Formation of tails in nonconservative kinetic models

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Outline

Pareto Tails in economy

- In various systems (described in terms of a density function) the density, for large times, has *overpopulated tails*.
- Vilfredo Pareto studied the distribution of income among people of different western countries and found an inverse power law for the distribution of wealth [V.Pareto, *Cours d'Economie Politique*, (1897)]. If $f(w)$ is the probability density function of agents with wealth w

$$F(w) = \int_w^\infty f(w_*)\,dw_* \sim w^{-\mu}.$$

The cooling of a granular gas

- A well–known phenomenon in the large–time behavior of the Boltzmann equation with dissipative interactions is the formation of *overpopulated tails* [Bobylev, A. V.; Gamba, I. M.; Panferov, V. A. (2004)].
- Exact results on the behavior of these tails have been obtained for simplified models, in particular for a gas inelastic Maxwell particles.

  In one dimension

$$f(w) = \frac{2}{\pi(1+v^2)^2}$$

  [Baldassarri A., Marini Bettolo Marconi U., Puglisi A. (2002)]
- Analogous results *(with no explicit self-similar solution)* hold in higher dimensions [Bobylev A.V., Cercignani C. (2003)].

Common features

- The previous first two examples have much in common.

Pareto tails

- In a strong economy the mean wealth $m(t)$ is increasing (only mass conservation).
- A time-independent density profile is obtained with the normalized density

$$f_n(w) = m(t) f(m(t) w, t)$$

of unit mean wealth.

Granular gases

- In a gas with dissipative interactions the temperature $E(t)$ is decreasing (mass and momentum are conserved).
- In this case one refers to a normalized density

$$f_n(w) = \sqrt{E(t)} f(\sqrt{E(t)} w, t)$$

of unit energy.

A reasonable conjecture

- It can be *reasonably* conjectured that the formation of overpopulated tails depends mainly on two facts:

- The kinetic system is based on binary collisions that generate convolution-like collisional operators.

- The collision is such that some conservation law in the kinetic system is missed at a microscopic level.

- Rescaling the solution to restore the conservation law is equivalent to multiplication by a independent variable.
- In probability theory it is well-known that the repeated multiplication of independent variables produces in the limit a lognormal distribution.

References

- Various applications of ideas and mathematical methods.

- L. Pareschi, G. Toscani, Self-similarity and power-like tails in nonconservative kinetic models, *J. Statist. Phys.* **124** 747-779 (2006)

- D. Matthes, G. Toscani, On steady distributions of kinetic models of conservative economies, *J. Statist. Phys.*, **130** 1087-1117 (2008)

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Details

- Introduce a one–dimensional kinetic model of Maxwell-Boltzmann type, with binary interactions

$$v^* = pv + qw, \quad w^* = qv + pw; \quad p > q > 0.$$

- The positive constants p and q represent the mixing parameters, i.e. the portion of the pre–collisional velocities (v, w) which generate the post–collisional ones (v^*, w^*).
- This *collision* can be used both for for molecular interactions (the velocities $(v, w) \in \boldsymbol{R}$) and economic interactions (the wealths $(v, w) \in \boldsymbol{R}_+$).

The Boltzmann equation I

- For molecular interactions, $f(v,t)$ denotes the distribution of particles with velocity $v \in \mathbb{R}$ at time $t \geq 0$. The kinetic model can be easily derived in a standard way.
- One obtains the Boltzmann type equation [Ben-Avraham D., Ben-Naim E., Lindenberg K., Rosas A. (2003)],

$$\frac{\partial f}{\partial t} = \int_{\mathbb{R}} \left(\frac{1}{J} f(v_*) f(w_*) - f(v) f(w) \right) dw$$

- (w_*, w_*) are the pre-collisional velocities that generate the couple (v, w) after the interaction. $J = p^2 - q^2$ is the Jacobian of the transformation of (v, w) into (w^*, w^*). Since $p > q$, the Jacobian J is positive.
- $J = 1$ only if $p = 1$ and $q = 0$ for which the collision operator vanishes.

The Boltzmann equation II

- Use the weak form

$$\frac{d}{dt}\int_{\mathbf{R}} \phi(v) f(v,t)\, dv = \int_{\mathbf{R}^2} f(v) f(w) (\phi(v^*) - \phi(v)) dv dw.$$

- Choosing $\phi(v) = v$ shows that

$$m(t) = \int_{\mathbf{R}} v f(v,t)\, dv = m(0) e^{(p+q-1)t}.$$

- If $m(0) = 0$, $m(t) = 0$. In this case, taking $\phi(v) = v^2$

$$E(t) = \int_{\mathbf{R}} v^2 f(v,t)\, dv = e^{(p^2+q^2-1)t}.$$

- The second moment is not conserved, unless $p^2 + q^2 = 1$.

