in SC_d$ corresponds to writing that $G(0) \leq 1$, and the proof of existence of minimizer is then reduced to show that the identity $G(\lambda) = 1$ is solvable.

In order to achieve this result, we will study in detail the behavior of the function $G(\lambda)$. First, we will prove that $G(\lambda)$ is increasing in λ.

Indeed, set $h(s) = (\Phi')^{-1}(s)$ and use the change of coordinates $\rho = \frac{|u|^2}{2}$. Expression (56) then becomes

$$G(\lambda) = K \frac{\left[\int_0^{+\infty} \rho^{d/2} h(-\rho - \lambda)\, d\rho\right]^d}{\left[\int_0^{+\infty} \rho^{d/2-1} h(-\rho - \lambda)\, d\rho\right]^{d+2}}.$$

with $K = m^{d+2}/[(2\mathcal{E})^{-d} {B_d}^{-2}]$.

Computing

$$\frac{d}{d\lambda} \ln G(\lambda) = -d \frac{\int_0^{+\infty} \rho^{d/2} h'(-\rho - \lambda)\, d\rho}{\int_0^{+\infty} \rho^{d/2} h(-\rho - \lambda)\, d\rho} + (d+2) \frac{\int_0^{+\infty} \rho^{d/2-1} h'(-\rho - \lambda)\, d\rho}{\int_0^{+\infty} \rho^{d/2-1} h(-\rho - \lambda)\, d\rho}\ , \tag{57}$$

and integrating by parts in the denominators of each term we obtain that right hand side of (57) becomes

$$\frac{d(d+2)}{2}\left[-\frac{\int_0^{+\infty}\rho^{d/2}h'(-\rho-\lambda)\,d\rho}{\int_0^{+\infty}\rho^{d/2+1}h'(-\rho-\lambda)\,d\rho}+\frac{\int_0^{+\infty}\rho^{d/2-1}h'(-\rho-\lambda)\,d\rho}{\int_0^{+\infty}\rho^{d/2}h'(-\rho-\lambda)\,d\rho}\right]. \tag{58}$$

Finally, since by (21) h is increasing in λ, then using Hölder's inequality we obtain

$$\left(\int_0^{+\infty}\rho^{d/2}h'(-\rho-\lambda)\,d\rho\right)^2<\int_0^{+\infty}\rho^{d/2-1}h'(-\rho-\lambda)\,d\rho\cdot\int_0^{+\infty}\rho^{d/2+1}h'(-\rho-\lambda)\,d\rho,$$

which proves that (58) is positive, and in particular $G'(\lambda)>0$.

Second, let us examine the behavior of $G(\lambda)$ for large values of λ. We use the notation $f_\lambda(u)=(\Phi')^{-1}(-|u|^2/2-\lambda)$. Then formula (56) reads

$$G(\lambda):=\frac{m^{d+2}\left[\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du\right]^d}{(2\mathcal{E})^d{B_d}^2\left[\int_0^{+\infty}|u|^{d-1}f_\lambda(u)\,du\right]^{d+2}}. \tag{59}$$

We note that by Hölder inequality

$$\int_0^{+\infty}|u|^{d-1}f_\lambda(u)\,du\le\left(\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du\right)^{(d-1)/(d+1)}\left(\int_0^{+\infty}f_\lambda(u)\,du\right)^{2/(d+1)},$$

which implies

$$G(\lambda)\ge\frac{m^{d+2}\left[\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du\right]^{2/(d+1)}}{(2\mathcal{E})^d{B_d}^2\left[\int_0^{+\infty}f_\lambda(u)\,du\right]^{2(d+2)/(d+1)}}. \tag{60}$$

By the monotonicity of the function $\Phi'(\cdot)$ (non-decreasing), $f_\lambda(u)\le f_\lambda(0)$. Hence, for every $R>0$

$$\int_0^{+\infty}f_\lambda(u)\,du\le\int_0^R f_\lambda(u)\,du+\frac{1}{R^{d+1}}\int_R^{+\infty}|u|^{d+1}f_\lambda(u)\,du\le$$

$$\int_0^R f_\lambda(0)\,du+\frac{1}{R^{d+1}}\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du=f_\lambda(0)R+\frac{1}{R^{d+1}}\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du.$$

