2D MOTION WITH LARGE DEFORMATIONS

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Abstract. We study the motion of a solid with large deformations. The solid may be loaded on its surface by needles, rods, beams, plates, ... It turns out that it is wise to choose a third gradient theory for the body. The stretch matrix of the polar decomposition has to be symmetric. This is an internal constraint which introduces a reaction stress in the Piola-Kirchhoff-Boussinesq stress. By use of a Galerkin approximation, combined with suitable a priori estimates and a passage to the limit, we prove that there exists a motion which solves a variational formulation of the complete equations of mechanics, at least locally in time. This motion may be interrupted by crushing resulting in a discontinuity of velocity with respect to time, i.e., an internal collision.

Questo articolo è dedicato
alla memoria di Enrico Magenes,
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1. Introduction

Lagrangian mechanics produces predictive theories of the motion of continuous media. For rigid solids which do not interact or interact smoothly, the equations and their solutions are given within the analytical mechanics theories, see [9]. For rigid solids which interact, for instance which collide, equations are complemented by the collision theory [6].

For deformable solids (see [2, 4, 10, 12]), the Lagrangian theory is not achieved. We give a mechanical description of the motion together with a mathematical analysis of the related equations. In particular, we investigate the motion of a deformable solid assuming the solid is fixed on a part of its boundary and that there are neither self-collision nor self-contact and neither collisions nor contact with obstacles. Indeed, in the case when self collisions or collisions with obstacles are included, the equations should result from the coupling of this theory with the collision theory, while smooth self-contact and smooth contact with an obstacle need a slight sophistication of the present theory.

In this paper, as a first step and for the sake of simplicity, we consider $2D$ problems and choose the value of all the mechanical constants but one equal to 1. Hence, the motion of a $2D$ solid is investigated during a finite time interval $(0, T)$. At time $t = 0$, the solid is

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assumed to be located in a smooth domain $\mathcal{D}_a$. The motion is described by the function

$$(a,t) \to \Phi(a,t) \in \mathbb{R}^2, \quad (a,t) \in \mathcal{D}_a \times [0,T]\,,$$
$$a = \Phi(a,0),$$

where time $T > 0$ is given.

The paper is organized as follows. Some properties of 2×2 matrices are recalled in Section 2 for the reader convenience. The mechanical theory is fully developed from Section 3 to Section 7.

The equations of motion result from the principle of virtual power. We point out that surface loads can be applied to the solid by needles, wires and curvilinear beams. Thus, in order to account for these loads, it is wise to have a third gradient power of the interior forces: this is introduced and detailed in Section 3.

We are considering the two classical equations of motion of mechanics: the linear momentum equation and the angular momentum and look for two unknowns. Indeed, in addition to the function Φ, a reaction ensuring an internal constraint is introduced when defining the constitutive laws. The kinematic relationships relate function Φ to the quantities which describe the deformations, that is, the stretch matrix $\mathbf{W}$ and the rotation matrix $\mathbf{R}$, see Section 4.

The constitutive laws are deduced in Section 5 from schematic and simple free energy and pseudo-potential of dissipation, which account for all the mechanical properties. One of these properties is the symmetry of the stretch matrix which is an internal constraint. Reaction stress $\mathbf{A}$ to this internal constraint is the other unknown of the problem, as specified in Section 6.

The usual indetermination on reaction $\mathbf{A}$ is solved by the equations of motion (resulting from the kinematic relationships and the constitutive laws). In particular, the angular momentum equation of motion leads to the reaction matrix $\mathbf{A}$: referring to Section 7, let us point out that, in dimension 2, this equation has a simple structure. The equations are solved within a suitable variational framework. More precisely, we show that that there exists a motion satisfying a weak version of the resulting nonlinear PDE system, at least locally in time, i.e., in some time interval $(0, \hat{T})$, with $0 < \hat{T} \leq T$.

Our main result is shown by applying a Galerkin discretization scheme, combined with suitable a priori estimates and passage to the limit: the proof is given in Section 8.

The fact that the existence result is local in time is justified by the possibility of the motion to be interrupted by crushing which is caused by discontinuity of velocity (with respect to time), i.e. an internal collision: this is explained in Section 9. These results have been previously announced in [3].

2. Properties of 2×2 Matrices

For the sake of clarity, we summarize in this section some useful results concerning properties of 2×2 matrices. Let $\mathcal{M}$ be the linear space of 2×2 matrices, endowed with scalar product

$$\mathbf{A} : \mathbf{B} = A_{ij} B_{ij} = \mathrm{tr}\,(\mathbf{A}\mathbf{B}^T).$$

The subspaces $\mathcal{S} \subset \mathcal{M}$ of the symmetric matrices and $\mathcal{A} \subset \mathcal{M}$ of the antisymmetric matrices are orthogonal. In $\mathcal{M}$, for $0 < \alpha < 1$ (whose physical meaning will be specified in

Section 5) , we introduce the sets

$$\mathcal{M}_\alpha = \left\{ \mathbf{F} \in \mathcal{M}, \quad \det \mathbf{F} > \alpha^2, \quad \sqrt{\operatorname{tr} \mathbf{F}^T\mathbf{F} + 2\det \mathbf{F}} > 2\alpha \right\}, \tag{1}$$

$$C_\alpha = \left\{ \mathbf{B} \in \ \mathcal{M} \ \middle| \ \operatorname{tr} \mathbf{B} \geq 2\alpha, \ \det \mathbf{B} \geq \alpha^2 \right\}. \tag{2}$$

The set C_α is the set of matrices with the sum of their eigenvalues larger than 2α and the product of their eigenvalues larger than α^2. The interior set $\mathring{C}_\alpha$ is defined by

$$\operatorname{tr} \mathbf{B} > 2\alpha, \quad \det \mathbf{B} > \alpha^2.$$

and $\mathring{C}_\alpha \cap \mathcal{S}$ is an open set of $\mathcal{S}$. The meaning and the role of $\mathcal{M}_\alpha$ and $\mathcal{C}_\alpha$ will be clear in the following. Note that $\mathbf{I} \in C_\alpha$ ($\mathbf{I}$ being the identity matrix). Moreover, we have

$$1 > \gamma > \alpha \implies C_\gamma \subset C_\alpha, \quad \mathbf{I} \in C_\gamma.$$

Now, let us recall some properties of 2×2 matrices.

Proposition 1. *Let* $\mathbf{F} \in \ \mathcal{M}$*, such that* $\det \mathbf{F} > 0$*, then there exist a unique matrix* $\mathbf{W} \in \mathcal{S}$ *and a unique direct orthogonal matrix* $\mathbf{R}$*, such that* $\mathbf{F} = \mathbf{R}\mathbf{W}$*.*

The proof is detailed, e.g. in [7], and refers to a fairly classical result. In particular, let us recall that a direct orthogonal matrix satisfies

$$\mathbf{R} \in \mathcal{M}, \quad \det \mathbf{R} = 1, \quad \mathbf{R}\mathbf{R}^T = \mathbf{I}.$$

Hence, we investigate functions

$$\mathbf{F} \to \mathbf{W}(\mathbf{F}), \quad \mathbf{F} \to \mathbf{R}(\mathbf{F}),$$

as well as

$$\mathbf{F} \to \operatorname{tr} \mathbf{W}(\mathbf{F}), \quad \det \mathbf{W}(\mathbf{F}), \quad \frac{1}{\det \mathbf{W}(\mathbf{F})}, \quad \frac{1}{\operatorname{tr} \mathbf{W}(\mathbf{F})},$$

defined in $\mathcal{M}_\alpha$.

Proposition 2. *Functions* $\mathbf{F} \to \mathbf{W}(\mathbf{F})$*,* $\mathbf{F} \to \mathbf{W}^{-1}(\mathbf{F})$*,* $\mathbf{F} \to \mathbf{R}(\mathbf{F})$*, and*

$$\mathbf{F} \to \operatorname{tr} \mathbf{W}(\mathbf{F}), \quad \det \mathbf{W}(\mathbf{F}), \quad \frac{1}{\det \mathbf{W}(\mathbf{F})}, \quad \frac{1}{\operatorname{tr} \mathbf{W}(\mathbf{F})}.$$

are C^∞ *on* $\mathcal{M}_\alpha$*. Moreover, if* $\mathbf{F} : \mathcal{D}_a \times (0,T) \to \mathcal{M}_\alpha$ *satisfies* $|\mathbf{F}(a,t)| = |\mathbf{W}(a,t)| \leq c$ *(i.e., the norm of* $\mathbf{F}$ *is uniformly bounded), all the derivatives of the previous functions with respect to* $\mathbf{F}$ *are globally bounded, i.e., uniformly bounded with respect to* $(a,t) \in \mathcal{D}_a \times (0,T)$*.*

Proof. The set $\mathcal{M}_\alpha$ is open in $\mathcal{M}$. We have

$$\det \mathbf{W} = \det \mathbf{F}, \quad \operatorname{tr} \mathbf{W} = \sqrt{\operatorname{tr} \mathbf{F}^T\mathbf{F} + 2\det \mathbf{F}}. \tag{3}$$

Functions

$$\mathbf{F} \to \operatorname{tr} \mathbf{W}(\mathbf{F}), \quad \det \mathbf{W}(\mathbf{F}), \quad \frac{1}{\det \mathbf{W}(\mathbf{F})}, \quad \frac{1}{\operatorname{tr} \mathbf{W}(\mathbf{F})}$$

are C^∞ functions of $\mathbf{F}$ from $\mathcal{M}_\alpha$ into $\mathbb{R}$. Moreover, we have with the Hamilton-Cayley theorem

$$\mathbf{W} = \frac{1}{\sqrt{\operatorname{tr} \mathbf{F}^T\mathbf{F} + 2\det \mathbf{F}}} \left\{ -\mathbf{F}^T\mathbf{F} + \det(\mathbf{F})\mathbf{I} \right\}, \tag{4}$$

which is a C^∞ functions of $\mathbf{F}$ from $\mathcal{M}_\alpha$ into $\mathcal{S}\cap\mathring{C}_\alpha$. We have also

$$\mathbf{W}^{-1} = \frac{1}{\det \mathbf{W}} \left\{ (\mathbf{W})\mathbf{I} - \mathbf{W} \right\}, \tag{5}$$

which is also a C^∞ functions of $\mathbf{F}$ from $\mathcal{M}_\alpha$ into $\mathcal{S}$. Note that

$$\mathbf{R} = \mathbf{F}\mathbf{W}^{-1},$$

is also a C^∞ functions of $\mathbf{F}$ from $\mathcal{M}_\alpha$ into $\mathcal{M}$. If the norm of $\mathbf{F}$ in $\mathcal{M}$ is bounded, the computation of the derivatives of $\mathbf{W}$ and $\mathbf{R}$ with respect to the $F_{i\alpha}$ show that they are globally bounded with respect to a and t, i.e., uniformly bounded with respect to a and t:

$$\left| \frac{\partial^k \mathbf{W}}{\partial \mathbf{F}^k} (\mathbf{F}(a,t) \right| \leq c.$$

□

3. The Model and the Principle of Virtual Power

3.1. **The Unknowns.** The unknowns of the problem are the position function $\Phi(a,t) \in \mathbb{R}^2$ and a reaction matrix $\mathbf{A}(a,t) \in \mathcal{A}$, which activates to guarantee the symmetry of stretch matrix $\mathbf{W}(a,t) \in \mathcal{S}$. This matrix is introduced as a reaction to an internal constraint. This reaction yields its value to be given by the equations of motion and not by the constitutive laws. Thus we are going to have two unknowns and two equations.

3.2. **Which Gradient Theory?** The 2D solid can be loaded on its surface by curvilinear beams. The velocities of the beams are equal to the traces on the solid surface of the solid body velocities. Principle of virtual power for beams requires the second order space derivatives on the surface. Thus it is convenient to have a third gradient body theory which insures that the trace of the second gradient is defined on the surface of the solid. Note also that the 2D solid may also be loaded by needles. The velocity of the tip of the needle is equal to the trace of the body velocity. The trace of the zero gradient is of course defined on the surface of the solid if the trace of the second gradient is defined. We have the same property for the trace of the first gradient for loads applied by wires.