The Boltzmann equation III

- If $p^2 + q^2 \neq 1$, the energy grows (or decreases). The large-time behavior is well-described by scaling the solution

$$g(v,t) = \sqrt{E(t)} f\left(v\sqrt{E(t)}, t\right).$$

- This scaling implies that $\int v^2 g(v,t) = 1$ for all $t \geq 0$.
- $g = g(v,t)$ satisfies the equation

$$\frac{d}{dt}\int_{\boldsymbol{R}} \phi(v) g(v,t)\, dv =$$

$$\int_{\boldsymbol{R}^2} g(v)g(w)(\phi(v^*) - \phi(v))dvdw +$$

$$\frac{1}{2}\left(p^2 + q^2 - 1\right)\int_{\boldsymbol{R}} \phi(v)\frac{\partial}{\partial v}\left(vg\right)\, dv$$

The Boltzmann equation IV

- Since $v^* - v = (p-1)v + qw$, using a second order Taylor expansion of $\phi(v^*)$ around v

$$\int_{\mathbf{R}^2} g(v)g(w)(\phi(v^*) - \phi(v))dvdw =$$

$$\int_{\mathbf{R}^2} g(v)g(w)\left((p-1)v + qw\right)\phi'(v)dvdw +$$

$$\frac{1}{2}\int_{\mathbf{R}^2} g(v)g(w)\left((p-1)v + qw\right)^2 \phi''(v)dvdw + R(p,q).$$

- The remainder is

$$R(p,q) = \frac{1}{2}\int_{\mathbf{R}^2} \left((p-1)v + qw\right)^2 \left(\phi''(\tilde{v}) - \phi''(v)\right) g(v)g(w)dv\,dw$$

The Boltzmann equation V

- The scaled density $g(v,t)$ satisfies

$$\frac{d}{dt}\int_{\mathbf{R}} \phi(v) g(v,t)\,dv + \frac{1}{2}\left((p-1)^2 + q^2\right)\int_{\mathbf{R}} \phi'(v) v g(v)\,dv =$$
$$\frac{1}{2}\int_{\mathbf{R}} g(v)\left((p-1)^2 v^2 + q^2\right)\phi''(v)dv + R(p,q).$$

- Set

$$\tau = q^2 t, \quad h(v,\tau) = g(v,t).$$

- Then, $g_0(v) = h_0(v)$, and $h(v,\tau)$ satisfies

$$\frac{d}{d\tau}\int_{\mathbf{R}} \phi(v) h(v,\tau)\,dv + \frac{1}{2}\left(\left(\frac{p-1}{q}\right)^2 + 1\right)\int_{\mathbf{R}} \phi'(v) v h(v)\,dv =$$
$$\frac{1}{2}\int_{\mathbf{R}} h(v)\left(\left(\frac{p-1}{q}\right)^2 v^2 + 1\right)\phi''(v)dv + \frac{1}{q^2} R(p,q).$$

- Suppose $R(p,q)/q^2$ small for small values of q. Take $p = p(q)$ such that

$$\lim_{q\to 0} \frac{p(q)-1}{q} = \lambda,$$

- The Boltzmann equation is well–approximated by the equation (in weak form)

$$\frac{d}{d\tau}\int_{\mathbf{R}} \phi(v)h(v,\tau)\,dv + \frac{1}{2}\left(\lambda^2+1\right)\int_{\mathbf{R}} \phi'(v)vh(v)\,dv = \frac{1}{2}\int_{\mathbf{R}} h(v)\left(\lambda^2 v^2+1\right)\phi''(v)dv.$$

- This is the weak form of the Fokker-Planck equation

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2}{\partial v^2}\left((1+\lambda^2 v^2)h\right) + \left(1+\lambda^2\right)\frac{\partial}{\partial v}(vh)\right),$$

Steady states

- Let us reassume the procedure shortly:

Boltzmann versus Fokker-Planck

- $f(v,t)$ solves the Boltzmann equation.
- Mass and momentum are conserved. The second moment $E(t)$ is varying with time.
- $g(v,t) = \sqrt{E(t)} f\left(v\sqrt{E(t)}, t\right)$ is a solution with time-independent energy.
- $h(v,\tau) = g(v,t)$, with $\tau = q^2 t$ solves the Fokker-Planck equation for $q << 1$ and $(p-1)/q \to \lambda$.

