

# Kinetic models of Maxwell type. A brief history

## Part II

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*Porto Ercole, June 8-10 2008*

*Summer School METHODS AND MODELS OF KINETIC THEORY*



# Outline

- 1 Maxwell molecules and the Fourier transform
  - Bobylev's trick
  - Fourier based probability metrics
  - Equivalence between probability metrics
  
- 2 About the use of Fourier transform
  - Uniqueness for true Maxwell molecules
  - Decreasing functionals



# Fourier transform version

- Introduce the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(v) e^{-i\xi \cdot v} dv$$

- Use the weak form (Maxwell's equation of transfer) with  $\phi(v) = e^{-i\xi \cdot v}$

$$\int_{\mathbb{R}^{2n} \times S^{n-1}} B \left( \frac{q \cdot n}{|q|} \right) f(v) f(w) (e^{-i\xi \cdot v^*} - e^{-i\xi \cdot v}) dv dw dn =$$

$$\int_{\mathbb{R}^{2n}} f(v) f(w) e^{-i\frac{1}{2}\xi \cdot (v+w)} \int_{S^{n-1}} B \left( \frac{q \cdot n}{|q|} \right) (e^{-i\xi \cdot \frac{|q|n}{2}} - e^{-i\xi \cdot \frac{q}{2}}) dv dw dn.$$

- We used  $q = v - w$ ,

$$v^* = \frac{1}{2}(v + w + |q|n), \quad v = \frac{1}{2}(v + w + q).$$



# Fourier transform version II

- The identity holds

$$\int_{S^{n-1}} B\left(\frac{q \cdot n}{|q|}\right) \left( e^{-i\xi \cdot \frac{|q|n}{2}} - e^{-i\xi \cdot \frac{q}{2}} \right) dn =$$

$$\int_{S^{n-1}} B\left(\frac{\xi \cdot n}{|\xi|}\right) \left( e^{-i|\xi| \frac{q \cdot n}{2}} - e^{-i\xi \cdot \frac{q}{2}} \right) dn$$

- The left-hand side is an **isotropic function** (invariant under rotation) of  $\xi$  and  $q$ .
- It depends only on  $|\xi|$ ,  $|q|$  and  $\xi \cdot q$ .
- The value of the integral **can not change by interchanging the directions of  $\xi$  and  $q$**  (but not their absolute magnitudes!)



## Fourier transform version III

- We end up with the identity

$$\int_{\mathbf{R}^{2n} \times S^{n-1}} B\left(\frac{q \cdot n}{|q|}\right) f(v)f(w)(e^{-i\xi \cdot v^*} - e^{-i\xi \cdot v}) dv dw dn =$$

$$\int_{\mathbf{R}^{2n}} f(v)f(w)e^{-i\frac{1}{2}\xi \cdot (v+w)} \int_{S^{n-1}} B\left(\frac{\xi \cdot n}{|\xi|}\right) \left(e^{-i|\xi|\frac{q \cdot n}{2}} - e^{-i\xi \cdot \frac{q}{2}}\right) dv dw dn =$$

$$\int_{S^{n-1}} B\left(\frac{\xi \cdot n}{|\xi|}\right) f(v)f(w) \left(e^{-iv \cdot \frac{\xi + |\xi|n}{2}} - e^{-iv \cdot \frac{\xi - |\xi|n}{2}} - e^{-i\xi \cdot v}\right) dn.$$

- Fourier version [A.V. Bobylev *Dokl. Akad. Nauk. SSSR* (1975)]

$$\frac{\partial \hat{f}}{\partial t} = \int_{S^{n-1}} B\left(\frac{\xi \cdot n}{|\xi|}\right) \left[ \hat{f}\left(\frac{\xi + |\xi|n}{2}\right) \hat{f}\left(\frac{\xi - |\xi|n}{2}\right) - \hat{f}(\xi)\hat{f}(0) \right] dn.$$



## Main properties

- Set

$$\xi_+ = \frac{\xi + |\xi|n}{2}, \quad \xi_- = \frac{\xi - |\xi|n}{2}.$$

- Note that  $|\xi_+|^2 + |\xi_-|^2 = |\xi|^2$
- Conservation of mass, momentum and energy move to

$$\widehat{f}(\xi = 0, t) = 1, \quad \nabla \widehat{f}(\xi, t)_{\xi=0} = 0, \quad \Delta \widehat{f}(\xi, t)_{\xi=0} = -3.$$

- Simplified Fourier version

$$\frac{\partial \widehat{f}}{\partial t} = \int_{S^{n-1}} B \left( \frac{\xi \cdot n}{|\xi|} \right) \left[ \widehat{f}(\xi_+) \widehat{f}(\xi_-) - \widehat{f}(\xi) \right] dn.$$