3.3. **The Third Gradient Theory.** Power of the interior forces involves third order space derivatives introducing a new interior force $\mathbf{Z}$, which is a stress taking into account the effects of the spatial variation of the Laplacian of the velocity

$$-\int_{\mathcal{D}_a} \left\{ \mathbf{\Pi} : \operatorname{grad} \vec{V} + \mathbf{Z} : \operatorname{grad} \Delta\vec{V} \right\} da + \int_{\mathcal{D}_a} \frac{1}{2} \left\{ \mathbf{M} : \widehat{\mathbf{\Omega}} - \Lambda_\alpha : \widehat{\mathbf{\Omega}}_{,\alpha} \right\} da,$$

with

$$\mathbf{Z} : \operatorname{grad} \Delta\vec{V} = \mathbf{Z}_{i\delta} V_{i,\beta\beta\delta}.$$

Here $\mathbf{\Pi}$ is the Piola-Kirchhoff-Boussinesq stress tensor, $\mathbf{M}$ represents the momentum, and $\mathbf{\Lambda}$ is the momentum flux vector. The virtual velocities $\vec{V}$ and virtual angular velocities $\widehat{\mathbf{\Omega}}$ are independent. Quantity

$$\operatorname{grad} \Delta\vec{V} = \operatorname{grad}(\operatorname{div}(\operatorname{grad} \vec{V}))$$

quantifies the variation of the Laplacian of the velocity of deformation $\Delta\vec{V}$ with respect to space. One may say it quantifies the diffusion of the velocity of deformation. The dual quantity is a stress (represented by the matrix $\mathbf{Z}_{i\delta}$) the physical meaning of which is given by the boundary conditions and the equation of motion.

Note that

$$\mathbf{\Pi} : \mathrm{grad}\,\vec{V} \quad \text{and} \quad \mathbf{Z} : \mathrm{grad}\,\Delta\vec{V}$$

are bilinear forms in $\mathcal{M} \times \mathcal{M}$ and

$$\frac{1}{2}\,\mathbf{M} : \widehat{\mathbf{\Omega}}, \quad \frac{1}{2}\,\Lambda_\alpha : \widehat{\mathbf{\Omega}}_{,\alpha}$$

are bilinear forms in $\mathcal{A} \times \mathcal{A}$ and in $\mathcal{A}^2{\times}\mathcal{A}^2$. We denote by

$$\Lambda :: G = \Lambda_\alpha : G_\alpha,$$

the scalar product in $\mathcal{A}^2{\times}\mathcal{A}^2$.

3.4. **The Linear Momentum Equation.** It results from integration by parts. We have

$$-\int_{\mathcal{D}_a} \left\{\mathbf{\Pi} : \mathrm{grad}\,\vec{V} + \mathbf{Z} : \mathrm{grad}\,\Delta\vec{V}\right\} da = -\int_{\mathcal{D}_a} \{\mathbf{\Pi}_{i\alpha} V_{i,\alpha} + \mathbf{Z}_{i\delta} V_{i,\beta\beta\delta}\}\, da$$

$$= \int_{\mathcal{D}_a} \mathbf{\Pi}_{i\alpha,\alpha} V_i + \mathbf{Z}_{i\delta,\beta\beta\delta} V_i\, da$$

$$-\int_{\partial\mathcal{D}_a} \mathbf{\Pi}_{i\alpha} N_\alpha V_i d\Gamma - \int_{\partial\mathcal{D}_a} \mathbf{Z}_{i\delta} V_{i,\beta\delta} N_\beta d\Gamma + \int_{\partial\mathcal{D}_a} \mathbf{Z}_{i\delta,\beta} V_{i,\delta} N_\beta d\Gamma - \int_{\partial\mathcal{D}_a} \mathbf{Z}_{i\delta,\beta\beta} V_i N_\delta d\Gamma$$

$$= \int_{\mathcal{D}_a} \{\mathrm{div}\,\mathbf{\Pi} + \mathrm{div}\,(\Delta\mathbf{Z})\} \cdot \vec{V}\, da$$

$$-\int_{\partial\mathcal{D}_a} \mathbf{\Pi}\vec{N} \cdot \vec{V} d\Gamma - \int_{\partial\mathcal{D}_a} \mathbf{Z} : \frac{\partial}{\partial N}(\mathrm{grad}\,\vec{V}) d\Gamma + \int_{\partial\mathcal{D}_a} \frac{\partial\mathbf{Z}}{\partial N} : \mathrm{grad}\,\vec{V} d\Gamma - \int_{\partial\mathcal{D}_a} (\Delta\mathbf{Z})\vec{N} \cdot \vec{V} d\Gamma,$$

where $\vec{N}$ is the normal vector to the boundary. This formula gives the physical meaning of $\Delta\mathbf{Z}$, which is a stress defined in the volume with a trace $(\Delta\mathbf{Z})\vec{N}$ on the boundary, which is a classical force which equilibrates with classical force $\mathbf{\Pi}\vec{N}$, the exterior force. Quantity $\partial\mathbf{Z}/\partial N$ is a surface stress which equilibrates an exterior stress. Quantity $\mathbf{Z}$ is a double stress which works with the normal variation of the velocity of deformation $\partial(\mathrm{grad}\,\vec{V})/\partial N$. The equation of motion follows:

$$\frac{d\vec{U}}{dt} = \mathrm{div}\,\mathbf{\Pi} + \mathrm{div}\,(\Delta\mathbf{Z}) + \vec{f}, \ \text{ in } \mathcal{D}_a,$$

where

$$\vec{U} = \frac{d\Phi}{dt} = \dot{\Phi}$$

is the actual velocity. It is combined with suitable initial and boundary conditions.

3.4.1. *The Initial Conditions.* The initial velocity is null

$$\Phi(a,0) = a, \quad \vec{U}(a,0) = \frac{d\Phi}{dt}(a,0) = 0.$$

3.4.2. *The Boundary Conditions.* Let Γ_0, Γ_1 be a partition of $\partial\mathcal{D}_a$. The solid is clamped on part Γ_0 to an immobile support; on the other part Γ_1 no surface force is applied and there is no surface deformation.

On Γ_0. We have

$$\dot{\Phi} = 0, \quad \mathrm{grad}\,\dot{\Phi} = 0, \quad \frac{\partial}{\partial N}\mathrm{grad}\dot{\Phi} = 0.$$

On Γ_1. We have

$$\operatorname{grad} \dot{\Phi} = 0.$$

All the exterior forces are assumed to be null except the body force $\vec{f}$.

3.5. **The Angular Momentum Equation.** It is

$$\operatorname{div} \Lambda + \mathbf{M} = 0, \text{ in } \mathcal{D}_a,$$

combined with suitable boundary condition.

3.5.1. *The Boundary Condition.* On $\partial\mathcal{D}_a$, we have

$$\mathbf{\Omega} = 0,$$

where $\mathbf{\Omega}$ is the actual angular velocity.

4. Kinematic Relationships

They relate the stretch matrix $\mathbf{W}$ and rotation matrix $\mathbf{R}$ to the gradient matrix $\mathbf{F}$ of the kinematically admissible position Φ. A kinematically admissible position is differentiable and $\det \mathbf{F} > 0$, with $\mathbf{F} = \operatorname{grad} \Phi$. This condition is a local impenetrability condition. Let us recall that we have assumed there are no self-collision neither self-contact during the motion which may produce a non local interpenetration. Thus the non local impenetrability condition is satisfied, see [7, 8].

We have

Proposition 3. *If position Φ is kinematically admissible, there are unique matrices $\mathbf{W}$ and $\mathbf{R}$ which satisfy relationships ($\operatorname{grad} \Phi = \mathbf{F}$)*

$$\mathbf{W}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{R} = \mathbf{F}\mathbf{W}^{-1}.$$

Proof. See Proposition 1. □

In the sequel, the constitutive laws imply that the kinematically admissible positions Φ are such that matrices $\mathbf{W}(\mathbf{F}) \in C_\alpha$

$$\mathbf{W}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{W} \in C_\alpha, \tag{6}$$

$$\mathbf{R} = \mathbf{F}\mathbf{W}^{-1}. \tag{7}$$

Let us recall our notation

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}}, \quad \dot{\mathbf{R}} = \frac{d\mathbf{R}}{dt}, \quad \dot{\mathbf{W}} = \frac{d\mathbf{W}}{dt}, \quad \vec{U} = \frac{d\Phi}{dt} = \dot{\Phi}.$$

Note that we have

$$\mathbf{R}^T \operatorname{grad} \dot{\Phi} = \dot{\mathbf{W}} + \mathbf{R}^T \mathbf{\Omega}\mathbf{F}. \tag{8}$$

Remark 4. *We may denote*

$$\mathbf{W} = \sqrt{\mathbf{F}^T\mathbf{F}}.$$

5. Free Energy and Pseudo-Potential of Dissipation

The opposite of the actual power of the internal forces, using internal constraint (8)

$$\begin{aligned}&\Big\{\mathbf{\Pi} : \mathrm{grad}\, \vec{U} + \mathbf{Z} : \mathrm{grad}\, \Delta\vec{U}\Big\} - \frac{1}{2}\left\{\mathbf{M} : \mathbf{\Omega} - \Lambda_\alpha : \mathbf{\Omega}_{,\alpha}\right\}\\ &= \mathbf{R}^T\mathbf{\Pi} : \dot{\mathbf{W}} + \frac{1}{2}\left(\mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T - \mathbf{M}\right) : \mathbf{\Omega} + \frac{1}{2}\Lambda :: \mathrm{grad}\, \mathbf{\Omega} + \mathbf{Z} : \mathrm{grad}\, \Delta\vec{U},\end{aligned}$$

is the sum of bilinear forms. In particular, bilinear form

$$\mathbf{R}^T\mathbf{\Pi} : \dot{\mathbf{W}},$$

is a bilinear form on $\mathcal{M} \times \mathcal{M}$ because $\mathbf{\Pi}$ and $\mathbf{R}^T\mathbf{\Pi}$ are elements of $\mathcal{M}$. Thus the symmetry of stretch matrix $\mathbf{W}$ or of its velocity $\dot{\mathbf{W}}$, is an internal constraint and has to be accounted for. We use the free energy to satisfy this internal constraint. No such internal constraint appears in the other three bilinear forms,

$$\frac{1}{2}\left(\mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T - \mathbf{M}\right) : \mathbf{\Omega}, \quad \frac{1}{2}\Lambda :: \mathrm{grad}\, \mathbf{\Omega}., \quad \mathbf{Z} : \mathrm{grad}\, \Delta\vec{U}$$

which are bilinear forms on $\mathcal{A} \times \mathcal{A}$, on $\mathcal{A}^2 \times \mathcal{A}^2$ and on $\mathcal{M} \times \mathcal{M}$, without any constraint on $\mathbf{\Omega}$ and on $\mathrm{grad}\, \Delta\vec{U}$.

