Kinetic and Hydrodynamic Models of Flocking Phenomena

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Kinetics and statistical methods for complex particle systems

Outline

Birds flocking

Flock of fish

Continuum decription of flocking

- Discuss the modeling of hydrodynamic equations for *flocking phenomena*.
- Look for a macroscopic description by the equations for fluid dynamics, modified to account for birds type interactions.
- Problem close to the hydrodynamic description of a granular gas (dissipation substituted by adapting velocities).
- Only discrete and kinetic models present in the literature.

The Cucker–Smale model

- The recent mathematical work of Cucker and Smale [F. Cucker, S. Smale, (2007)], connected with the emergent behaviors on flocks, obtained a noticeable resonance in the mathematical community.
- The goal was to prove, in agreement with observations, that under some initial conditions, for example on their positions and velocities, the state of the flock converges to one in which all birds fly with the same velocity.
- Main hypothesis which justifies the behavior of the population is that every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds.

The Cucker–Smale model II

- Given a population of k birds, at time $t \in \mathbb{N}$, and for i-th bird,

$$v_i(t+1) - v_i(t) = \sum_{i=1}^{k} a_{ij} \left(v_j(t) - v_i(t) \right).$$

- The weights a_{ij} quantify the way the birds influence each other.

$$a_{ij} = \frac{K}{\left(\sigma^2 + \|x_i - x_j\|^2\right)^{\beta}}$$

- K, $\sigma > 0$ and $\beta \geq 0$ are fixed constants.

The Cucker–Smale result

- For $x, v \in \boldsymbol{E}^k$, denote

$$\Gamma(x) = \frac{1}{2} \sum_{i \neq j} \|x_i - x_j\|^2,$$

- and

$$\Lambda(v) = \frac{1}{2} \sum_{i \neq j} \|v_i - v_j\|^2.$$

- Then, when $\beta < 1/2$, it is proven that there exists a constant B_0 such that $\Gamma(x(t)) \leq B_0$ for all $t \in \mathbb{R}_+$, while $\Lambda(v(t))$ converges towards zero as $t \to \infty$, and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x}_{ij}$, for all $i, j \leq k$.

Nonlinear friction models

- A kinetic description of Cucker and Smale flocking model, obtained recently by [S-Y Ha, E. Tadmor, 2008].
- The kinetic model

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(x, v, t) = \nabla_v \cdot [f(x, v, t)\,(\nabla_v W * H * f)(x, v, t)]$$

- The flux reads

$$(\nabla_v W * H * f)(x, v) = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dw \frac{v - w}{(1 + |x - y|^2)^{\beta}} f(y, w).$$

Large-time behavior

- The large-time behavior obtained by studying the (energy) functional

$$F(f)(t) = \frac{1}{2} \int dx\, dy\, dv\, dw \frac{(v-w)^2}{(1+|x-y|^2)^\beta} f(x,v,t) f(y,w,t).$$

- Provided $\beta < 1/2$, the energy functional decays to zero, while the support in space and velocity remains bounded in time.
- The resulting density is concentrating in the velocity variable.

Boltzmann models

- A Boltzmann type description of Cucker and Smale flocking model, obtained recently by [J.A. Carrillo, M. Fornasier, J. Rosado, G. T., 2008].
- The *collisional rule*

$$v^* = (1 - \gamma a(|x-y|))v + \gamma a(|x-y|)w,$$
$$w^* = \gamma a(|x-y|)v + (1 - \gamma a(|x-y|))w.$$

- As in Cucker-Smale model

$$a(|x-y|) = \frac{K}{(1+|x-y|^2)^{\beta}}$$

Boltzmann models II

- The Boltzmann equation

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(x, v, t) = Q(f, f)(x, v, t),$$

- The collision integral

$$Q(f, f)(x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} dy\, dw \left(\frac{1}{J} f(x, v_*) f(y, w_*) - f(x, v) f(y, w)\right).$$

- (w_*, w_*) are the pre-collisional velocities that generate the couple (v, w) after the interaction. $J = (1 - 2\gamma a)^d$ is the Jacobian of the transformation of (v, w) into (v^*, w^*).