# Main properties II

- The Maxwellian equilibrium

$$\widehat{M}(\xi) = e^{-|\xi|^2/2}$$

- Since  $|\xi_+|^2 + |\xi_-|^2 = |\xi|^2$ , immediate to prove that  $f(t) * M_\delta$  satisfies the equation.

$$\widehat{f}_\delta(\xi) = \widehat{f}(\xi) \widehat{M}_\delta(\xi),$$

- 

$$\widehat{f}_\delta(\xi_+) \widehat{f}_\delta(\xi_-) = \widehat{f}(\xi_+) \widehat{f}(\xi_-) \widehat{M}_\delta(\xi_+) \widehat{M}_\delta(\xi_-) = \widehat{f}(\xi_+) \widehat{f}(\xi_-) e^{-\delta|\xi|^2/2}.$$



## Fourier version of Kac model

- Kac model takes the form

$$\frac{\partial \hat{f}}{\partial t} = \frac{1}{2\pi} \int_{[0,2\pi]} \hat{f}(\xi \cos \theta) \hat{f}(\xi \sin \theta) d\theta - \hat{f}(\xi)$$

- Wild sum of the solution  $\hat{f}(t)$  represented by

$$\hat{f}(\xi, t) = e^{-t} \sum_{k=0}^{\infty} (1 - e^{-t})^k \hat{f}_{(k)}(\xi)$$

- The functions  $\hat{f}_{(k)}$  are given by recursion

$$\hat{f}_{(k+1)}(\xi) = \frac{1}{k+1} \sum_{i=1}^k \frac{1}{2\pi} \int_{[0,2\pi]} \hat{f}_{(i)}(\xi \cos \theta) \hat{f}_{(k-i)}(\xi \sin \theta) d\theta, .$$



## Fourier based Metric

- Given two probability measures  $f, g \in \mathcal{P}(\mathbf{R}^n)$ , for any  $s > 0$ , the **Fourier based metric**  $d_s$  is defined as

$$d_s(f, g) = \sup_{\xi \in \mathbf{R}^n} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s}$$

- Let  $\mathcal{P}_s(\mathbf{R}^n)$  the class of all probability distributions  $f \in \mathcal{P}(\mathbf{R}^n)$  such that

$$\int_{\mathbf{R}^n} |v|^s df(v) < \infty.$$

- Given any two probability measures  $f, g \in \mathcal{P}_s(\mathbf{R}^n)$  with  $s > 0$  with **equal moments up to  $[s]$**  if  $s \notin \mathbb{N}$ , or **equal moments up to  $s - 1$**  if  $s \in \mathbb{N}$ , then  $d_s(f, g)$  is finite.



# History

- Introduced only recently in  
[E. Gabetta, G. Toscani, W. Wennberg *J. Statist. Phys.* (1995)]
- The case  $s = 2$  used in  
[E. Gabetta, G. Toscani, W. Wennberg *J. Statist. Phys.* (1995)],  
to prove **uniqueness of the non cut-off Boltzmann equation** for  
Maxwell molecules.
- A further application of the general case  $s > 0$  to the finding of  
**Berry–Essen type bounds in the central limit theorem** for a  
stable law given in  
[T. Goudon, S. Junca, G. Toscani *Monatsh. Math.* (2002)].
- Various applications to dissipative Maxwell molecules  
[M. Bisi, J.A. Carrillo, G. Toscani *J. Statist. Phys.* (2005-2006)]



# Properties

- **Interpolation of metrics:** Given any two probability measures  $f, g \in \mathcal{P}_s(\mathbf{R}^n)$  such that  $d_s$  is finite

$$d_p(f, g) \leq 2 \left( \frac{s-p}{2p} \right)^{p/s} \frac{s}{s-p} [d_s(f, g)]^{p/s}, \quad p < s.$$

- **Control of moments:** Given any two probability measures  $f, g \in \mathcal{P}_s(\mathbf{R}^n)$  with  $s \in \mathbb{N}$ , for all multi-indices  $\beta$  with  $|\beta| = s$

$$\left| \int_{\mathbf{R}^n} v^\beta df(v) - \int_{\mathbf{R}^n} v^\beta dg(v) \right| \leq C d_s(f, g).$$

- **Scaling:** Given any two probability measures  $f, g \in \mathcal{P}_s(\mathbf{R}^n)$  with  $d_s$  finite

$$d_s(\mathcal{S}_\theta[f], \mathcal{S}_\theta[g]) = \theta^{-s/2} d_s(f, g).$$



## Properties II

- **Convexity:** Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_s(\mathbf{R}^n)$  such that  $d_s$  is finite and  $\alpha$  in  $[0, 1]$

$$d_s(\alpha f_1 + (1-\alpha)f_2, \alpha g_1 + (1-\alpha)g_2) \leq \alpha d_s(f_1, g_1) + (1-\alpha)d_s(f_2, g_2).$$

- **Additivity with respect to convolution:** Given  $f_1, f_2, g_1$  and  $g_2$  in  $\mathcal{P}_s(\mathbf{R}^n)$

$$d_s(f_1 * f_2, g_1 * g_2) \leq d_s(f_1, g_1) + d_s(f_2, g_2)$$

- **Superadditivity with respect to convolution** holds for any value of the index  $s$

[J.A. Carrillo, G. Toscani *Riv. Mat. Univ. Parma* (2007)].