The state variables quantify the deformation properties of the material. We choose the stretch matrix $\mathbf{W}$, an objective spatial variation of the rotation matrix, that is $(\mathrm{grad}\, \mathbf{R})\mathbf{R}^T$, and the third gradient deformation $\mathrm{grad}\, \Delta\Phi$. The quantities which describe the evolution of the material are the stretch matrix velocity $\dot{\mathbf{W}}$ and the gradient of the angular velocity $\mathrm{grad}\, \mathbf{\Omega}$. All these quantities measure the mechanical influence of a material point on its neighbourhood. We derive the constitutive laws from the free energy and the pseudo-potential of dissipation which account for the whole physical properties, in particular for the internal constraint. The schematic and simple free energy we choose is

$$\begin{aligned}\Psi(\mathbf{B}, \mathrm{grad}\, \Delta\hat{\Phi}, \|\mathrm{grad}\, \widehat{\mathbf{R}}\|^2) &= \frac{1}{2}\|\mathbf{B} - \mathbf{I}\|^2\\ &+ \frac{1}{2}\|\mathrm{grad}\, \Delta\hat{\Phi}\|^2 + \hat{\Psi}(\mathbf{B}) + I_{\mathcal{S}}(\mathbf{B}) + \frac{1}{4}\|\mathrm{grad}\, \widehat{\mathbf{R}}\|^2,\end{aligned}$$

where $\hat{\Phi}$ is a position function, $\mathbf{B}$ is a matrix of $\mathcal{M}$, $\widehat{\mathbf{R}}$ is a matrix of $\mathcal{M}$, and

$$\|\mathbf{B}\|^2 = \mathbf{B} : \mathbf{B}, \quad \|\mathrm{grad}\, \Delta\hat{\Phi}\|^2 = \hat{\Phi}_{i,\alpha\beta\beta}\hat{\Phi}_{i,\alpha\delta\delta}.$$

The function $I_{\mathcal{S}}$ is the indicator function of subspace $\mathcal{S}$ of $\mathcal{M}$, [11], [5]. The schematic and simple pseudo-potential of dissipation we choose is

$$D(\dot{\mathbf{B}}, \mathrm{grad}\, \widehat{\mathbf{\Omega}}) = \frac{1}{2}\|\dot{\mathbf{B}}\|^2 + \frac{1}{4}\|\mathrm{grad}\, \widehat{\mathbf{\Omega}}\|^2.$$

Let us note, it is also possible to have $I_{\mathcal{S}}(\dot{\mathbf{B}})$ in the pseudo-potential of dissipation, in order to have matrix $\dot{\mathbf{W}}$ symmetric.

5.1. **The Function $\hat{\Psi}(\mathbf{B})$ Approximation of the Indicator Function of C_α.** Quantity $\hat{\Psi}(\mathbf{B})$ in the free energy accounts for the resistance of the material to flattening or to crushing. It makes impossible all the principal stretches of matrix $\mathbf{W}$ to be small at the same time, i.e., all the principal stretches cannot be lower than $\alpha > 0$. Parameter α quantifies this resistance to flattening. Function $\hat{\Psi}(\mathbf{B})$ is a smooth approximation from the interior of the indicator function of the set C_α in $\mathcal{M}$. Let $I^{\det}(x)$ be a decreasing non negative smooth approximation of the indicator function of $[\alpha^2, \infty)$ from the interior, i.e., such that $I^{\det}(x) = \infty$ if $x \le \alpha^2$ (for instance, $1/(x-\alpha^2)$ for $x > \alpha^2$). Let $I^{\mathrm{tr}}(x)$ be a decreasing non negative smooth approximation of the indicator function of $[2\alpha, \infty)$ from the interior, i.e., such that such $I^{\mathrm{tr}}(x) = \infty$ if $x \le 2\alpha$. Then function $\hat\Psi$ may be defined by

$$\hat\Psi : \mathcal{M} \to [0,\infty], \qquad \hat\Psi(\mathbf{B}) := \begin{cases} I^{\det}(\det \mathbf{B}) + I^{\mathrm{tr}}(\mathrm{tr}\,\mathbf{B}), & if\ \mathbf{B} \in \mathring{C}_\alpha, \\ \infty, & if\ \mathbf{B} \notin \mathring{C}_\alpha. \end{cases} \tag{9}$$

Note that we may split free energy into two parts

$$\Psi(\mathbf{B}, \mathrm{grad}\,\Delta\hat\Phi, \left\|\mathrm{grad}\,\widehat{\mathbf{R}}\right\|^2) = \bar\Psi(\mathbf{B}, \mathrm{grad}\,\Delta\hat\Phi, \left\|\mathrm{grad}\,\widehat{\mathbf{R}}\right\|^2) + I_{\mathcal{S}}(\mathbf{B}),$$

where

$$\bar\Psi(\mathbf{B}, \mathrm{grad}\,\Delta\hat\Phi, \left\|\mathrm{grad}\,\widehat{\mathbf{R}}\right\|^2) = \frac{\mathbf{1}}{\mathbf{2}}\left\|\mathbf{B}-\mathbf{I}\right\|^{\mathbf{2}} + \frac{1}{2}\left\|\mathrm{grad}\,\Delta\hat\Phi\right\|^2 + \hat\Psi(\mathbf{B}) + \frac{1}{4}\left\|\mathrm{grad}\,\widehat{\mathbf{R}}\right\|^2$$

is a smooth function in $\mathring{C}_\alpha$.

The following result holds.

Proposition 5. *If the functions $I^{\det}(x)$ and $I^{\mathrm{tr}}(x)$ are C^∞ functions for $x > \alpha^2$ and for $x > 2\alpha$, then free energy $\mathbf{F} \to \hat\Psi(\mathbf{W}(\mathbf{F})) : \mathcal{M}_\alpha \to \mathbb{R}$ is a C^∞ function of $\mathbf{F}$. Matrix $\mathbf{W}$ commutes with matrix $\frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W})$*

$$\mathbf{W}\frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W}) = \frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W})\mathbf{W}.$$

If $\mathbf{W} \in \mathring{C}_\gamma$ with $1 > \gamma > \alpha$ and $|\mathbf{W}(a,t)| \le c$, all the derivatives with respect to $\mathbf{F}$ of $\frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W})$ are globally bounded. Function $\hat\Psi$ satisfies $\hat\Psi(\mathbf{I}) < \infty$, and

$$\hat\Psi(\mathbf{W}) < \infty \ \Leftrightarrow\ \mathbf{W} \in \mathring{C}_\alpha.$$

Remark 6. *We may choose function $\hat\Psi$ such that $\hat\Psi(\mathbf{I}) = 0$.*

Proof. We have

$$\frac{d\det\mathbf{B}}{d\mathbf{B}} = \mathrm{cof}\ \mathbf{B} = \det(\mathbf{B})\mathbf{B}^{-T}, \qquad \frac{d\mathrm{tr}\,\mathbf{B}}{d\mathbf{B}} = \mathbf{I}.$$

We get

$$\begin{aligned} \frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W}) &= \frac{dI^{\det}}{d\det\mathbf{B}}(\det\mathbf{W})\frac{d\det\mathbf{B}}{d\mathbf{B}}(\mathbf{W}) + \frac{dI^{\mathrm{tr}}}{d\mathrm{tr}\,\mathbf{B}}(\mathrm{tr}\,\mathbf{W})\frac{d\mathrm{tr}\,\mathbf{B}}{d\mathbf{B}}(\mathbf{W}) \\ &= \left(\frac{dI^{\det}}{d\det\mathbf{B}}(\det\mathbf{W})\right)(\det\mathbf{W})\mathbf{W}^{-\mathbf{1}} + \left(\frac{dI^{\mathrm{tr}}}{d\mathrm{tr}\,\mathbf{B}}(\mathrm{tr}\,\mathbf{W})\right)\mathbf{I}, \end{aligned} \tag{10}$$

which commutes with $\mathbf{W}$. We have

$$\det\mathbf{W} = \det\mathbf{F}, \quad \mathrm{tr}\,\mathbf{W} = \sqrt{\mathrm{tr}\,\mathbf{F}^T\mathbf{F} + 2\det\mathbf{F}}.$$

Due to Proposition 2, these functions are C^∞ function of $\mathbf{F}$ in $\mathcal{M}_\alpha$. Because $\frac{dI^{\det}}{dx}(x)$ is a is a C^∞ function of x if $x > \gamma^2 > \alpha^2$ and $\frac{dI^{\mathrm{tr}}}{dy}(x)$ is a C^∞ function of x if $x > 2\gamma > 2\alpha$, $\frac{dI^{\det}}{dx}(\det\mathbf{W}(a,t))$ and $\frac{dI^{\mathrm{tr}}}{dy}(\mathrm{tr}\,\mathbf{W}(a,t))$ are C^∞ functions of $\mathbf{F}$. They are bounded in $\mathbb{R}$ because $|\mathbf{W}| \le c$. It results that $\frac{d\hat\Psi}{d\mathbf{B}}(\mathbf{W})$ is a C^∞ function of $\mathbf{F}$ in $\mathring{C}_\gamma$ and its derivatives with respect to $\mathbf{F}$ are globally bounded. □

6. The Clausius-Duhem Inequality and the Constitutive Laws

The Clausius-Duhem inequality is

$$\frac{d\Psi}{dt}(\mathbf{W}, \operatorname{grad}\Delta\Phi, \left\|\operatorname{grad}\mathbf{R}\right\|^2) \leq \left\{\mathbf{\Pi} : \operatorname{grad}\vec{U} + \mathbf{Z} : \operatorname{grad}\Delta\vec{U}\right\} - \frac{1}{2}\left\{\mathbf{M} : \mathbf{\Omega} - \Lambda_\alpha : \mathbf{\Omega}_{,\alpha}\right\}.$$

It involves actual quantities. Thus these quantities satisfy the kinematic relationships. By using internal constraint (8) which give

$$\operatorname{grad}\vec{U} = \mathbf{R}\dot{\mathbf{W}} + \mathbf{\Omega}\mathbf{F},$$

we get the equivalent inequality

$$\begin{aligned} &\frac{d\Psi}{dt}(\mathbf{W}, \operatorname{grad}\Delta\Phi, \left\|\operatorname{grad}\mathbf{R}\right\|^2) \\ &\leq \mathbf{R}^T\mathbf{\Pi} : \dot{\mathbf{W}} + \mathbf{\Pi}\mathbf{F}^T : \mathbf{\Omega} + \mathbf{Z} : \operatorname{grad}\Delta\vec{U} - \frac{1}{2}\left\{\mathbf{M} : \mathbf{\Omega} - \Lambda_\alpha : \mathbf{\Omega}_{,\alpha}\right\} + \mathbf{Z} : \operatorname{grad}\Delta\vec{U} \\ &= \mathbf{R}^T\mathbf{\Pi} : \dot{\mathbf{W}} + \frac{1}{2}\left(\mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T - \mathbf{M}\right) : \mathbf{\Omega} + \frac{1}{2}\Lambda : \operatorname{grad}\mathbf{\Omega} + \mathbf{Z} : \operatorname{grad}\Delta\vec{U}. \end{aligned}$$

We have

$$\begin{aligned} &\frac{d\Psi}{dt}(\mathbf{W}, \operatorname{grad}\Delta\Phi, \left\|\operatorname{grad}\mathbf{R}\right\|^2) \\ &= \frac{d\bar{\Psi}}{dt}(\mathbf{W}, \operatorname{grad}\Delta\Phi, \left\|\operatorname{grad}\mathbf{R}\right\|^2) + \partial\mathcal{I}_{\mathcal{S}}(\mathbf{W}) : \dot{\mathbf{W}} \\ &= \left\{(\mathbf{W} - \mathbf{I}) + \frac{d\hat{\Psi}}{d\mathbf{B}}(\mathbf{W}) + \mathbf{A}\right\} : \dot{\mathbf{W}} + \operatorname{grad}\Delta\Phi :: \operatorname{grad}\Delta\vec{U} + \frac{(\operatorname{grad}\mathbf{R})\mathbf{R}^T}{2} : \operatorname{grad}\mathbf{\Omega}, \end{aligned}$$

with $\mathbf{A} \in \partial\mathcal{I}_{\mathcal{S}}(\mathbf{W}) = \mathcal{A}$. As matrix $\mathbf{W}$ is symmetric, we find out that

$$\begin{aligned} 0 \leq &\left\{\mathbf{R}^T\mathbf{\Pi} - (\mathbf{W} - \mathbf{I}) - \frac{d\hat{\Psi}}{d\mathbf{B}}(\mathbf{W}) + \mathbf{A}\right\} : \dot{\mathbf{W}} + \frac{1}{2}\left(\mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T - \mathbf{M}\right) : \mathbf{\Omega} \\ &+ \frac{1}{2}\left(\Lambda - (\operatorname{grad}\mathbf{R})\mathbf{R}^T\right) : \operatorname{grad}\mathbf{\Omega} + (\mathbf{Z} - \operatorname{grad}\Delta\Phi) :: \operatorname{grad}\Delta\vec{U}, \end{aligned}$$

where

$$\left((\operatorname{grad}\mathbf{R})\mathbf{R}^T\right)_{ij\alpha} = R_{i\beta,\alpha}R_{\beta j}.$$

The pseudo-potential of dissipation we have chosen assumes dissipation with respect to stretch velocity $\dot{\mathbf{W}}$ and the spatial variation of the angular velocity $\mathbf{\Omega}$. Then, we obtain

- the constitutive laws

$$\mathbf{\Pi} = \mathbf{R}(\mathbf{S} + \mathbf{A}), \quad \mathbf{S} \in \mathcal{S}, \quad \mathbf{A} \in \mathcal{A}, \tag{11}$$

$$\mathbf{S} = (\mathbf{W} - \mathbf{I}) + \frac{d\hat{\Psi}}{d\mathbf{B}}(\mathbf{W}) + \dot{\mathbf{W}}, \quad \mathbf{A} \in \partial\mathcal{I}_{\mathcal{S}}(\mathbf{W}) = \mathcal{A}. \tag{12}$$

  Note that the constitutive law for stress $\mathbf{A}$

$$\mathbf{A} \in \partial I_{\mathcal{S}}(\mathbf{W}) \tag{13}$$

  means that matrix $\mathbf{A}$ is antisymmetric. Stress $\mathbf{A}$ ensures that the stretch matrix $\mathbf{W}$ is symmetric. This reaction matrix is an important quantity of the theory.