The steady state of Fokker-Planck

- The Fokker-Planck equation has a unique stationary state with mass one and momentum zero.
- The steady state is

$$M_\lambda(v) = c_\lambda \left(\frac{1}{1+\lambda^2 v^2} \right)^{\frac{3}{2}+\frac{1}{2\lambda^2}},$$

  c_λ constant.
- The steady state has overpopulated tails. The size depends of λ.

Remarkable cases I

- The conservative case $p^2 + q^2 = 1$. $p = \sqrt{1 - q^2}$ implies $\lambda = 0$ as unique possible value.
- In the limit one then obtains the linear Fokker–Planck equation

$$\frac{\partial h}{\partial \tau} = \frac{1}{2} \left(\frac{\partial^2 h}{\partial v^2} + \frac{\partial}{\partial v} (vh) \right) .$$

- The stationary solution $M(v)$ is the Maxwell density

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2},$$

- Same limit if $p^2 + q^2 > 1$, $p < 1$, since $\lambda \to 0$ as $p \to 1$.

Remarkable cases II

- If $p = 1 - q$, the kinetic model reduces to the model for granular dissipative collisions introduced and studied in [Baldassarri A., Marini Bettolo Marconi U., Puglisi A. (2002)] as a one–dimensional caricature of the Maxwell–Boltzmann equation.
- In the limit $\lambda = 1$ and the stationary state of the Fokker-Planck equation is

$$M_1(v) = \frac{2}{\pi} \left(\frac{1}{1+v^2} \right)^2 .$$

- This solution also solves the Boltzmann equation for any value of the parameter $q < 1/2$.

Economy

- The wealth situation $w \in \boldsymbol{R}_+$ can be treated likewise.
- Main differences

$$g(v,t) = m(t) f\left(m(t) v, t\right),$$

- The correct scaling for small values of the parameter q is now

$$\tau = qt, \quad h(v,\tau) = g(v,t),$$

- In the limit one then obtains the Fokker–Planck equation

$$\frac{\partial h}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial v^2}\left(v^2 h\right) + \frac{\partial}{\partial v}\left((v-m)h\right).$$

Steady states of economy

- The Fokker-Planck equation has a unique stationary state of unit mass (Γ-distribution)

$$M_\lambda(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\mathrm{e}\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}}$$

$$\mu = 1 + \frac{2}{\lambda} > 1$$

- Same steady state in [Bouchaud J.P., Mézard M. (2000)]
- Pareto power law tails for large v's.

Fourier transform version

- In some regime there is formation of overpopulated tails.
- What about kinetic models?
- Main tool: Use Fourier transform

$$\frac{\partial \widehat{f}(\xi,t)}{\partial t} = \widehat{f}(p\xi)\widehat{f}(q\xi) - \widehat{f}(\xi)\widehat{f}(0).$$

$$\widehat{f}(0) = 1, \widehat{f}'(0) = 0, \widehat{f}''(0) = -1,$$

- Consider the d_s-metric

$$d_s(f,g) = \sup_{\xi \in \boldsymbol{R}} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s}$$

$s = m + \alpha$, m integer and $0 \leq \alpha < 1$.

First results I

- Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation, with initial values of momentum zero and unit energy.

Theorem

If for some $2 \le s \le 3$, $d_s(f_{1,0}, f_{2,0})$ is bounded, for all times $t \ge 0$,

$$d_s(f_1(t), f_2(t)) \le e^{(p^s+q^s-1)t} d_s(f_{1,0}, f_{2,0}).$$

In case $p^s + q^s - 1 < 0$ the distance d_s is contracting exponentially in time.

- Uniqueness of solutions follows.