## Equivalence in one dimension

- Equivalence between the Wasserstein and the Fourier based metrics Easy to check with **explicitly computable constants** in the one-dimensional case.
- The general case studied in [J.A. Carrillo, G. Toscani *Riv. Mat. Univ. Parma* (2007)].
- The **one-dimensional analysis** takes advantage of the **explicit expression of Wasserstein metric**. This is particularly evident in case  $p = 1$ ,

$$W_1(f, g) = \int_0^1 |F^{-1}(\rho) - G^{-1}(\rho)| d\rho.$$

- The integral is the measure of the area between the two distribution functions  $F$  and  $G$ ,

$$W_1(f, g) = \int_{\mathbb{R}} |F(v) - G(v)| dv.$$



## Equivalence in one dimension

- Suppose  $f, g \in \mathcal{P}_2(\mathbb{R}^n)$  with equal mean value such that, if  $\alpha > 0$

$$M_{2+\alpha} = \max \left\{ \int_{\mathbb{R}} |v|^{2+\alpha} f(v) dv, \int_{\mathbb{R}} |v|^{2+\alpha} g(v) dv \right\} < \infty$$

- By **Markov inequality**, if  $X$  is a random variable with law  $f$  (or  $g$ )

$$P(|X| > \epsilon) \leq \frac{\mathbb{E}[X^{2+\alpha}]}{\epsilon^{2+\alpha}}.$$

- This implies

$$\lim_{R \rightarrow \infty} R^2 (F(-R) + 1 - F(R)) = \lim_{R \rightarrow \infty} R^2 P(|X| > R) = 0.$$



## Equivalence in one dimension II

- Integrate by parts to get the inequality

$$\int_{|v| \geq R} |v| |F(v) - G(v)| dv \leq \int_{|v| \geq R} \frac{v^2}{2} (f(v) + g(v)) dv.$$

- In fact

$$\int_{-\infty}^{-R} |v| |F(v) - G(v)| dv \leq - \int_{-\infty}^{-R} v(F(v) + G(v)) dv$$

- and

$$\int_R^{\infty} |v| |F(v) - G(v)| dv \leq \int_R^{\infty} v((1 - F(v)) + (1 - G(v))) dv.$$

- If  $f, g \in \mathcal{P}_2(\mathbb{R}^n)$ , with no extra moments bounded, **same inequality** by standard approximation arguments.



## Equivalence in one dimension III

- Then

$$\begin{aligned}
 & \int_{\mathbb{R}} |F(v) - G(v)| \, dv \\
 & \leq \int_{-R}^R |F(v) - G(v)| \, dv + \frac{1}{R} \int_{|v| \geq R} |v| |F(v) - G(v)| \, dv \\
 & \leq (2R)^{1/2} \left( \int_{\mathbb{R}} |F(v) - G(v)|^2 \, dv \right)^{1/2} + \frac{M_2}{R},
 \end{aligned}$$

- where

$$M_2 = \max \left\{ \int_{\mathbb{R}} |v|^2 f(v) \, dv, \int_{\mathbb{R}} |v|^2 g(v) \, dv \right\}.$$



## Equivalence in one dimension IV

- Hence

$$\begin{aligned}
 \int_{\mathbf{R}} |F(v) - G(v)|^2 dv &= \frac{1}{2\pi} \int_{\mathbf{R}} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|^2}{|\xi|^2} d\xi \\
 &\leq \frac{1}{2\pi} \left( \int_{-R}^R \frac{|\hat{f}(\xi) - \hat{g}(\xi)|^2}{|\xi|^2} d\xi + 2 \int_{|\xi| \geq R} \frac{1}{|\xi|^2} d\xi \right) \\
 &\leq \frac{1}{2\pi} \left( \int_{-R}^R |\xi|^2 d_2(f, g)^2 d\xi + \frac{2}{R} \right) \\
 &= \frac{1}{\pi} \left( \frac{R^3}{3} d_2(f, g)^2 + \frac{1}{R} \right).
 \end{aligned}$$