Position Φ and reaction matrix $\mathbf{A}$ are the main unknowns of the problem. We have

$$\mathbf{A} : \dot{\mathbf{W}} = 0,$$

because $\dot{\mathbf{W}}$ is symmetric and $\mathbf{A}$ is antisymmetric. As expected, reaction $\mathbf{A}$ is a workless reaction, *un vincolo perfetto* in Italian. Constitutive law (13) does not give more information besides the antisymmetry of $\mathbf{A}$. As usual, the indetermination on workless reaction $\mathbf{A}$ is solved by the equations of motion. Stress $\mathbf{S}$ is symmetric. Stress $d\hat{\Psi}/d\mathbf{B}$ is the impenetrability reaction which intervenes to avoid flattening of the material. Stress $\dot{\mathbf{W}}$ is dissipative. The constitutive law (12) implies that $\mathbf{W} \in \mathring{C}_\alpha$ because function $\hat{\Psi}(\mathbf{B})$ is differentiable for $\mathbf{B} = \mathbf{W}$.

- It is impossible to have dissipation with respect to the angular velocity $\mathbf{\Omega}$ which is non objective. Then constitutive law for $\mathbf{M}$ is the usual relationship

$$\mathbf{M} = \mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T; \tag{14}$$

- the constitutive law for momentum flux Λ

$$\Lambda = (\operatorname{grad} \mathbf{R})\mathbf{R}^T + \operatorname{grad} \mathbf{\Omega}, \tag{15}$$

is dissipative, dissipation resulting from $\operatorname{grad} \mathbf{\Omega}$;

- the constitutive law for $\mathbf{Z}$

$$\mathbf{Z} = \operatorname{grad} \Delta\Phi, \tag{16}$$

is non dissipative.

Remark 7. *It is easy to verify that the constitutive laws are objective, as it follows for instance from results of* [1]. *The dissipated power*

$$0 \leq \dot{\mathbf{W}} : \dot{\mathbf{W}} + \frac{1}{2} \operatorname{grad} \mathbf{\Omega} :: \operatorname{grad} \mathbf{\Omega},$$

which intervenes in the entropy balance

$$\frac{\partial s}{\partial t} + \operatorname{div} \vec{Q} = \frac{1}{T} \left\{ \dot{\mathbf{W}} : \dot{\mathbf{W}} + \frac{1}{2} \operatorname{grad} \mathbf{\Omega} :: \operatorname{grad} \mathbf{\Omega} - \operatorname{grad} T \cdot \vec{Q} \right\},$$

is an objective scalar (s the entropy, T the temperature and $\vec{Q} = (Q_\alpha)$ the entropy flux vector, are objective quantities behaving like scalars). There is no dissipation with respect to $\mathbf{\Omega}$ and $\operatorname{grad} \Delta\vec{U}$ which are not objective. In case we assume that $\mathbf{\Omega}$, the rotation with respect to the support Γ_0, is a quantity which describes the deformation of the system made of the solid and the immobile obstacle, matrix $\mathbf{\Omega}$ becomes objective.

The constitutive laws are such that

$$\dot{\mathbf{W}} : \dot{\mathbf{W}} + \frac{1}{2} \operatorname{grad} \mathbf{\Omega} :: \operatorname{grad} \mathbf{\Omega}$$
$$= \left\{ \mathbf{R}^T\mathbf{\Pi} - (\mathbf{W} - \mathbf{I}) - \frac{d\hat{\Psi}}{d\mathbf{B}}(\mathbf{W}) + \mathbf{A} \right\} : \dot{\mathbf{W}} + \frac{1}{2} \left(\mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T - \mathbf{M} \right) : \mathbf{\Omega}$$
$$+ \frac{1}{2} \left(\Lambda - (\operatorname{grad} \mathbf{R})\mathbf{R}^T\right) : \operatorname{grad}\mathbf{\Omega} + (\mathbf{Z} - \operatorname{grad} \Delta\Phi) :: \operatorname{grad} \Delta\vec{U},$$

which is non negative, proving that the Clausius-Duhem inequality is satisfied.

7. The Equations

For the sake of clarity, we make precise the PDE system resulting from the model, using kinematic relationships, the equations of motion and the constitutive laws plus the boundary and initial conditions. We look for Φ and $\mathbf{A}$ solving the following equations

$$\frac{d^2\Phi}{dt^2} = \operatorname{div}\mathbf{\Pi} + \operatorname{div}(\Delta\mathbf{Z}) + \vec{f}, \text{ in } \mathcal{D}_a,$$

$$\operatorname{div}((\operatorname{grad}\mathbf{R})\mathbf{R}^T) + \Delta\mathbf{\Omega} + \mathbf{R}\left\{\mathbf{AW} + \mathbf{WA} + \dot{\mathbf{W}}\mathbf{W} - \mathbf{W}\dot{\mathbf{W}})\right\}\mathbf{R^T} = \mathbf{0}, \text{ in } \mathcal{D}_a,$$

with boundary conditions

$$\dot{\Phi} = 0, \quad \operatorname{grad}\dot{\Phi} = 0, \quad \frac{\partial}{\partial N}\left(\operatorname{grad}\dot{\Phi}\right) = 0, \text{ on } \Gamma_0,$$

$$\operatorname{grad}\dot{\Phi} = 0 \text{ and no exterior force is applied, on } \Gamma_1,$$

in the time interval $(0, T)$, and initial conditions

$$\Phi(a, 0) = a, \quad \frac{d\Phi}{dt}(a, 0) = 0,$$

where

$$\mathbf{F} = \operatorname{grad}\Phi, \quad \mathbf{W} = \sqrt{\mathbf{F}^T\mathbf{F}}, \quad \mathbf{R} = \mathbf{F}\mathbf{W}^{-1},$$

$$\mathbf{\Pi} = \mathbf{R}(\mathbf{S} + \mathbf{A}),$$

$$\mathbf{S} = (\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\Psi}}{\partial\mathbf{B}}(\mathbf{W}), \quad \mathbf{A} \in \partial I_{\mathcal{S}}(\mathbf{W}),$$

$$\mathbf{Z} = \operatorname{grad}\Delta\Phi,$$

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T.$$

Note that the initial and boundary conditions for Φ give the initial condition for $\mathbf{R}$, $\mathbf{R}(a, 0) = \mathbf{I}$, and a Dirichlet boundary condition $\mathbf{R} = \mathbf{I}$ on $\partial\mathcal{D}_a$, see Proposition 10.

Remark 8. *Let us point out that we may replace* $\mathbf{W}$ *by* $\mathbf{W}(\Phi) = \sqrt{\mathbf{F}^T\mathbf{F}}$ *and* $\mathbf{R}$ *by* $\mathbf{R}(\mathbf{\Phi}) = \mathbf{F}\mathbf{W}^{-1}$ *in the previous equations. Thus, we actually get two equations for the two unknowns, position* Φ *and antisymmetric reaction matrix* $\mathbf{A}$.

7.1. The 2D Equations.

In dimension 2, the direct orthogonal matrix $\mathbf{R}$ is defined by an angle θ,

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{17}$$

and the antisymmetric matrix $\mathbf{A}$ may be specified with the help of a function z:

$$\mathbf{A} = z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus, choosing Φ (from which one finds θ thanks to (17) and the definition of $\mathbf{F}$) and function z as unknowns, we are allowed to recover the following equivalent angular momentum system

$$\Delta\dot{\theta} + \Delta\theta + z(w_{11} + w_{22}) + (\dot{w}_{11} - \dot{w}_{22})w_{12} + \dot{w}_{12}(w_{22} - w_{11}) = 0, \text{ in } \mathcal{D}_a, \tag{18}$$

$$\dot{\theta} = 0, \text{ on } \partial\mathcal{D}_a. \tag{19}$$

Note that the properties of free energy $\hat{\Psi}(\mathbf{W})$ ensure that (cf. Proposition 5) matrix $\mathbf{W}$ commutes with matrix $\partial\hat{\Psi}/\partial\mathbf{W}$.

7.2. **Properties of Angular Momentum Equation.** Let us comment that equation (18) actually leads to the reaction $\mathbf{A}$, once θ is determined by $\operatorname{grad}\Phi$. Indeed, due to the fact that, by definition of C_α, $(w_{11}+w_{22}) = \operatorname{tr}\mathbf{W} > 2\alpha$, it easily follows that equation (18) has a unique solution z in $\mathbb{R}$ as it is stated by the following proposition.

Proposition 9. *If $\mathbf{W} \in C_\alpha$ is known and sufficiently smooth, system (18)–(19) gives a unique z (and thus $\mathbf{A}$) depending on $\mathbf{R}$, Ω, $\mathbf{W}$ and $\dot{\mathbf{W}}$ (i.e., θ, $\dot{\theta}$, $\mathbf{W}$ and $\dot{\mathbf{W}}$).*

We can also prove that

Proposition 10. *Boundary condition $\mathbf{\Omega}= \mathbf{0}$ on $\partial\mathcal{D}_a$, which is equivalent to $\mathbf{R} = \mathbf{I}$, is satisfied if*

$$\operatorname{grad}\dot{\Phi} = 0, \quad \textit{on } \partial\mathcal{D}_a.$$

Proof. If $\operatorname{grad}\dot{\Phi} = 0$, we have $\operatorname{grad}\Phi = \mathbf{I}$ because $\operatorname{grad}\Phi = \operatorname{grad} a = \mathbf{I}$ at time $t = 0$. Then, we have that

$$\dot{\mathbf{R}} = \operatorname{grad}\dot{\Phi}\mathbf{W}^{-1} + \operatorname{grad}\Phi\frac{d}{dt}\mathbf{W}^{-1} = 0,$$

owing to

$$\dot{\mathbf{W}} = \frac{d}{dt}\sqrt{\mathbf{F}^T\mathbf{F}} = \frac{d\mathbf{I}}{dt} = 0,$$

which also gives

$$\frac{d}{dt}\mathbf{W}^{-1} = 0.$$

It results that

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T = 0.$$

Relationship $\mathbf{\Omega} = \mathbf{0}$ or $\dot{\mathbf{R}} = 0$ is equivalent to $\mathbf{R} = \mathbf{I}$ due to the initial condition. □

Let us note that the boundary condition for the angular momentum equation results from the boundary conditions for the linear momentum.