Boltzmann models III

- Povzner [A.Y. Povzner, 1962] proposed a modified Boltzmann collision operator considering a smearing process for the pair collisions.
- This modified Povzner collision operator looks as follows

$$Q_P(f,f)(x,v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} dydw \, B \, \left(f(x,v_*)f(y,w_*) - f(x,v)f(y,w) \right).$$

- The kernel $B = B(x-y, v-w)$, while

$$v^* = v - (v-w) \cdot n(x-y)n(x-y)$$
$$w^* = w + (v-w) \cdot n(x-y)n(x-y)$$

Boltzmann models IV

- The model of Carrillo, Fornasier, Rosado and G.T. is of Povzner type, with the addition of dissipation in a collision.
- The degree of dissipation depends of the distance between birds.
- For a small degree of dissipation (small γ) the collision integral is close to the nonlinear friction operator considered by Ha and Tadmor.
- Difficult to obtain a hydrodynamic description.

The analogies with dissipative gases

- Deeper understanding of macroscopic equations from kinetic theory. Precise description of the evolution of materials composed of many small discrete grains.
- Valid when the mean free path of the grains is much larger than the typical particle size.
- Starting point Boltzmann-like equations for partially inelastic rigid spheres. Corrections to take into account statistical correlation among particles.
- Main problem to pass to macroscopic equations: No classical Maxwellian equilibrium.
- Alternative: look for a substitute: the homogeneous cooling state, fundamental self-similar solution of the Boltzmann equation.

Closure problem

- Two possible way of closure.

Weak inelasticity

- Deviations of the coefficient of restitution from unity of the same order of magnitude as the Knudsen number.
- The equilibrium is still a local Maxwellian.
- Validity of Boltzmann H-theorem to justify the passage to hydrodynamics.
- Valid for any variation in space

Cooling state closure

- The zero order approximation of the solution is constituted by the so–called homogeneous cooling state
- Requires a detailed theory of the homogeneous cooling state. Difficult for a general coefficient of restitution.
- Correct only for small spatial variations.

Essential references

- Macroscopic equations: P.K. Haff, "Grain flow as a fluid-mechanical phenomenon," (1983); A. Goldshtein, and M. Shapiro, "Mechanics of collisional motion of granular materials. Part 1. General hydrodynamic equations," (1995).
- Homogeneous cooling state: N.V. Brilliantov, and T. Pöschel, "Self–diffusion in granular gases," (2000); "Hydrodynamics of Granular Gases of viscoelastic particles," (2002).
- Weak inelasticity: G. Toscani, "Kinetic and hydrodynamic models of nearly elastic granular flows," (2004); M. Bisi, G. Spiga, G. Toscani "Grad's equations and hydrodynamics for weakly inelastic granular flows" (2005).

Binary interactions

- The birds entering into collision have positions and velocities (x, v) and (y, w). $n = (x - y)/|x - y|$ is the unit vector along the $x - y$ direction.
- The post collision velocities (v^*, w^*) satisfy

$$(v^* - w^*) \cdot n = -(1 - 2\gamma a(|x - y|))\,(v - w) \cdot n\,.$$

- Assuming conservation of momentum, the change of velocity for the colliding particles is

$$v^* = v - (1 - \gamma a)[(v - w) \cdot n]\, n\,, \quad w^* = w + (1 - \gamma a)[(v - w) \cdot n]\, n.$$

- Elastic collisions (Povzner equation) if $\gamma = 0$. In general, γ increases with *increasing* interaction among birds.

The birds collision

- In the picture, birds are diminishing the relative velocity along the $x - y$ direction.
- The dissipativity depends on the relative distance. *Collisions with small relative distance are more dissipative*.
- The choice of $a(|x - y|)$ in accord with Cucker–Smale model

$$a(|x - y|) = \frac{K}{(1 + |x - y|^2)^{\beta}}$$

- This rule can be easily generalized changing the unit vector n.