- Optimizing over  $R$

$$\int_{\mathbf{R}} |F(v) - G(v)|^2 dv \leq \frac{4}{3\pi} \sqrt{d_2(f, g)}.$$



# From $d_2$ to $W_1$

- Putting together the previous bounds, we obtain

## Theorem (From $d_2$ to $W_1$ )

Given  $f, g \in \mathcal{P}_2(\mathbb{R})$  such that  $d_2(f, g)$  is bounded, then

$$W_1(f, g) \leq \left( \frac{18M_2}{\pi} \right)^{1/3} d_2(f, g)^{1/6}$$

with

$$M_2 = \max \left\{ \int_{\mathbb{R}} v^2 df(v), \int_{\mathbb{R}} v^2 dg(v) \right\}.$$



## Equivalence in one dimension V

- Let, for  $\alpha > 0$

$$M_{2+\alpha} = \max \left\{ \int_{\mathbb{R}} |v|^{2+\alpha} df(v), \int_{\mathbb{R}} |v|^{2+\alpha} dg(v) \right\} < \infty$$

- We have

$$\begin{aligned} & \int_{\mathbb{R}^2} |v - w|^2 d\Pi^*(v, w) \\ & \leq \int_{|v-w| < R} |v - w|^2 d\Pi^*(v, w) + \frac{1}{R^\alpha} \int_{|v-w| > R} |v - w|^{2+\alpha} d\Pi^*(v, w) \\ & \leq R \int_{|v-w| < R} |v - w| d\Pi^*(v, w) + \frac{2^{2+\alpha} M_{2+\alpha}}{R^\alpha} \\ & = R W_1(f, g) + \frac{2^{2+\alpha} M_{2+\alpha}}{R^\alpha}. \end{aligned}$$



## From $d_2$ to $W_2$ in one-d

- Using the previous estimate we obtain

### Corollary (From $d_2$ to $W_2$ in one-d)

Given  $f, g \in \mathcal{P}_2(\mathbb{R}^n)$  such that

$$M_{2+\alpha} = \max \left\{ \int_{\mathbb{R}} |v|^{2+\alpha} f \, dv, \int_{\mathbb{R}} |v|^{2+\alpha} g \, dv \right\} < \infty$$

with  $\alpha > 0$ , then

$$W_2(f, g) \leq C_\alpha d_2(f, g)^{\alpha/[6(1+\alpha)]} M_2^{\alpha/[3(1+\alpha)]} M_{2+\alpha}^{1/(1+\alpha)}$$

with

$$C_\alpha = 2^{(2+\alpha)/(1+\alpha)} \left( \alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)} \right) \left( \frac{18}{\pi} \right)^{\alpha/[3(1+\alpha)]} .$$



# From Wasserstein to Fourier metric

- **Difficult** to extend the result to **higher dimensions**
- Suppose  $f$  and  $g$  with the **same mean**.
- Let  $\Pi^*(v, w)$  the optimal plan between  $f$  and  $g \in \mathcal{P}_2(\mathbb{R}^n)$  for the Euclidean Wasserstein distance  $W_2$ . Since they have equal mean velocity, then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} (v - w) d\Pi^*(v, w) = 0,$$

- We can write

$$\hat{f}(\xi) - \hat{g}(\xi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( e^{-iv \cdot \xi} - e^{-iw \cdot \xi} + i\xi \cdot (v - w) \right) d\Pi^*(v, w).$$



## From Wasserstein to Fourier metric II

- We can estimate the integrand as

$$\begin{aligned}
 |e^{-iv \cdot \xi} - e^{-iw \cdot \xi} + i\xi \cdot (v - w)| &\leq |e^{-iw \cdot \xi} (e^{-i(v-w) \cdot \xi} - 1 + i\xi \cdot (v - w))| \\
 &\quad + |(e^{-iw \cdot \xi} - 1)i\xi \cdot (v - w)| \\
 &\leq \frac{1}{2} |\xi \cdot (v - w)|^2 + |\xi| \|w\| |\xi \cdot (v - w)| \leq \frac{1}{2} |\xi|^2 |v - w|^2 + |\xi|^2 \|w\| |v - w|
 \end{aligned}$$

- Apply Taylor's formula to  $e^{-i(v-w) \cdot \xi}$  up to order 2 and mean value theorem for the function  $e^{-iw \cdot \xi}$ .

$$\begin{aligned}
 \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^2} &\leq \int_{\mathbf{R}^n \times \mathbf{R}^n} \left( \frac{1}{2} |v - w|^2 + |w| |v - w| \right) d\Pi^*(v, w) \\
 &\leq \frac{1}{2} W_2^2(f, g) + \left( \int_{\mathbf{R}^n} |v|^2 df(v) \right)^{1/2} W_2(f, g)
 \end{aligned}$$



## From $W_2$ to $d_2$

- We obtain

### Theorem (From $W_2$ to $d_2$ )

Given  $f, g \in \mathcal{P}_2(\mathbf{R}^n)$  with equal mean value, then

$$d_2(f, g) \leq \frac{1}{2} W_2^2(f, g) + M^{1/2} W_2(f, g)$$

where

$$M = \min \left( \int_{\mathbf{R}^n} |v|^2 df(v), \int_{\mathbf{R}^n} |v|^2 dg(v) \right).$$

or in probabilistic terms,

$$d_2(f, g) \leq \frac{1}{2} W_2^2(f, g) + \min (\mathbb{E}[X^2], \mathbb{E}[Y^2])^{1/2} W_2(f, g)$$

for  $X$  and  $Y$  with laws  $f$  and  $g$  respectively such that  $\mathbb{E}[X] = \mathbb{E}[Y]$ .