First results II

- The Fourier transform of $g(v,t)$

$$\widehat{g}(\xi) = \widehat{f}\left(\frac{\xi}{\sqrt{E(t)}}\right),$$

- plus the property

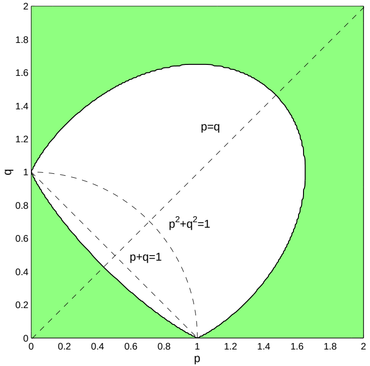
$$\sup_{\xi \in \mathbf{R}} \frac{|\widehat{f_1}(a\xi) - \widehat{f_2}(a\xi)|}{|\xi|^s} = a^s d_s(f_1, f_2),$$

- imply

$$d_s(g_1(t), g_2(t)) \le \mathrm{e}^{\left[(p^s+q^s-1)-\frac{s}{2}(p^2+q^2-1)\right]t} d_s(f_{1,0}, f_{2,0}).$$

The domain of convergence

- $\min_{\delta} \left\{ p^{2+\delta} + q^{2+\delta} - 1 - \frac{2+\delta}{2} \left(p^2 + q^2 - 1 \right) \right\} < 0$



More on $\mathcal{S}_{p,q}$

- The behavior of $\mathcal{S}_{p,q}(\delta)$ when $p^2 + q^2 = 1$ is clear. $p < 1$ and $q < 1$

$$\mathcal{S}_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 < 0,$$

- Same conclusion when $p^2 + q^2 > 1$, while both $p < 1$ and $q < 1$.

Lemma

Given $\lambda > 0$, if $p^2 + q^2 < 1$, define $p = 1 - \lambda q$. Then, provided $q < \min\left\{1/(1+\lambda), (2\lambda)/(1+\lambda^2)\right\}$ there exists a interval $I_- = (0, \bar{\delta}_-(q))$ such that $\mathcal{S}_{p,q}(\delta) < 0$ for $\delta \in I_-$. If $p^2 + q^2 > 1$, and $p = 1 + \lambda q$ there exists a interval $I_+ = (0, \bar{\delta}_+(q))$ such that $\mathcal{S}_{p,q}(\delta) < 0$ for $\delta \in I_+$.

Consequences

- Large-time behavior of the solution

Theorem

Let $g_1(t)$ and $g_2(t)$ be two solutions of the Boltzmann equation, corresponding to initial values $f_{1,0}$ and $f_{2,0}$ of zero momentum and unit energy. Then, there exists a constant $\bar{\delta} > 0$ such that, if $2 < s < 2 + \bar{\delta}$, for all times $t \geq 0$,

$$d_s(g_1(t), g_2(t)) \leq \mathrm{e}\left\{-C_s t\right\} d_s(f_{1,0}, f_{2,0}).$$

$C_s = -\mathcal{S}_{p,q}(s-2)$ is strictly positive, and the distance d_s is contracting exponentially in time.

Evolution of moments

- Suppose

$$\int_{\mathbb{R}} |v|^{2+\delta} g_0(v)\, dv = m_\delta < \infty.$$

- One shows that, for any $\delta < \bar{\delta}$

$$\int_{\mathbb{R}} |v|^{2+\delta} g(v,t)\, dv \le m_\delta + \frac{B_{p,\delta}}{|\mathcal{S}_{p,q}(\delta)|} < \infty.$$

- Can we draw conclusions on the large–time convergence of the solution?

Convergence to self-similarity I

- Existence of a uniform bound on moments implies tightness of probability densities $\{g(v,t)\}_{t\geq 0}$ (*Prokhorov theorem*).
- Any sequence $\{g(v,t_n)\}_{n\geq 0}$ contains an infinite subsequence which converges weakly to some probability measure g_∞.
- g_∞ possesses moments of order $2+\delta$, for $0<\delta<\bar{\delta}$.

Choosing $f_{0,1}(v) = f_0(v)$, and $f_{0,2}(v) = f(v,T)$ shows that $d_s(f(t), f(t+T))$ converges exponentially to zero. The d_s-distance between subsequences converges to zero as soon as $\mathcal{S}_{p,q}(s-2)<0$.

- Is the unique limit g_∞ a stationary solution to the kinetic equation?

Convergence to self-similarity II

- $g(v,t)$ and g_∞ have the same mass, momentum and energy.
- $g(v,t)$ and g_∞ possess moments of order $2+\delta$, for $0<\delta<\bar{\delta}$.
- $s = 2+\delta$ implies

$$d_s\left(Q(g(t),g(t)),Q(g_\infty,g_\infty)\right) \le (p^s+q^s+1)\, d_s\left(g(t),g_\infty\right).$$

- $Q(g(t),g(t))$ converges weakly towards $Q(g_\infty,g_\infty)$.
- For all $\phi \in C_0^1(\mathbb{R})$,

$$\int_{\boldsymbol{R}} \phi(v) Q(g(t),g(t))(v)\, dv \to \int_{\boldsymbol{R}} \phi(v) Q(g_\infty,g_\infty)(v)\, dv.$$

Convergence to self-similarity III

- For all $\phi \in C_0^1(\mathbb{R})$,

$$\int_{\boldsymbol{R}} \phi(v) \left\{ \frac{\partial}{\partial v} (v g_\infty(v)) - Q(g_\infty, g_\infty)(v) \right\} dv = 0.$$

- g_∞ is the unique stationary solution to the kinetic equation.