7.3. **Variational Formulation of the Equations and Existence of a Solution.** Actually, we are not able to solve directly the PDE system we have summarized at the beginning of this section, due to a lack of regularity of the solutions. Thus, we introduce a weak version as it is stated in the following. However, let us point out that our weak formulation may be read as a mechanical duality between forces and velocities. To this aim, let us define the space of the virtual velocities

$$\mathcal{V}(T) = \Big\{\vec{\varphi} \in L^2(0,T;H^3(\mathcal{D}_a)) : \ \frac{d\vec{\varphi}}{dt} \in L^2(0,T;L^2(\mathcal{D}_a)),$$
$$\vec{\varphi} = 0, \ \operatorname{grad}\vec{\varphi} = 0, \ \frac{\partial}{\partial N}(\operatorname{grad}\vec{\varphi}) = 0, \ \text{on } \Gamma_0, \ \ \operatorname{grad}\vec{\varphi} = 0, \ \text{on } \Gamma_1\Big\},$$

and the spaces of the virtual angular velocities

$$\mathcal{V}_{rv}(T) = \Big\{\hat{\Omega} \in L^2(0,T;H^1(\mathcal{D}_a)) : \ \hat{\Omega} \in \mathcal{A}, \ \ \hat{\Omega} = 0, \ \text{on } \partial\mathcal{D}_a\Big\},$$

or

$$\mathcal{V}_r(T) = \left\{ \xi \in L^2(0,T;H^1(\mathcal{D}_a)) : \ \xi = 0, \text{ on } \partial\mathcal{D}_a \right\},$$

with

$$\text{for every } \hat{\Omega} \in \mathcal{V}_{rv}(T) \text{ there is a unique } \xi \in \mathcal{V}_r(T)$$
$$\text{such that} \quad \hat{\Omega} = \xi \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and viceversa.}$$

For the sake of simplicity, we are using the same symbol X for a Banach space and any power of it. We use the notation $\langle\langle \cdot , \cdot \rangle\rangle$ for the duality pairing between $\mathcal{V}(T)$ and $\mathcal{V}'(T)$, while the duality between $\mathcal{V}_{rv}(T)$ and $\mathcal{V}_r(T)$ and their dual spaces is realized by $\int_0^T \langle \cdot , \cdot \rangle \, dt$, where $\langle \cdot , \cdot \rangle$ is the usual duality pairing between the Hilbert space $H_0^1(\mathcal{D}_a)$ and its dual space. Finally, note that, as usual, $L^2(\mathcal{D}_a)$ is identified with its dual space. The variational formulation of our problem is the following.

Problem (P). We look for the pair (Φ, z) fulfilling

$$\Phi \in L^\infty(0,T;H^3(\mathcal{D}_a)) \cap H^1(0,T;H^1(\mathcal{D}_a)) \cap W^{1,\infty}(0,T;L^2(\mathcal{D}_a)), \tag{20}$$

$$\frac{d^2\Phi}{dt^2} \in \mathcal{V}'(T), \qquad z \in \mathcal{V}_r'(T), \tag{21}$$

$$\Phi(a,0) = a, \quad \frac{d\Phi}{dt}(a,0) = 0, \quad a \in \mathcal{D}_a, \tag{22}$$

$$(\Phi - a) \in \mathcal{V}(T), \quad \text{and for all } \ \vec{\varphi} \in \mathcal{V}(T),$$

$$\begin{aligned} \langle\langle \frac{d^2\Phi}{dt^2}, \vec{\varphi} \rangle\rangle &+ \int_0^T \int_{\mathcal{D}_a} \mathbf{R} \left\{ (\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}) \right\} : \mathrm{grad}\ \vec{\varphi} \, da \, d\tau \\ &+ \int_0^T \frac{1}{2} \langle \mathbf{A}, \mathbf{R}^T \,\mathrm{grad}\ \vec{\varphi} - (\mathrm{grad}\, \vec{\varphi})^T \mathbf{R} \rangle \, d\tau \\ &+ \int_0^T \int_{\mathcal{D}_a} \mathrm{grad}\, \Delta\Phi : \mathrm{grad}\, \Delta\vec{\varphi} \, da \, d\tau = \int_0^T \int_{\mathcal{D}_a} \vec{f} \cdot \vec{\varphi} \, da \, d\tau, \end{aligned} \tag{23}$$

and such that there exists a function θ with

$$\dot{\theta} \in \mathcal{V}_r(T), \quad \theta(a,0) = 0, \quad a \in \mathcal{D}_a, \quad \text{and for all } \ \xi \in \mathcal{V}_r(T),$$

$$\begin{aligned} \int_0^T \int_{\mathcal{D}_a} \left\{ \mathrm{grad}\, (\dot{\theta} + \theta) \cdot \mathrm{grad}\, \xi \right\} da \, d\tau &= \int_0^T \langle z, (w_{11} + w_{22}) \xi \rangle \, d\tau \\ &+ \int_0^T \int_{\mathcal{D}_a} \big((\dot{w}_{11} - \dot{w}_{22}) w_{12} + \dot{w}_{12}(w_{22} - w_{11}) \big) \xi \, da \, d\tau, \end{aligned} \tag{24}$$

where $\hat{\Psi}$ is specified by (9), and

$$\mathbf{W} = \sqrt{\mathbf{F}^T\mathbf{F}}, \quad \mathbf{F} = \mathrm{grad}\, \Phi, \quad \mathbf{R}\mathbf{W} = \mathbf{F}, \tag{25}$$

$$\mathbf{A} = z \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{26}$$

Remark 11. *Concerning the variational equalities* (23) *and* (24)*, we point out that, due to the resulting regularities of* $\mathbf{R}$, $\mathbf{W}$ *and* $\vec{\varphi}$, ξ, *actually* $\mathbf{R}^T \mathrm{grad}\, \vec{\varphi} - (\mathrm{grad}\, \vec{\varphi})^T \mathbf{R}$ *belongs to* $\mathcal{V}_{rv}$ *and* $(w_{11} + w_{22})\xi$ *is in* $\mathcal{V}_r$. *As* $\mathbf{A}$ *is antisymmetric, it turns out that*

$$\mathbf{A} : \mathbf{R}^T \,\mathrm{grad}\ \vec{\varphi} = \mathbf{A} : \frac{\mathbf{R}^T \,\mathrm{grad}\ \vec{\varphi} - (\mathrm{grad}\, \vec{\varphi})^T \mathbf{R}}{2}. \tag{27}$$

Note also that in mechanical parlance the above duality pairings may be understood as integrals.

By use of a suitable Galerkin approximation combined with a priori estimates – passage to the limit techniques (see the next section for details), we can prove the following local in time existence result.

Theorem 12. *Assuming that $\vec{f}$ belongs to $L^\infty(0,T;L^2(\mathcal{D}_a))$, there exists $\hat{T}$ with $0<\hat{T}\le T$, such that the Problem (P) admits a solution in $(0,\hat{T})$. Moreover, the estimates*

$$\|\Phi\|_{L^\infty(0,\hat{T};H^3(\mathcal{D}_a))\cap H^1(0,\hat{T};H^1(\mathcal{D}_a))\cap W^{1,\infty}(0,\hat{T};L^2(\mathcal{D}_a))}\le c,$$
$$\Big\|\frac{d^2\Phi}{dt^2}\Big\|_{\mathcal{V}'(\hat{T})}\le c,$$
$$\|\mathbf{W}\|_{L^\infty(0,\hat{T};H^2(\mathcal{D}_a))\cap H^1(0,\hat{T};L^2(\mathcal{D}_a))}\le c,$$
$$\|\hat{\Psi}(\mathbf{W})\|_{L^\infty(0,\hat{T};L^1(\mathcal{D}_a))}\le c,$$
$$\|\theta\|_{L^\infty(0,\hat{T};H^2(\mathcal{D}_a))\cap H^1(0,\hat{T};H^1(\mathcal{D}_a))}\le c,$$
$$\|\mathbf{R}\|_{L^\infty(0,\hat{T};H^2(\mathcal{D}_a))\cap H^1(0,\hat{T};H^1(\mathcal{D}_a))}\le c,$$
$$\|z\|_{\mathcal{V}_r'(\hat{T})}+\|\mathbf{A}\|_{\mathcal{V}_{rv}'(\hat{T})}\le c,$$

hold for some positive constant c depending on T and on the data of the problem.

8. Proof of the Existence Result

In this section we give some details on the proof of the existence (and stability) result stated by Theorem 12. The outline of the proof is the following. We first apply a Galerkin approximation of our system, proving the existence of a solution for the finite dimensional approximated problem, at each discretized step, depending on a parameter n. Then, we perform some a priori estimates on the solutions, independently of the parameter n. We prove, in particular, that there is a time interval $(0,\hat{T})$ with $0<\hat{T}\le T$, in which, for each step n, there exists a solution to the discrete problem. Finally, we pass to the limit by use of weak(star)-strong compactness results and, after the identification of the limits for nonlinear terms in the system, we prove that the limit problem admits a (weak) solution, at least locally in time.

8.1. **The Galerkin Approximation.** Let

$$\Phi_n(a,t)=x_i(t)\vec{u}_i(a)+a,\quad 0\le i\le 2n+1,$$

with $\vec{u}_i\in\hat{\mathcal{V}}$ and where $\hat{\mathcal{V}}$ is the space of the velocities

$$\hat{\mathcal{V}}=\left\{\vec{\varphi}\in H^3(\mathcal{D}_a):\ \vec{\varphi}=0,\ \frac{\partial}{\partial N}(\operatorname{grad}\vec{\varphi})=0,\ \text{on }\Gamma_0,\ \ \operatorname{grad}\vec{\varphi}=0,\ \text{on }\partial\mathcal{D}_a\right\}.\tag{28}$$

The $\vec{u}_i(a)$ (actually belonging to $C^3(\mathcal{D}_a)$), $0\le i\le 2n+1$, span the linear space $\hat{\mathcal{V}}_n\subset\hat{\mathcal{V}}$. Note that $\hat{\mathcal{V}}_n\subset\hat{\mathcal{V}}_m$ for $n<m$ and that $\cup_{n\in\mathbb{N}}\hat{\mathcal{V}}_n$ is dense in $\hat{\mathcal{V}}$. We choose

$$\vec{u}_{2k}(a)=\begin{bmatrix}0\\ \varphi_k(a)\end{bmatrix},\quad \vec{u}_{2k+1}(a)=\begin{bmatrix}\varphi_k(a)\\ 0\end{bmatrix},\quad 0\le k\le n.$$

where $\varphi_k \in C^3(\mathcal{D}_a)$ are known. For $x(t) = (x_i(t))_{0\le i\le 2n+1}$ we define $\tilde{\Phi}_n(a, x(t)) = \Phi_n(a,t)$ and observe that the quantity $\left\|\tilde{\Phi}_n(a,x) - \vec{a}\right\|_{H^3(\mathcal{D}_a)}$ is a norm of x in $\mathbb{R}^{2n+2}$. We have

$$\dot{\Phi}_n(a,t) = \dot{x}_i(t)\vec{u}(a) =: \hat{\Phi}_n(a,\dot{x}),$$

and observe that $\left\|\hat{\Phi}_n(a,\dot{x})\right\|_{L^2(\mathcal{D}_a)}$ is a norm of $\dot{x}$ in $\mathbb{R}^{2n+2}$. We aim to approximate matrices $\mathbf{F}_n(a,t)$ and $\mathbf{W}_n(a,t)$ as well. We have

$$\operatorname{grad}\Phi_n(a,t) = \mathbf{F}_n(a,t) = \mathbf{I} + x_i(t)\operatorname{grad}\vec{u}_i(a) =: \tilde{\mathbf{F}}_n(a, x(t)).$$

Hence, we infer that

$$\mathbf{R}_n(a,t) = \mathbf{R}(\mathbf{F}_n(a,t)) = \mathbf{R}(\tilde{\mathbf{F}}_n(a,x(t))) =: \tilde{\mathbf{R}}_n(a,x(t)),$$

and let

$$\mathbf{W}_n(a,t) =: \tilde{\mathbf{W}}_n(a,x(t)).$$

Before introducing the discrete problem, we state some results on the approximation of functions. For the sake of simplicity, we do not report technical details of the proofs.