# The Boltzmann equation

- Assume  $n = 3$  for simplicity, but all the results can be **generalized readily to any dimension  $n \geq 2$** .
- Same proof for the one-dimensional Kac model.
- We shall study the Boltzmann equation with the conditions

$$\int df_0(v) dv = 1, \quad \int v df_0(v) = 0, \quad \int v^2 df_0(v) = 3;$$

- The conditions are preserved under the time-evolution of the Boltzmann equation.
- The Fourier transform of the equation

$$\partial_t \hat{f}(t, \xi) = \int_{S^2} \sigma \left( \frac{\xi \cdot n}{|\xi|} \right) \left[ \hat{f}(\xi^+) \hat{f}(\xi^-) - \hat{f}(\xi) \hat{f}(0) \right] dn,$$



# The Boltzmann equation II

- Note that

$$\xi^+ = \frac{\xi + |\xi|n}{2}, \quad \xi^- = \frac{\xi - |\xi|n}{2},$$

- and the initial conditions are

$$\widehat{f}(0) = 1, \quad \nabla \widehat{f}(0) = 0, \quad \nabla^2 \widehat{f}(0) = -3,$$

$$\widehat{f} \in C^2(\mathbb{R}^d).$$

- Note that  $\xi^+ + \xi^- = \xi$ , and  $|\xi^+|^2 + |\xi^-|^2 = |\xi|^2$ .
- We prove

## Theorem

Let  $F$  and  $G$  be two solutions of the Boltzmann equation. Then, for all time  $t \geq 0$ ,

$$d_2(f(t), g(t)) \leq d_2(f(0), g(0)).$$



# The Boltzmann equation III

- We mention three useful corollaries.

## Corollary

*Let  $f_0$  be a nonnegative measure with finite variance. Then, there exists a unique weak solution  $f(t)$  of the Boltzmann equation, such that  $f(0) = f_0$ .*

## Corollary

*Let  $f_0$  be a nonnegative measure with finite variance, and  $f(t)$  the associated solution of the Boltzmann equation. Let  $M$  be the Maxwellian distribution with the same mean vector and variance that  $f_0$ . Then  $d_2(f(t), M)$  is decreasing towards 0.*



# The Boltzmann equation IV

- The last result is concerned with the non cut-off equation

## Corollary

*If  $f_\varepsilon(t)$  is a sequence of approximate solutions of the Boltzmann equation, obtained by a standard cut-off procedure for instance, then  $f_\varepsilon$  converges weakly to  $f$ . This entails in particular that such results as **the decrease of the Fisher information**, or **the decrease of Wasserstein metric**, which are known to hold for the cut-off equation, also **hold for the non cut-off equation**.*



# The proof

- Let  $f$  and  $g$  be two solutions of the Boltzmann equation, and  $\widehat{f}$ ,  $\widehat{g}$  their Fourier transforms  
 [G. Toscani, C. Villani *J. Statist. Phys.*(1999)].
- It holds

$$\begin{aligned} \left| \frac{\widehat{f}(\xi^+) \widehat{f}(\xi^-) - \widehat{g}(\xi^+) \widehat{g}(\xi^-)}{|\xi|^2} \right| &\leq |\widehat{f}(\xi^+)| \left| \frac{\widehat{f}(\xi^-) - \widehat{g}(\xi^-)}{|\xi^-|^2} \right| \frac{|\xi^-|^2}{|\xi|^2} \\ &\quad + |\widehat{g}(\xi^-)| \left| \frac{\widehat{f}(\xi^+) - \widehat{g}(\xi^+)}{|\xi^+|^2} \right| \frac{|\xi^+|^2}{|\xi|^2} \\ &\leq \sup \left| \frac{\widehat{f} - \widehat{g}}{|\xi|^2} \right| \left( \frac{|\xi^-|^2 + |\xi^+|^2}{|\xi|^2} \right) = \sup \left| \frac{\widehat{f} - \widehat{g}}{|\xi|^2} \right|. \end{aligned}$$



# The proof II

- For cut-off molecules, denote by

$$S = \int_{S^2} \sigma(n \cdot e) \, dn$$

the total cross-section.