Theorem

Let $\delta > 0$ be such that $\mathcal{S}_{p,q}(\delta) < 0$. Let $g(v,t)$ be the weak solution of the Boltzmann equation corresponding to the initial density f_0 with finite moments of order $2+\delta$. Then $g(v,t)$ converges exponentially fast in Fourier metric towards the unique stationary solution $g_\infty(v)$, and

$$d_{2+\delta}(g(t), g_\infty) \leq d_{2+\delta}(f_0, g_\infty) \mathrm{e}\left\{-|\mathcal{S}_{p,q}(\delta)| t\right\}$$

Overpopulated tails

- Depending of p and q, g_∞ can have overpopulated tails.
- Look at the singular part of the Fourier transformed equation

$$-\frac{p^2+q^2-1}{2}\xi\frac{\partial\widehat{g}}{\partial\xi}+\widehat{g}(\xi)=\widehat{g}(p\xi)\widehat{g}(q\xi).$$

- Set $\widehat{g}(\xi)=1-|\xi|^2+A|\xi|^{2+\delta}+\dots$. $\widehat{g}(\xi)$ satisfies the equation at the order $2+\delta$ if $A\mathcal{S}_{p,q}(\delta)$.

Overpopulated tails in the stationary distribution are present in all cases in which there exists a $\delta=\bar{\delta}>0$ such that $\mathcal{S}_{p,q}(\bar{\delta})=0$.

Results in economy

- Set $F(v,t) = f(v,t)I(v \geq 0), \quad v \in \boldsymbol{R}$.
- Rewrite the Boltzmann equation

$$\frac{d}{dt}\int_{\boldsymbol{R}} \phi(v)F(v,t)\,dv = \int_{\boldsymbol{R}^2} F(v,t)F(w,t)(\phi(v^*) - \phi(v))dvdw.$$

- Scale the solution

$$G(v,t) = m(t)F\left(m(t)v, t\right)$$

- Write the Fourier transform of the Boltzmann equation. The key function is

$$\mathcal{R}_{p,q}(\delta) = p^{1+\delta} + q^{1+\delta} - 1 - \frac{1+\delta}{2}\left(p+q-1\right).$$

The role of $\mathcal{R}_{p,q}$

- The sign of $\mathcal{R}_{p,q}$ now determines the asymptotic behavior of the distance $d_s(g_1(t), g_2(t))$.

Lemma

Given a constant $\lambda > 0$, if $p + q < 1$, let us define $p = 1 - \lambda\sqrt{q}$. Then, provided $q < 1/\lambda^2$ there exists a interval $I_- = (0, \bar\delta_-(q))$ such that $\mathcal{R}_{p,q}(\delta) < 0$ for $\delta \in I_-$. If $p + q > 1$, and $p = 1 + \lambda\sqrt{q}$ there exists a interval $I_+ = (0, \bar\delta_+(q))$ such that $\mathcal{R}_{p,q}(\delta) < 0$ for $\delta \in I_+$. In the remaining cases, namely when $p + q = 1$ or $p + q > 1$ but $p < 1$, $\mathcal{R}_{p,q}(\delta) < 0$ for all $\delta > 0$.

Convergence

- The main result

Theorem

Let $\delta > 0$ be such that $\mathcal{R}_{p,q}(\delta) < 0$, and let $g_\infty(v)$ be the unique stationary solution. Let the initial value possess moments,

$$\int_{\boldsymbol{R}_+} |v|^{1+\delta} f_0(v)\, dv < \infty.$$

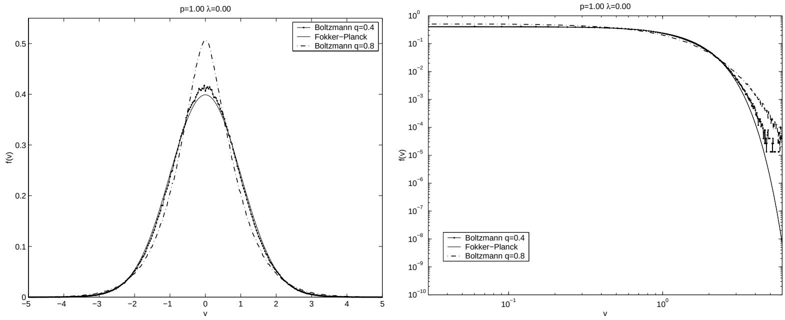
Then, $g(v,t)$ converges exponentially fast in Fourier metric towards $g_\infty(v)$, and the following bound holds