Details

- Introduce a kinetic model of Boltzmann type

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = G(\rho)\bar{Q}(f,f)(x,v,t),$$

- Collision operator

$$\bar{Q}(f,f) = \sigma^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B\left\{\chi f(x,v_*)f(y,w_*) - f(x,v)f(y,w)\right\} \, dy \, dw.$$

- $B = B(|x-y|)$, (v_*, w_*) pre collision velocities which results with (v,w) as post collision velocities. χ linked to the Jacobian of the transformation $dv_* dw_*$ into $dvdw$.

The Boltzmann equation I

- In weak form

$$< \varphi\,,\ \bar{Q}(f,f)(x,v) > = \sigma^2 \int_{\mathbb{R}^3} \varphi(v)\bar{Q}(f,f)(x,v)\,dv =$$

$$\sigma^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B\left(\varphi(v^*) - \varphi(v)\right) f(x,v) f(y,w) dv\, dw\, dy$$

- The post-collision velocity is

$$v^* = v - (1 - \gamma a(|x-y|))((v-w)\cdot n)n\,, \qquad n = \frac{x-y}{|x-y|}.$$

The Boltzmann equation II

- Elastic Povzner type collision with (v, w) as incoming velocities

$$v' = v - (q \cdot n)n\,, \quad w' = w + (q \cdot n)n.$$

- Connection with inelastic collision

$$v^* = v' + \gamma a(q \cdot n)n\,, \quad w^* = w' - \gamma a(q \cdot n)n.$$

- Obtain

$$v^* - v' = \gamma a\left(|x-y|\right)(q \cdot n)n.$$

The Boltzmann equation III

- Taylor expansion of $\varphi(v^*)$ around $\varphi(v')$

$$\varphi(v^*) = \varphi(v') + \gamma \nabla\varphi(v') \cdot a\left(|x-y|\right)(q\cdot n)n + \frac{1}{2}\gamma^2 \sum_{i,j} \frac{\partial^2\varphi(v')}{\partial v_i' \partial v_j'} a^2\left(|x-y|\right)(q\cdot n)^2 n_i n_j + \dots$$

- Nearly elastic interactions, $\gamma << 1$

$$< \varphi\,,\, \bar{Q}(f,f) > \;= \sigma^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|x-y|)\Big(\varphi(v') - \varphi(v) + \gamma\nabla\varphi(v')\cdot a\left(|x-y|\right)(q\cdot n)n\Big) f(x,v) f(y,w)\, dv\, dw\, dy = \;< \varphi\,,\, Q_{\mathrm{P}}(f,f) > \; + \gamma < \varphi\,,\, I(f,f) > .$$

The Boltzmann equation IV

- The Boltzmann operator sum of Povzner elastic

$$Q_P(f,f) = \sigma^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B\left\{f(x,v')f(y,w') - f(x,v)f(y,w)\right\} dw\, dy.$$

- and nonlinear friction operator

$$I(f,f)(x,v) = \sigma^2 \mathrm{div}_v \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n(q\cdot n) b\left(|x-y|\right) f(x,v')f(y,w') dw\, dy.$$

- $b(|x-y|) = B(|x-y|)a(|x-y|)$. Enskog–Boltzmann equation for flocking at the leading order

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = G(\rho) Q_P(f,f)(x,v,t) + G(\rho)\beta I(f,f)(x,v,t),$$

Exact computations

- Choosing $\varphi = 1$ the mass is a collision invariant.

$$< 1\,,\, I(f,f) > (x,t) \;= 0$$

- The moment correction

$$< v\,,\, I(f,f)(x,v,t) > = A(\rho,u)(x,t) =$$
$$\sigma^2 \int_{\mathbb{R}^3} dy\, a(|x-y|) n\, n \cdot (u(x,t) - u(y,t))\, \rho(x,t)\rho(y,t).$$