- With this position

$$\begin{aligned} & \partial_t \frac{(\hat{f} - \hat{g})}{|\xi|^2} + S \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^2} \\ &= \int \sigma \left( \frac{\xi \cdot n}{|\xi|} \right) \frac{\hat{f}(\xi^+) \hat{f}(\xi^-) - \hat{g}(\xi^+) \hat{g}(\xi^-)}{|\xi|^2} \, dn \\ & \leq S \sup \left| \frac{\hat{f} - \hat{g}}{|\xi|^2} \right|. \end{aligned}$$



# The proof III

- We set

$$h(t, \xi) = \frac{\widehat{f}(\xi) - \widehat{g}(\xi)}{|\xi|^2}.$$

- The preceding computation shows that

$$|\partial_t h - Sh| \leq S \|h\|_\infty.$$

- Gronwall's lemma proves at once that **for cut-off molecules,  $\|h(t)\|_\infty$  is nonincreasing.**
- For true Maxwell molecules,  $\sigma(\nu)$  is **singular like  $(1 - \nu)^{-5/4}$ .**
- This singularity corresponds to **grazing collisions**, i.e.  $\xi^+ \sim \xi$ ,  $\xi^- \sim 0$ .



# The proof IV

- Since it is nonintegrable,  $S = \infty$ .
- We split the collision integral according to  $|1 - \nu| \geq \varepsilon$  or  $|1 - \nu| < \varepsilon$ .
- For the first term, we use the preceding estimate.
- For the second, we use the fact that **the singularity is cancelled** by the vanishing of

$$\hat{f}(\xi^+) \hat{f}(\xi^-) - \hat{f}(\xi) \hat{f}(0)$$

for grazing collisions.



# The proof V

- Indeed, we write

$$\begin{aligned}
 & \left| \widehat{f}(\xi^+) \widehat{f}(\xi^-) - \widehat{f}(\xi) \widehat{f}(0) \right| \\
 & \leq |\widehat{f}(\xi^+)| |\widehat{f}(\xi^+) - \widehat{f}(\xi)| + |\widehat{f}(\xi)| |\widehat{f}(\xi^-) - \widehat{f}(0)| \\
 & \leq \sup_{|\eta| \leq \sup(|\xi|, |\xi^+|)} |\nabla \widehat{f}(\eta)| |\xi^+ - \xi| + \sup_{|\eta| \leq |\xi^-|} |\nabla \widehat{f}(\eta)| |\xi^-|.
 \end{aligned}$$

- Since  $|\xi^+|, |\xi^-| \leq |\xi|$ ,  $|D^2 \widehat{f}(\xi)| \leq 3$ , and  $\nabla \widehat{f}(0) = 0$ ,

$$\left| \widehat{f}(\xi^+) \widehat{f}(\xi^-) - \widehat{f}(\xi) \widehat{f}(0) \right| \leq C |\xi| |\xi^-| \leq C |\xi|^2 (1 - \nu)^{1/2},$$

- $C$  depends **only on the dimension**.



# The proof VI

- The integrand is bounded by  $C(1 - \nu)^{-3/4}$ , and thus **the integral is convergent, uniformly in  $\xi$  and in  $t$ .**
- Let us set

$$S_\varepsilon = \int_{S^2} 1_{|1 - n \cdot e| \geq \varepsilon} \sigma(n \cdot e) \, dn,$$

$$r_\varepsilon = \sup_{\xi, t} \left| \int_{S^2} 1_{|1 - \frac{\xi \cdot n}{|\xi|}| < \varepsilon} \sigma\left(\frac{\xi \cdot n}{|\xi|}\right) \left[ \widehat{f}(\xi^+) \widehat{f}(\xi^-) - \widehat{f}(\xi) \widehat{f}(0) \right] \, dn \right|.$$

- Then  $r_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and

$$|\partial_t h(\xi, t) - S_\varepsilon h(\xi, t)| \leq S_\varepsilon \|h\|_\infty(t) + r_\varepsilon.$$



# The proof VII

- This is equivalent to

$$\left| \partial_t \left( h(\xi, t) e^{S_\varepsilon t} \right) \right| \leq S_\varepsilon \|h(\cdot, t) e^{S_\varepsilon t}\|_\infty + r_\varepsilon e^{S_\varepsilon t}.$$

- Integrate from 0 to  $t$

$$|h(\xi, t)| e^{S_\varepsilon t} \leq |h(\xi, 0)| + \int_0^t d\tau \left( S_\varepsilon \|h(\cdot, \tau) e^{S_\varepsilon \tau}\|_\infty + r_\varepsilon e^{S_\varepsilon \tau} \right).$$