The following proposition defines a set $\mathring{C}^n_\alpha$ approximating $\dot{C}_\alpha \cap \mathcal{S}$.

Proposition 13. *Condition $\tilde{\mathbf{W}}_n(a,x) \in \mathring{C}_\alpha \cap \mathcal{S}$ is equivalent to $x \in \mathring{C}^n_\alpha$, where $\mathring{C}^n_\alpha$ is an open set of $\mathbb{R}^{2n+2}$ which contains vector $x = 0$.*

Now, let us comment on the functions $\tilde{\mathbf{W}}_n(a,x)$, $\tilde{\mathbf{R}}_n(a,x)$ approximating $\mathbf{W}(\mathbf{F})$, $\mathbf{R}(\mathbf{F})$. Due to Proposition 2 and the fact that matrix $\mathbf{F}$ is a linear function of vector x depending on a we can prove the following result.

Proposition 14. *If $\left\|\tilde{\Phi}_n(a,x) - \vec{a}\right\|_{H^3(\mathcal{D}_a)} \le c$ and $x \in \mathring{C}^n_\alpha$, functions $x \to \tilde{\mathbf{R}}_n(a,x)$ and $x \to \tilde{\mathbf{W}}_n(a,x)$ are C^∞ functions of x. All the derivatives with respect to x are globally bounded. In particular $x \to \tilde{\mathbf{R}}_n(a,x)$ and $x \to \tilde{\mathbf{W}}_n(a,x)$ are, uniformly with respect to a, Lipschitz functions of x.*

Finally, we state properties concerning the functions $\frac{\partial \tilde{\mathbf{R}}_n}{\partial t}(a,x,\dot{x})$ and $\frac{\partial \tilde{\mathbf{W}}_n}{\partial t}(a,x,\dot{x})$ that approximate $\dot{\mathbf{R}}(\mathbf{F},\dot{\mathbf{F}})$ and $\dot{\mathbf{W}}(\mathbf{F},\dot{\mathbf{F}})$.

Proposition 15. *If $\left\|\tilde{\Phi}_n(a,x) - \vec{a}\right\|_{H^3(\mathcal{D}_a)} \le c$, $\dot{\Phi}_n \in L^\infty(0,T_n; L^2(\mathcal{D}_a))$, i.e.,*

$$\left\|\hat{\Phi}_n(a,\dot{x})\right\|_{L^2(\mathcal{D}_a)} \le c \quad \text{and} \quad x \in \mathring{C}^n_\alpha,$$

functions $x,\dot{x} \to \frac{\partial \tilde{\mathbf{R}}_n}{\partial t}(a,x,\dot{x}) = \dot{\mathbf{R}}_n(a,t)$ and $x,\dot{x} \to \frac{\partial \tilde{\mathbf{W}}_n}{\partial t}(a,x,\dot{x}) = \dot{\mathbf{W}}_n(a,t)$ are C^∞ functions of $x,\dot{x}$. All the derivatives with respect to $x,\dot{x}$ are uniformly bounded with respect to a and $x,\dot{x}$. In particular $x,\dot{x} \to \frac{\partial \tilde{\mathbf{R}}_n}{\partial t}(a,x,\dot{x})$ and $x,\dot{x} \to \frac{\partial \tilde{\mathbf{W}}_n}{\partial t}(a,x,\dot{x})$ are, uniformly with respect to a, Lipschitz functions of $x,\dot{x}$.

Now, let us approximate function z as follows.

Proposition 16. *If $\left\|\tilde{\Phi}_n(a,x) - \vec{a}\right\|_{H^3(\mathcal{D}_a)} \le c$, $\dot{\Phi}_n \in L^\infty(0,T_n; L^2(\mathcal{D}_a))$, i.e.,*

$$\left\|\hat{\Phi}_n(a,\dot{x})\right\|_{L^2(\mathcal{D}_a)} \le c \quad \text{and} \quad x \in \mathring{C}^n_\alpha,$$

the relationship

$$z_n(a,t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{-1}{(w_{n11}(a,t)+w_{n22}(a,t))}\Big\{\Big(\mathrm{div}\,\big((\mathrm{grad}\,\mathbf{R}_n)\mathbf{R}_n^T\big)\Big)(a,t) + (\mathrm{div}\,(\mathrm{grad}\,\dot{\mathbf{R}}_n\mathbf{R}_n^T))(a,t))$$
$$+\,\mathbf{R}_n(a,t)\Big(\dot{\mathbf{W}}_n(a,t)\mathbf{W}_n(a,t)-\mathbf{W}_n(a,t)\dot{\mathbf{W}}_n(a,t)\Big)\mathbf{R}_n^T(a,t)\Big\} \tag{29}$$

defines a function

$$\tilde{z}(a,x(t),\dot{x}(t))\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that $\tilde{z}$ is a C^∞ function of x and $\dot{x}$. All the derivatives with respect to x and $\dot{x}$ are uniformly bounded with respect to a, x and $\dot{x}$. In particular, $\tilde{z}(a,x,\dot{x})$ is a Lipschitz function of x and $\dot{x}$ (uniformly with respect to to a).

To deal with the approximating function $\frac{d\hat{\Psi}}{d\mathbf{W}}(\tilde{\mathbf{W}}_n(a,x))$ of $\frac{d\hat{\Psi}}{d\mathbf{W}}(\mathbf{W})$ we exploit the following result.

Proposition 17. *If $\big\|\tilde{\Phi}_n(a,x)-\vec{a}\big\|_{H^3(\mathcal{D}_a)} \le c$ and $x \in \mathring{C}^n_\gamma$ with $\gamma > \alpha$, the function $\frac{d\hat{\Psi}}{d\mathbf{W}}(\mathbf{W_n}) = \frac{d\hat{\Psi}}{d\mathbf{W}}(\tilde{\mathbf{W}}_n)$ is a C^∞ function of $x \in \mathring{C}^n_\gamma$. All the derivatives with respect to x are globally bounded. In particular, $x \to \frac{d\hat{\Psi}}{d\mathbf{W}}(\tilde{\mathbf{W}}_n(a,x))$ is, uniformly with respect to a, a Lipschitz function of x.*

Analogously, we aim to approximate $\mathrm{grad}\,\Delta\Phi$.

Proposition 18. *If $\big\|\tilde{\Phi}_n(a,x)-\vec{a}\big\|_{H^3(\mathcal{D}_a)} \le c$ and $x \in \mathring{C}^n_\alpha$, the function*

$$x \to \mathrm{grad}\,\Delta\Phi_n(a,x) = (x_k \mathrm{grad}\,\Delta\vec{u}_k(a))$$

is, uniformly with respect to a, a linear Lipschitz function of x.

Now, we are in the position of introducing the finite dimensional approximated problem as follows (it is written in a suitable variational formulation and $\tilde{z}$ depends on x and $\dot{x}$).

$$\forall i, \quad 0 \le i \le 2n+1, \quad \int_{\mathcal{D}_a} \frac{d^2\Phi_n}{dt^2}.\vec{u}_i\,da$$
$$+\int_{\mathcal{D}_a}\tilde{\mathbf{R}}_n\left\{(\tilde{\mathbf{W}}_n-\mathbf{I}) + \frac{d\tilde{\mathbf{W}}_n}{dt}\dot{\mathbf{W}} + \frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\tilde{\mathbf{W}}_n) + \tilde{z}_n\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} : \mathrm{grad}\,\vec{u}_i$$
$$+\,\mathrm{grad}\,\Delta\Phi_n : \mathrm{grad}\,\Delta\vec{u}_i\,da = \int_{\mathcal{D}_a}\vec{f}\cdot\vec{u}_i\,da, \tag{30}$$

combined with (29). Note that (30) leads to

$$\begin{gathered} M\ddot{x} + G_1(x)\dot{x} + G_2(x) = f(t), \\ x(0) = 0,\ \dot{x}(0) = 0, \end{gathered} \tag{31}$$

where M is the mass matrix,

$$f(t) = (f_n^i(t)) = (\int_{\mathcal{D}_a}\vec{f}(\vec{a},t)\cdot\vec{u}_n^i(\vec{a})\,da),$$

and

$$G_1(x)\dot{x} + G_2(x) = \int_{\mathcal{D}_a} \left(\tilde{\mathbf{R}}_n \Big\{ (\tilde{\mathbf{W}}_{\mathbf{n}} - \mathbf{I}) + \frac{d\tilde{\mathbf{W}}_n}{dt} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\tilde{\mathbf{W}}_n) + z_n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Big\} : \operatorname{grad} \vec{u}_i \right.$$
$$\left. + \operatorname{grad} \Delta\Phi_{\mathbf{n}} : \operatorname{grad} \Delta\vec{u}_i \right) da.$$

Recall that we assume

$$\vec{f} \in L^\infty(0,T; L^2(\mathcal{D}_a)). \tag{32}$$

Let vectors $v \in \mathbb{R}^{2n+2}$, $\hat{v} \in \mathbb{R}^{2n+2}$ and define $\Phi_n(v) = v^i \vec{u}^i_n(\vec{a}) + \vec{a}$, $\hat{\Phi}_n(\hat{v}) = \hat{v}^i \vec{u}^i_n(\vec{a})$. The set

$$E = \Big\{ (v,\hat{v}) \in \left(\mathbb{R}^{2n+2}\right)^2 : \left\| \Phi_n(v) - \vec{a} \right\|_{H^3(\mathcal{D}_a)} < c_1,$$
$$\mathbf{W}_n(\Phi_n(v)) \in \mathring{C}_\gamma, \quad \left\| \hat{\Phi}_n(\hat{v}) \right\|_{L^2\ (\Omega_a)} < c_2 \Big\}, \tag{33}$$

with $1 > \gamma > \alpha$, is an open set which contains $(0,0)$.

Remark 19. *Let us recall that we have chosen the norms* $v \to \left\| \Phi_n(v) - \vec{a} \right\|_{H^3(\mathcal{D}_a)}$ *and* $\hat{v} \to \left\| \hat{\Phi}_n(\hat{v}) \right\|_{L^2(\mathcal{D}_a)}$ *in* $\mathbb{R}^{2n+2}$ *because they are going to be useful when dealing with the a priori estimates.*

We observe that the function $(v,\hat{v}) \to G_1(v)\hat{v} + G_2(v)$, is a Lipschitz function.

As a consequence of the above argument, the differential equation (31) has a solution in time interval $[0,T_n]$, with $0 < T_n \le T$, which satisfies

$$\left\| \Phi_n(t) - \vec{a} \right\|_{H^3\ (\Omega_a)} < c_1, \quad \left\| \dot{\Phi}_n(t) \right\|_{L^2\ (\Omega_a)} < c_2, \quad \mathbf{W}_n(\vec{a},t) \in \mathring{C}_\gamma.$$

Time T_n depends on c_1, c_2, n and γ.