$$d_{1+\delta}(g(t), g_\infty) \le d_{1+\delta}(f_0, g_\infty) e \left\{ -|\mathcal{R}_{p,q}(\delta)| t \right\} \tag{1}$$

- Depending of the mixing parameters p and q, the stationary solution g_∞ can have overpopulated tails.

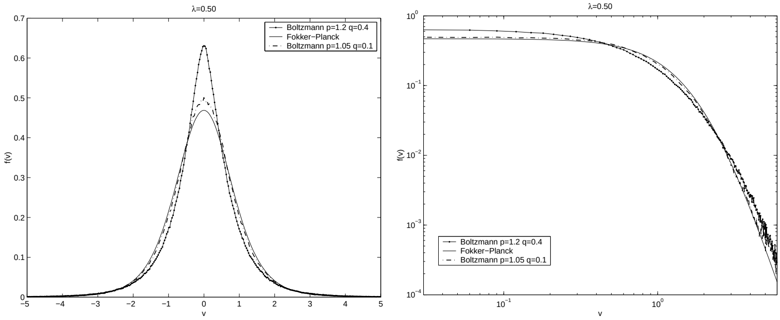
- Comparison of the self–similar stationary of the kinetic model with the stationary state of the Fokker-Planck model.
- Results for the kinetic model obtained by using Monte Carlo simulation. The method we adopted is based on Bird's time counter approach at each time step followed by a normalizing procedure according to the self-similar scaling.
- We use $N = 5000$ particles and perform several iterations until a stationary state is reached. Due to the slow convergence of the method near the tails some fluctuations are still present.
- First we consider the case $\lambda = 0$ for which the steady state of the Fokker-Planck asymptotic is the Gaussian .
- Next we simulate the formation of power laws for positive λ.

Exponential tails



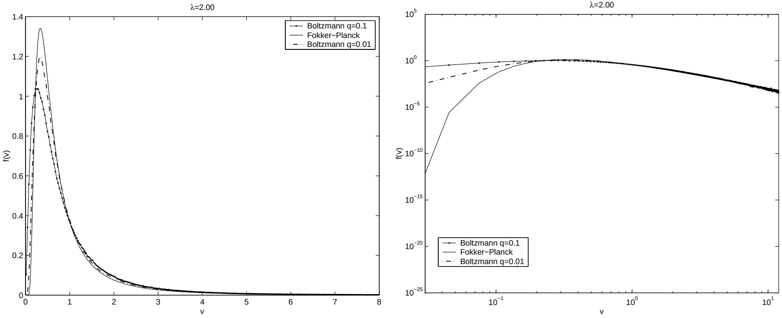
- **Figure 1** Asymptotic behavior for $\lambda = 0$ of the Fokker-Planck model and the Boltzmann model with $p = 1$ and $q = 0.4, 0.8$. Figure on the right is in loglog-scale.

Formation of power laws



- **Figure 2** Asymptotic behavior for $\lambda = 0.5$ of the Fokker-Planck model and the Boltzmann model for $p = 1.2$, $q = 0.4$ and $p = 1.05$, $q = 0.1$. Figure on the right is in loglog-scale.

Pareto tails



- **Figure 3** Asymptotic behavior for $\lambda = 2$ of the Fokker-Planck model and the Boltzmann model for $p = 1 - q + 2\sqrt{q}$, $q = 0.1$ and $q = 0.01$. Figure on the right is in loglog-scale.

Discussion

- We studied the large–time behavior of a one-dimensional kinetic model of Maxwell type.
- Two situations, depending wether the velocity variable can take values on $\boldsymbol{R}$ (nonconservative models of kinetic theory of rarefied gases) or in $\boldsymbol{R}_+$ (kinetic models of open economies).
- In both situations lack of conservation laws leads to situations in which the self–similar solution has overpopulated tails.
- Important in the case of economy, since it gives a elementary explanation of the formation of Pareto tails.
- How to extend the analysis to more realistic situations?