- Set  $H_\varepsilon(t) = \|h(\cdot, t) e^{S_\varepsilon t}\|_\infty$ . Then

$$H_\varepsilon(t) \leq H_\varepsilon(0) + \int_0^t r_\varepsilon e^{S_\varepsilon \tau} d\tau + \int_0^t S_\varepsilon H(\tau) d\tau.$$



# The proof VIII

- The generalized Gronwall inequality,

$$u(t) \leq \varphi(t) + \int_0^t \lambda(\tau) u(\tau) d\tau$$

- implies

$$u(t) \leq \varphi(0) e^{\left\{ \int_0^t \lambda(\tau) d\tau \right\}} + \int_0^t e^{\left\{ \int_\tau^t \lambda(\tau) d\tau \right\}} \frac{d\varphi}{d\tau} d\tau.$$

- Apply this inequality with  $\lambda(\tau) = S_\varepsilon$  and

$$\varphi(t) = H_\varepsilon(0) + \int_0^t r_\varepsilon e^{S_\varepsilon \tau} d\tau.$$



# The proof IX

- We obtain

$$H_\varepsilon(t) \leq H_\varepsilon(0)e^{S_\varepsilon t} + tr_\varepsilon e^{S_\varepsilon t},$$

- This corresponds to

$$\|h(\cdot, t)\|_\infty \leq \|h(\cdot, 0)\|_\infty + r_\varepsilon t.$$

- Letting  $\varepsilon$  going to 0,  $\|h(\cdot, t)\|_\infty \leq \|h(\cdot, 0)\|_\infty$ , i.e.

$$d_2(F(t), G(t)) \leq d_2(F(0), G(0)).$$



## Decreasing functionals

- Useful **application of Fourier transform** concerns **decreasing functionals**[A.V. Bobylev, G. Toscani *J. Math. Phys.* (1992)].
- Assume

$$S = \int_{S^2} \sigma(n \cdot e) dn = 1$$

- The discrete in time form

$$\frac{\tilde{f} - f}{\Delta t} = f \circ f - f.$$

- Is a finite difference scheme for of the Boltzmann equation.  $\tilde{f}$  and  $f$  approximate  $f(v, t + \Delta t)$  and  $(f(v, t)$ .



## Decreasing functionals II

- At each time step the solution is a **mixture** of the two probability densities  $f \circ f$  and  $f$ .

$$\tilde{f} = \Delta t f \circ f + (1 - \Delta t) f.$$

- For any convex functional

$$\Gamma(\tilde{f}) \leq \Delta t \Gamma(f \circ f) + (1 - \Delta t) \Gamma(f).$$

- The inequality

$$\Gamma(f \circ f) \leq \Gamma(f),$$

- Is a sufficient condition for the **monotonic behavior in time**

$$\Gamma(\tilde{f}) \leq \Gamma(f),$$



# Decreasing functionals in Fourier

- In Fourier version the condition reads

$$\widehat{\Gamma}(f \circ f) \leq \widehat{\Gamma}(f),$$

- where

$$\widehat{f \circ f}(\xi) = \int \sigma \left( \frac{\xi \cdot n}{|\xi|} \right) \widehat{f}(\xi^+) \widehat{f}(\xi^-) dn$$

- In the **isotropic** case

$$\widehat{f \circ f}(\xi) = \int_0^\pi \sin\theta \sigma(\cos\theta) \widehat{f} \left( |\xi| \frac{\cos\theta}{2} \right) \widehat{f} \left( |\xi| \frac{\sin\theta}{2} \right) d\theta$$

- **Necessary condition**

$$\widehat{\Gamma} \left( \widehat{f} \left( |\xi| \frac{\cos\theta}{2} \right) \widehat{f} \left( |\xi| \frac{\sin\theta}{2} \right) \right) dn \leq \widehat{\Gamma}(\widehat{f}(|\xi|))$$



## Planar case

- In the planar case

$$\xi_+ = |\xi| \frac{m+n}{2}, \quad \xi_- = |\xi| \frac{m-n}{2}.$$

- Fix the unit vectors

$$m = m(\theta) = (\cos \theta, \sin \theta), \quad n = n(\psi) = (\cos \psi, \sin \psi), \quad \psi = \theta + \gamma.$$

- We obtain

$$\xi_+ = |\xi| \cos \frac{\gamma}{2} \left\{ \cos \left( \theta + \frac{\gamma}{2} \right), \sin \left( \theta + \frac{\gamma}{2} \right) \right\}$$
$$\xi_- = |\xi| \sin \frac{\gamma}{2} \left\{ \sin \left( \theta + \frac{\gamma}{2} \right), -\cos \left( \theta + \frac{\gamma}{2} \right) \right\}$$