Next, optimizing over R we get

$$\int_0^{+\infty}f_\lambda(u)\,du\le A_d[f_\lambda(0)]^{(d+1)/(d+2)}\left(\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du\right)^{1/(d+2)} \tag{61}$$

where the constant A_d is explicitly computable. Inequality (61) can be rewritten as

$$\int_0^{+\infty}|u|^{d+1}f_\lambda(u)\,du\ge B_d\left(\frac{1}{f_\lambda(0)}\right)^{d+1}\left(\int_0^{+\infty}f_\lambda(u)\,du\right)^{d+2} \tag{62}$$

Using inequality (62) into (60) we obtain

$$G(\lambda)\ge\frac{m^{d+2}}{(2\mathcal{E})^d{B_d}^2}\,(B_d)^{2/(d+1)}\left(\frac{1}{f_\lambda(0)}\right)^2. \tag{63}$$

Since

$$\lim_{\lambda\to\infty}f_\lambda(0)=0,$$

inequality (63) shows that $\lim_{\lambda\to+\infty} G(\lambda) = +\infty$, which proves the existence of a unique solution λ to the equation $G(\lambda) = 1$. □

Remark. If $M_c \to \infty$ while $\mathcal{E}_c$ remains finite, then by the sub-criticality condition (46) all the pairs $(m, \mathcal{E}) \in (0, \infty)^2$ lie in the sub-critical ensemble SC_d.

Remark. Condition (47), which defines the sub-critical ensemble, can be rewritten as

$$\frac{m^{d+2}}{\mathcal{E}^d} \le \frac{m_c^{d+2}}{\mathcal{E}_c^d} = C_d. \tag{64}$$

Hence, a pair of values $(m, \mathcal{E})$ violates the sub-critical condition (64) not only when the mass m is above the critical mass, but, most interestingly, when the mass m is below the critical mass, while the energy $\mathcal{E}$ is below a suitable value. In the physically relevant case of the Bose-Einstein distribution, there is no regular Bose-Einstein state also for particles density below the critical value, provided the energy is suitably small. We remark that this relationship between mass and energy has not been observed in [4].

Acknowledgement: The authors thank the second reviewer for the very useful comments and corrections as well as generous suggestions for improvement of this work presentation. We have followed this valuable advices. In particular we are specially thankful for the original suggestion for the two sided control of the entropy variation with a growth rate depending of the super-linear drift function. Such estimate will be left for future work.

REFERENCES

[1] J.A. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani and A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*. Monatsh. Math. **133** (2001) 1–82.

[2] S. Chapman and T. Cowling, The mathematical theory of non- uniform gases. Third edition. Cambridge University Press, 1970.

[3] F. Demengel and R. Temam, *Convex functions of a measure and applications*. Jndiana Math. J. **33** (1984), 673-709

[4] M. Escobedo,S. Mischler and M.A. Valle, *Entropy maximization problem for quantum relativistic particles*. Bull. Soc. math. France, **133** (1) (2005), 87-120.

[5] R.T. Glassey and W.A. Strauss, *Asymptotic stability of the relativistic Maxwellian*, Publ. Res. Inst. Math. Sciences, Kyoto University, **29** (1993).

[6] G. Kaniadakis and P. Quarati, *Kinetic equation for classical particles obeying an exclusion principle*. Phys. Rev. E **48** (1993), 4263-4270.

[7] G. Kaniadakis and P. Quarati, *Classical model of bosons and fermions*. Phys. Rev. E **49** (1994), 5103-5110.

[8] X. Lu, *A modified Boltzmann equation for Bose-Einstein particles : isotropic solutions and long-time behavior*. J. Statist. Phys. **98** (2000), 1335-1394.

E-mail address: gamba@math.utexas.edu
E-mail address: giuseppe.toscani@unipv.it