- The momentum is conserved only globally

$$\int_{\mathbb{R}^3} dx\, < v\,,\, I(f,f)(x,v,t) > = 0$$

Exact computations II

- The energy correction (for f isotropic in the velocity variable)

$$< \frac{1}{2}v^2 \,,\, I(f,f)(x,v,t) > = -B(\rho, u, T)(x,t) =$$

$$\sigma^2 \int_{\mathbb{R}^3} b(|x-y|)\rho(x,t)\rho(y,t)\,\Big[n\cdot u(x,t)\, n\cdot u(y,t) -$$

$$\left(\frac{1}{3}|u(y,t)|^2 + T(y,t)\right)\Big]\, dy.$$

- The contribution of the energy is negative

Euler equations

- Small mean free path for the Povzner equation (Povzner $\cong$ Boltzmann)

$$G(\rho) = \frac{1}{\epsilon} g(\rho)$$

- ψ collision invariant, $\psi = 1, v, \frac{1}{2}v^2$

$$\int_{\mathbb{R}^3} \psi(v) \left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f - g(\rho) \frac{\gamma}{\epsilon} I(f,f)(x,v,t) \right) dv = \frac{1}{\epsilon} g(\rho) \int_{\mathbb{R}^3} \psi(v) Q_{\mathrm{P}}(f,f)(x,v,t)\, dv = 0, \quad (1)$$

- Closure by assuming f to be the locally Maxwellian function. $\longrightarrow$ $I(f,f) = I(M,M)$

Euler equations for flocking

- let $\epsilon \to 0$, $\gamma/\epsilon = \lambda$. [Lachowicz, Pulvirenti ARMA (1990)]
- Euler equations (in divergence form) for density $\rho(x,t)$, bulk velocity $u(x,t)$ and temperature $T(x,t)$ for flocking

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) &= 0 \\
\frac{\partial}{\partial t}(\rho u_i) + \mathrm{div}(\rho u u_i + \rho T e_i) &= \lambda\, g(\rho) A_i(\rho, u) \\
\frac{\partial}{\partial t}\left(\rho(\frac{2}{3}T + \frac{1}{2}u^2)\right) + \mathrm{div}\left(\rho u(\frac{1}{2}u^2 + \frac{5}{2}T)\right) &= -\lambda\, g(\rho) B(\rho, u, T)
\end{aligned}
$$

- e_i is the component of the unit vector e in the i-th direction.

Steady states

- Possible steady state if $A(\rho, u) = 0$ and $B(\rho, u, T) = 0$.
- $A(\rho, u) = 0$ if $\rho(x, t) = 0$ or $u(x, t) = \bar{u}$.
- Given a (bounded) domain $\mathcal{D} \subseteq \mathbb{R}^3$, we choose $\rho(x, t) = 0$ outside $\mathcal{D}$, and $\rho(x, t) = \bar{\rho}, u(x, t) = \bar{u}$ inside $\mathcal{D}$.
- In addition $T(x, t) = 0$ inside $\mathcal{D}$. This implies $B(\rho, u, T) = 0$.
- All birds contained in a bounded domain, flying with constant speed and constant density inside is a steady state of Euler equations for flocking.

Conclusions

- We discussed the modeling of flocking phenomena.
- It has been shown that various kinetic models can be constructed on the basis of the discrete model proposed by Cucker and Smale, in which birds adapt their velocity to that of the other birds.
- The kinetic models by Ha and Tadmor and Carrillo, Fornasier, Rosado and G.T. are not suitable to construct a reasonable hydrodynamics.
- Analogies with granular gases, and the use of Povzner modification of Boltzmann equation allow a powerful approach, based on the closure with respect to a locally Maxwellian function.
- Numerical experiments are in progress.

References for Flocking

- Discrete models: F. Cucker and S. Smale, "On the mathematics of emergence" (2007); "Emergent behavior in flocks" (2007).
- Kinetic models: S-Y Ha and E. Tadmor , "From particle to kinetic and hydrodynamic description of flocking" *Kinetic Related Models* (2008); J.A. Carrillo, M.Fornasier, J. Rosado and G. Toscani, "Asymptotic Flocking Dynamics for the kinetic Cucker-Smale model" (2009).
- Hydrodynamics: M. Fornasier and G. Toscani, "Povzner equation and hydrodynamic models of flocking" (2009)