# Planar case II

- Easy computations show that

$$\widehat{f \circ f}(|\xi|, \theta) = \int_0^\pi A(\gamma) \widehat{f}\left(|\xi| \cos \frac{\gamma}{2}, \theta + \frac{\gamma}{2}\right) \widehat{f}\left(|\xi| \sin \frac{\gamma}{2}, \theta + \frac{\gamma - \pi}{2}\right) d\gamma,$$

- where

$$A(\gamma) = [\sigma(\cos \gamma) + \sigma(\cos(\pi - \gamma))], \quad \int_0^\pi A(\gamma) d\gamma = 1$$

- **Necessary condition**

$$\widehat{\Gamma} \left[ \widehat{f}\left(|\xi| \cos \frac{\gamma}{2}, \theta + \frac{\gamma}{2}\right) \widehat{f}\left(|\xi| \sin \frac{\gamma}{2}, \theta + \frac{\gamma - \pi}{2}\right) \right] \leq \widehat{\Gamma} \left[ \widehat{f}(|\xi|, \theta) \right]$$



# Convolution inequalities

- Both isotropic and planar cases are **inequalities for convolutions**

$$\widehat{\Gamma} \left[ \widehat{f}(a|\xi|) \widehat{g}(b|\xi|) \right] \leq a^2 \widehat{\Gamma}[\widehat{f}(|\xi|)] + b^2 \widehat{\Gamma}[\widehat{g}(|\xi|)], \quad a^2 + b^2 = 1.$$

- In the physical space

$$\Gamma \left[ a^{-n} f(a^{-1}v) * b^{-n} g(b^{-1}v) \right] \leq a^2 \Gamma(f) + b^2 \Gamma(g), \quad a^2 + b^2 = 1.$$

- In many cases, the equality sign holds **if and only if**  $f$  is the Maxwellian distribution  $M$ .
- This inequality is usually called **super-additivity** for convolutions.



# Superadditivity

- Both Wasserstein metric  $W_2$  and the Fourier based metric  $d_2$  are **super-additive** with respect to **rescaled convolutions**.
- Let  $\{X_0, Y_0\}, \{X_1, Y_1\}$  be two independent pairs of random variables, and let  $f_i$  (resp.  $g_i$ ) be the laws of  $X_i$  (resp.  $Y_i$ ),  $i = 0, 1$ .
- For  $0 < \lambda < 1$ , let  $f_\lambda$  (resp.  $g_\lambda$ ) be the law of  $\sqrt{\lambda}X_0 + \sqrt{1-\lambda}X_1$  (resp.  $\sqrt{\lambda}Y_0 + \sqrt{1-\lambda}Y_1$ ), i.e.

$$f_\lambda(v) = \frac{1}{\lambda^{n/2}} f_0\left(\frac{v}{\sqrt{\lambda}}\right) * \frac{1}{(1-\lambda)^{n/2}} f_1\left(\frac{v}{\sqrt{1-\lambda}}\right).$$



# Superadditivity II

- Properties of convexity and additivity imply

$$W_2^2(f_\lambda, g_\lambda) \leq \lambda W_2^2(f_0, g_0) + (1 - \lambda) W_2^2(f_1, g_1),$$

- and

$$d_2(f_\lambda, g_\lambda) \leq \lambda d_2(f_0, g_0) + (1 - \lambda) d_2(f_1, g_1).$$

- Superadditivity is also known for **convex functionals** (relative entropies), like **Boltzmann's relative entropy**

$$H(f|M^f) = \int_{R^n} f(v) \log \frac{f(v)}{M^f(v)} dv,$$



# Superadditivity III

- The **same property** holds for the **relative Fisher information** [A. Stam *Inform. Control* (1959)],

$$I(f|M^f) = \int_{\mathbb{R}^n} \left| \nabla \log f(v) - \nabla \log M^f(v) \right|^2 f(v) dv.$$

- The inequality for the  $H$ -functional is a consequence of Shannon's entropy power inequality.
- By means of the **relative entropy**  $H$ , one can define the so-called  **$H$ -neighborhoods**. Even if those do not define a topological space, in the usual sense, their **topological structure** is **finer than the metric topology defined by the  $L^1$ -distance**,

$$\int_{\mathbb{R}^n} |f(v) - g(v)| dv \leq \sqrt{2H(f|g)},$$

which is the so-called Csiszar-Kullback inequality.



# Superadditivity IV

- The proof in [A.V. Bobylev, G. Toscani *J. Math. Phys.* (1992)] included also the axially symmetric case

$$\widehat{f}(\xi) = \widehat{f}\left(|\xi|, \frac{\xi}{|\xi|} \cdot m\right).$$

- In this case super-additivity implies monotonicity.
- The proof in [A.V. Bobylev, G. Toscani *J. Math. Phys.* (1992)] did not include the full case  $n = 3$ .
- The **decreasing property** of Fisher information in any dimension was solved by C. Villani [C. Villani *J. Math. Pures Appl.* (1998)].