8.2. **The a Priori Estimates.** We perform suitable a priori estimates on the approximated problem given by (29), (30) (written in terms of z_n and Φ_n), actually not depending on n. We point out that, for the moments, the existence of a solution for the discrete problem is given, for each n, in $(0,T_n)$. Now, we test (29) by $(w_{n11} + w_{n22})\dot{\theta}_n)/2$, and (30) (written for z_n and Φ_n) by $\dot{\Phi}_n$, and integrate over $(0,t)$. We add the two relationships. Some terms cancel and we can integrate by parts in time and get

$$\frac{1}{2}\int_{\mathcal{D}_a} \dot{\Phi}_n^2(t)\, da + \int_{\mathcal{D}} \left\{ \frac{1}{2} \left\| \mathbf{W}_n(t) - \mathbf{I} \right\|^2 + \hat{\Psi}(\mathbf{W}_n(t)) \right\} da + \int_0^t \|\dot{\mathbf{W}}_n\|^2_{L^2(\mathcal{D}_a)} d\tau$$
$$+ \frac{1}{2}\int_{\mathcal{D}_a} \left\| \operatorname{grad} \Delta\Phi_n(t) \right\|^2 da$$
$$+ \frac{1}{2}\int_{\mathcal{D}_a} \left\| \operatorname{grad} \theta_n(t) \right\|^2 da + \int_0^t \| \operatorname{grad} \dot{\theta}_n \|^2_{L^2(\mathcal{D}_a)} d\tau$$
$$= \int_0^t \int_{\mathcal{D}_a} \vec{f} \cdot \dot{\Phi}_n \, da\, d\tau + \int_{\mathcal{D}_a} \hat{\Psi}(\mathbf{I})\, da. \tag{34}$$

Remark 20. *Note that*

$$\operatorname{grad} \Delta\Phi_n = \Delta \operatorname{grad} \Phi_n = (\Phi_n)_{i,\alpha\beta\beta} = (\Phi_n)_{i,\beta\beta\alpha}.$$

Thus $\Delta(\operatorname{grad} \Phi_n)$ *is bounded in* $L^\infty(0,T_n; L^2(\mathcal{D}_a))$. *With boundary condition* $\operatorname{grad} \Phi_n = \mathbf{I}$ *on* $\partial\mathcal{D}_a$ *we deduce that* $\operatorname{grad} \Phi_n$ *is bounded in* $L^\infty(0,T_n; H^2(\mathcal{D}_a))$.

Remark 21. *Relationship (34) is the integral with respect to time of the principle of virtual power where the velocities are the actual ones.*

We directly get the following estimates

$$
\begin{gathered}
\|\Phi_n\|_{W^{1,\infty}(0,T_n;L^2(\mathcal{D}_a))\cap L^\infty(0,T_n;H^3(\mathcal{D}_a))} \le c, \\
\|\mathbf{W}_n\|_{H^1(0,T_n;L^2(\mathcal{D}_a))} \le c, \\
\|\hat{\Psi}(\mathbf{W}_n)\|_{L^\infty(0,T_n;L^1(\mathcal{D}_a))} \le c, \\
\|\theta_n\|_{H^1(0,T_n;H^1(\mathcal{D}_a))} \le c.
\end{gathered}
$$

Moreover, with the a priori estimates and the above propositions (see, in particular, Proposition 2), it results that

$$
\begin{gathered}
\|\Phi_n\|_{H^1(0,T_n;H^1(\mathcal{D}_a))} \le c, \\
\|\theta_n\|_{L^\infty(0,T_n;H^2(\mathcal{D}_a))} \le c, \\
\|\mathbf{W}_n\|_{L^\infty(0,T_n;H^2(\mathcal{D}_a))} \le c, \\
\|\mathbf{R}_n\|_{L^\infty(0,T_n;H^2(\Omega_a))\cap H^1(0,T_n;H^1(\Omega_a))} \le c.
\end{gathered}
$$

Indeed, these a priori estimates follows by the fact that once Φ_n is bounded, as $\mathbf{W}_n$ and $\mathbf{R}_n$ are smooth functions of $\operatorname{grad}\Phi_n$, they are bounded, too.

Actually, the bounds do not depend on n, but (for the moment) they may depend on T_n. However, we will show in a while that we can find some uniform bounds on a interval $(0,\hat{T})$ for the solutions, with $\hat{T}$ independent of n. Hence, we are allowed to write

$$
z_n = -\frac{\Delta\dot{\theta}_n + \Delta\theta_n}{(w_{n11}+w_{n22})} - \frac{(\dot{w}_{n11}-\dot{w}_{n22})w_{n12} + \dot{w}_{n12}(w_{n22}-w_{n11})}{(w_{n11}+w_{n22})},
$$

where

$$
0 < \frac{1}{(w_{n11}+w_{n22})} \le \frac{1}{2\alpha}.
$$

Let us test the previous equality by $\xi \in \mathcal{V}_r$, by inferring that

$$
\begin{aligned}
\int_0^t \int_{\mathcal{D}_a} z_n \xi \, da\, d\tau &= \int_0^t \int_{\Omega_a} (\operatorname{grad}\dot{\theta}_n + \operatorname{grad}\theta_n)\cdot \operatorname{grad}\frac{\xi}{(w_{n11}+w_{n22})}\, da\, d\tau \\
&\quad - \int_0^t \int_{\mathcal{D}_a} \frac{(\dot{w}_{n11}-\dot{w}_{n22})w_{n12} + \dot{w}_{n12}(w_{n22}-w_{n11})}{(w_{n11}+w_{n22})}\xi\, da\, d\tau,
\end{aligned}
$$

where

$$
\operatorname{grad}\frac{\xi}{(w_{n11}+w_{n22})} = \frac{\operatorname{grad}\xi}{(w_{n11}+w_{22})} - \xi\frac{\operatorname{grad}(w_{n11}+w_{n22})}{(w_{n11}+w_{n22})^2}.
$$

Moreover, in view of $(w_{n11}+w_{n22}) > 2\alpha$, as the term $\operatorname{grad}(w_{n11}+w_{n22})$ is bounded in $L^\infty(0,T_n;H^1(\mathcal{D}_a))$, we have

$$
\Big\|\operatorname{grad}\frac{\xi}{(w_{n11}+w_{n22})}\Big\|_{L^2(0,T_n;L^2(\mathcal{D}_a))} \le c\|\xi\|_{L^2(0,T_n;H^1(\mathcal{D}_a))}.
$$

We also have

$$\left|\int_0^t\int_{\mathcal{D}_a} \frac{(\dot w_{n11}-\dot w_{n22})w_{n12}+\dot w_{n12}(w_{n22}-w_{n11})}{(w_{n11}+w_{n22})}\xi\, da\, d\tau\right|$$
$$\leq \int_0^t\int_{\mathcal{D}_a} \frac{|\dot w_{n11}-\dot w_{n22}|\,|w_{n12}|+|\dot w_{n12}|\,|w_{n22}-w_{n11}|}{2\alpha}\,|\xi|\; da\, d\tau$$
$$\leq c\big\|w_n\big\|_{L^\infty(0,T_n;L^\infty(\Omega_a))}\big\|\dot w_n\big\|_{L^2(0,T_n;L^2(\Omega_a))}\big\|\xi\big\|_{L^2(0,T_n;H^1(\Omega_a))} \leq c\big\|\xi\big\|_{L^2(0,T_n;H^1(\Omega_a))}.$$

Since $(\operatorname{grad}\dot\theta+\operatorname{grad}\theta)$ is bounded in L^2, we deduce that

$$\left|\int_0^t\int_{\mathcal{D}_a} z_n\xi\, da\, d\tau\right| \leq c\big\|\xi\big\|_{L^2(0,T_n;H^1(\mathcal{D}_a))},$$

so that z_n is bounded in $\mathcal{V}_r'(T_n)$, and $\mathbf{A}_n$ in $\mathcal{V}_{rv}'(T_n)$.

Now, let us point out that the a priori estimates on Φ_n and $\dot\Phi_n$ show that T_n does not depend on c_1 and c_2 (see (33)) provided they are taken large enough. We let $\hat\Psi$ satisfy $\hat\Psi(\mathbf{I})=0$. Then, from (34) it follows that

$$\frac{1}{2}\int_{\mathcal{D}_a}\dot\Phi_n^2(t)\, da \leq \int_0^t\int_{\mathcal{D}_a}\vec f\cdot\dot\Phi_n\, da\, d\tau,$$

which gives, along with (32) and the Gronwall lemma,

$$\big\|\dot\Phi_n(t)\big\|_{L^2(\mathcal{D}_a)} \leq ct.$$

With this estimate, from (34) we infer

$$\frac{1}{2}\,\big\|\Delta(\operatorname{grad}\Phi_n(t))\big\|^2_{L^2(\mathcal{D}_a)}\, da \leq \int_0^t\int_{\mathcal{D}_a}\vec f\cdot\dot\Phi_n\, da\, d\tau \leq ct^2.$$

We know that

$$\operatorname{grad}\Phi_n(t)=\mathbf{I},\quad \text{on } \partial\mathcal{D}_a,$$

and therefore

$$|\mathbf{F}_n(a,t)-\mathbf{I}| = |\operatorname{grad}\Phi_n(a,t)-\mathbf{I}|$$
$$\leq c\big\|\Delta(\operatorname{grad}\Phi_n(t))\big\|_{L^2(\mathcal{D}_a)} = c\big\|\Delta(\operatorname{grad}\Phi_n(t)-\mathbf{I})\big\|_{L^2(\mathcal{D}_a)} \leq ct.$$

Then, for $t<t_1$ we have

$$1+ct \geq |\mathbf{F}_n(a,t)| \geq 1-ct \geq \delta > 0,$$
$$|\det\mathbf{F}_n(a,t)-\det\mathbf{I}| = |\det\mathbf{F}_n(a,t)-1| \leq c\,|\mathbf{F}_n(a,t)-\mathbf{I}| \leq ct,$$

because the derivatives of $\det\mathbf{F}$ with respect to $\mathbf{F}$ are bounded. Thus, for $t<t_2<t_1$ we see that

$$\det\mathbf{F}_n(a,t)=\det\mathbf{W}_n(a,t) \geq 1-ct \geq \gamma^2 \geq \alpha^2.$$

As a consequence, for $t<t_2$ we have $\det\mathbf{F}\geq\gamma^2$ and

$$\operatorname{tr}\mathbf{W}=\sqrt{\operatorname{tr}\mathbf{F}^T\mathbf{F}+2\det\mathbf{F}},\qquad |\operatorname{tr}\mathbf{W}-\operatorname{tr}\mathbf{I}| = |\operatorname{tr}\mathbf{W}-2| \leq c\,|\mathbf{F}_n(a,t)-\mathbf{I}| \leq ct,$$

due to the boundedness of the derivatives of $\operatorname{tr}\mathbf{W}$ with respect to $\mathbf{F}$ (note that $\operatorname{tr}\mathbf{F}^T\mathbf{F}>0$). Then, we obtain

$$\operatorname{tr}\mathbf{W} \geq 2-ct \geq 2\gamma \geq 2\alpha \quad \text{ for } 0\leq t<t_3<t_2<t_1$$

and for $t<t_3=\hat T$ we have $x_n(t)\in\mathring{C}^n_\gamma$. Thus, there is a solution to the approximated problem (31) up to time $\hat T$ which is independent of n, c_1 and c_2.

Proposition 22. *If assumption (32) is satisfied, there exists $\hat{T} > 0$ such that the approximated problems* (31) *have solutions up to the time $\hat{T}$.*

Remark 23. *As for $t < t_2$ there holds*

$$\operatorname{tr} \mathbf{W} = \sqrt{\operatorname{tr} \mathbf{F}^T \mathbf{F} + 2 \det \mathbf{F}} \geq \sqrt{2 \det \mathbf{F}} \geq \sqrt{2}\, \gamma,$$

it is clear that to ensure the property $\operatorname{tr} \mathbf{W} > 2\alpha$ *we may need to have $t < t_3 \leq t_2$.*

8.3. **Convergence Results.** Using well-known weak, weak*, and strong compactness result on the interval $(0, \hat{T})$, we get the following convergence results, holding at least for subsequences,

$$\Phi_n \rightharpoonup^* \Phi \ \text{ in } W^{1,\infty}(0,\hat{T}; L^2(\mathcal{D}_a)) \cap H^1(0,\hat{T}; H^1(\mathcal{D}_a)) \cap L^\infty(0,\hat{T}; H^3(\mathcal{D}_a)),$$
$$\Phi_n \to \Phi \ \text{ in } C([0,\hat{T}]; H^2(\mathcal{D}_a)),$$
$$\theta_n \rightharpoonup^* \theta \ \text{ in } H^1(0,\hat{T}; H^1(\mathcal{D}_a)) \cap L^\infty(0,\hat{T}; H^2(\Omega_a)),$$
$$\theta_n \to \theta \ \text{ in } C([0,\hat{T}]; H^1(\mathcal{D}_a)),$$
$$\mathbf{R}_n \rightharpoonup^* \mathbf{R} \ \text{ in } L^\infty(0,\hat{T}; H^2(\Omega_a)) \cap H^1(0,\hat{T}; H^1(\Omega_a)),$$
$$\mathbf{R}_n \to \mathbf{R} \ \text{ in } C^0([0,\hat{T}]; H^1(\mathcal{D}_a)),$$
$$\mathbf{W}_n \rightharpoonup^* \mathbf{W} \ \text{ in } H^1(0,\hat{T}; L^2(\mathcal{D}_a)) \cap L^\infty(0,\hat{T}; H^2(\Omega_a)),$$
$$\mathbf{W}_n \to \mathbf{W} \ \text{ in } C^0([0,\hat{T}]; H^1(\mathcal{D}_a)),$$
$$\mathbf{A}_n \rightharpoonup \mathbf{A} \ \text{ in } \mathcal{V}'_{rv}(\hat{T}).$$

Note that if $\vec{\varphi} \in \mathcal{V}$, then we infer that $\mathbf{R}_n^T \operatorname{grad} \vec{\varphi} - (\operatorname{grad} \vec{\varphi})^T \mathbf{R}_n \in \mathcal{V}_{rv}$ or that the non-null element of this antisymmetric matrix is an element of $\mathcal{V}_r$, because

$$\mathbf{R}_n^T \operatorname{grad} \vec{\varphi} - (\operatorname{grad} \vec{\varphi})^T \mathbf{R}_n = 0, \ \text{ on } \partial\mathcal{D}_a.$$

Thus, as $\mathbf{R}_n$ converges strongly in $C^0([0,\hat{T}]; H^1(\mathcal{D}_a))$ and, on the other hand, $\mathbf{A}_n$ converges weakly in $\mathcal{V}'_{rv}(\hat{T})$, the integral

$$\int_0^t \int_{\mathcal{D}_a} \mathbf{R}_n \mathbf{A}_n : \operatorname{grad} \vec{\varphi}\, da\, d\tau = \frac{1}{2} \int_0^t \langle \mathbf{A}_n, \mathbf{R}_n^T \operatorname{grad} \vec{\varphi} - (\operatorname{grad} \vec{\varphi})^T \mathbf{R}_n \rangle d\tau$$

converges too.

Now, we introduce a definition of the limit process.

Definition 24. *We say that the exterior force is not extreme or does not tend to crush the solid if $\hat{\Psi}(\mathbf{W})$ is globally bounded, i.e., if there exists $c > 0$ such that*

$$\forall (a,t), \quad \hat{\Psi}(\mathbf{W}(a,t)) \leq c.$$

We show below that we can select a motion, as long as this assumption is satisfied.

Proposition 25. *Assume that the exterior force $\vec{f}$ satisfies* (32)*; then there exists $\hat{T} > 0$ such that the exterior force is not extreme in the time interval $\left[0, \hat{T}\right]$.*

Proof. From Proposition 22, we have that

$$\det \mathbf{W}(a,t) \geq \gamma^2 \geq \alpha^2, \qquad \operatorname{tr} \mathbf{W}(a,t) \geq 2\gamma \geq 2\alpha.$$

Then, it follows that

$$\hat{\Psi}(\mathbf{W}(a,t)) = I^{\det}(\det \mathbf{W}(a,t)) + I^{\mathrm{tr}}(\mathrm{tr}\,\mathbf{W}(a,t)) \leq I^{\det}(\gamma^2) + I^{\mathrm{tr}}(2\gamma) = c.$$

□

Note that assumption (32) is satisfied by any practical loading which does not crush too much the solid.

Proposition 26. *For all* (a,t) *fulfilling* $\hat{\Psi}(\mathbf{W}(a,t)) \leq c$ *and* $|\mathbf{W}(a,t)| \leq c$, *there hold*

$$\left| \mathbf{R}_n \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}_n)(a,t) \right| \leq \hat{c} \quad \textit{for some constant } \hat{c} > 0. \tag{35}$$

Proof. If $\hat{\Psi}(\mathbf{W}(a,t)) \leq c$, there exist γ such that $tr\mathbf{W}_n > 2\gamma > 2\alpha$ and $\det \mathbf{W}_n > \gamma^2 > \alpha^2$. Then (35) follows from formula (10) and the fact that $\mathbf{W}_n(a,t)$ is bounded. □

The property (35) is satisfied in the time interval $[0,\hat{T}]$ as we have that

$$\mathrm{tr}\,\mathbf{W}_n > 2\gamma > 2\alpha, \quad \det \mathbf{W}_n > \gamma^2 > \alpha^2, \quad \text{and} \quad |\mathbf{W}(a,t)| \leq c.$$

Thanks to relationship (35) and using the approximated linear momentum equation of motion, we get for $\vec{\varphi}_p \in L^2(0,T;\hat{\mathcal{V}}_n)$

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^{\hat{T}} \int_{\mathcal{D}_a} \frac{d^2\Phi_n}{dt^2} \cdot \vec{\varphi}_p \, da\, d\tau \\
&= - \int_0^{\hat{T}} \int_{\mathcal{D}_a} \mathbf{R}\Big\{ \mathbf{W} - \mathbf{I} + \dot{\mathbf{W}} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}) \Big\} : \mathrm{grad}\,\vec{\varphi}_p \, da\, d\tau \\
&\quad + \int_0^{\hat{T}} \frac{1}{2} \langle \mathbf{A}, \mathbf{R}^T \mathrm{grad}\,\vec{\varphi}_p - \mathrm{grad}\,\vec{\varphi}_p^T \mathbf{R} \rangle d\tau \\
&- \int_0^{\hat{T}} \int_{\mathcal{D}_a} \mathrm{grad}\,\Delta\Phi : \mathrm{grad}\,\Delta\vec{\varphi}_p \, da\, d\tau + \int_0^{\hat{T}} \int_{\mathcal{D}_a} \vec{f} \cdot \vec{\varphi}_p \, da\, d\tau = \langle\langle B, \vec{\varphi}_p \rangle\rangle,
\end{aligned}$$

where B is the element of the dual space of $\mathcal{V}(T)$ defined by

$$\begin{aligned}
\langle\langle B, \vec{\varphi} \rangle\rangle &= - \int_0^{\hat{T}} \int_{\mathcal{D}_a} \mathbf{R}\Big\{ \mathbf{W} - \mathbf{I} + \dot{\mathbf{W}} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}) \Big\} : \mathrm{grad}\,\vec{\varphi} \, da\, d\tau \\
&\quad + \int_0^{\hat{T}} \frac{1}{2} \langle \mathbf{A}, \mathbf{R}^T \mathrm{grad}\,\vec{\varphi} - \mathrm{grad}\,\vec{\varphi}^T \mathbf{R} \rangle d\tau \\
&- \int_0^{\hat{T}} \int_{\mathcal{D}_a} \mathrm{grad}\,\Delta\Phi : \mathrm{grad}\,\Delta\vec{\varphi} \, da\, d\tau + \int_0^{\hat{T}} \int_{\mathcal{D}_a} \vec{f} \cdot \vec{\varphi} \, da\, d\tau.
\end{aligned}$$

This result is not sufficient for the convergence of $d^2\Phi_n/dt^2$ because we cannot exchange the limits $n \to \infty$ and $p \to \infty$ with $\lim_{p\to\infty} \vec{\varphi}_p = \vec{\varphi}$. We couple this result with the convergence for $d\Phi_n/dt$ and point out that $\vec\varphi = \vec\varphi_p + (\vec\varphi - \vec\varphi_p)$. We have

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d^2\Phi_n}{dt^2}\cdot\vec\varphi\,da\,d\tau &= \lim_{n\to\infty}\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d^2\Phi_n}{dt^2}\cdot(\vec\varphi_p + (\vec\varphi - \vec\varphi_p))\,da\,d\tau\\ &= \langle\langle B, \vec\varphi_p\rangle\rangle - \lim_{n\to\infty}\left\{\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d\Phi_n}{dt}\cdot\frac{d(\vec\varphi - \vec\varphi_p)}{dt}\,da\,d\tau\right\}\\ &\qquad + \lim_{n\to\infty}\left\{\int_{\mathcal{D}_a}\frac{d\Phi_n}{dt}(\hat T)\cdot(\vec\varphi(\hat T) - \vec\varphi_p(\hat T))\,da\right\};\end{aligned}$$

and, with the a priori estimates on $d\Phi_n/dt$,

$$\begin{aligned}&-\lim_{n\to\infty}\left\{\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d\Phi_n}{dt}\cdot\frac{d(\vec\varphi - \vec\varphi_p)}{dt}\,da\,d\tau\right\}\\ &\quad = -\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d\Phi}{dt}\cdot\frac{d(\vec\varphi - \vec\varphi_p)}{dt}\,da\,d\tau = \langle\langle\vec C, (\vec\varphi - \vec\varphi_p)\rangle\rangle\end{aligned}$$

and

$$\left|\int_{\mathcal{D}_a}\frac{d\Phi_n}{dt}(\hat T)\cdot(\vec\varphi(\hat T) - \vec\varphi_p(\hat T))\,da\right| \le \left\|\frac{d\Phi_n}{dt}\right\|_{L^\infty(0,T;L^2(\mathcal{D}_a))}\left\|\vec\varphi - \vec\varphi_p\right\|_{\mathcal{V}},$$

where $\vec C$ belongs to the dual space of $\mathcal{V}(T)$. Then, we infer that

$$\left|\lim_{n\to\infty}\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d^2\Phi_n}{dt^2}\cdot\vec\varphi\,da\,d\tau - \langle\langle B, \vec\varphi_p\rangle\rangle\right| \le c\left\|\vec\varphi - \vec\varphi_p\right\|_{\mathcal{V}},$$

for all p, whence

$$\left|\lim_{n\to\infty}\int_0^{\hat T}\int_{\mathcal{D}_a}\frac{d^2\Phi_n}{dt^2}\cdot\vec\varphi\,da\,d\tau - \langle\langle B, \vec\varphi\rangle\rangle\right| = 0,$$

and $\frac{d^2\Phi_n}{dt^2} \to B$, weakly in the dual space of $\mathcal{V}(T)$. Then $d^2\Phi_n/dt^2$ is bounded in the dual space of $\mathcal{V}(T)$ and $\dot\Phi_n$ converges strongly in $C([0,T]; L^2(\mathcal{D}_a))$.

Thus, we are now in the position of passing to the limit in (23), (24) (written for n) and solve Problem (P). Of course, the kinematic relationship grad $\Phi = \mathbf{RW}$ is satisfied.

Remark 27. *Due to regularities we have obtained, and by a comparison in* (23), *the term*

$$\langle\langle\frac{d^2\Phi}{dt^2}, \vec\varphi\rangle\rangle$$

is linear and bounded with respect to $\vec\varphi$ varying in $L^2(0,\hat T;\hat{\mathcal{V}})$, where $\hat{\mathcal{V}}$ is defined in (28). *Then, by using a density argument in* (23), *it is not difficult to check that*

$$\left\|\frac{d^2\Phi}{dt^2}\right\|_{L^2(0,\hat T;\hat{\mathcal{V}})} \le c,$$

and that the variational in equality (23) *can be equivalently rewritten as*

$$
\begin{aligned}
&{}_{\hat{\mathcal{V}}'}\langle \frac{d^2\Phi}{dt^2}(t), \vec{\varphi}\rangle_{\hat{\mathcal{V}}} + \int_{\mathcal{D}_a} \mathbf{R}\left\{(\mathbf{W}-\mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\mathbf{W})\right\}(t) : \operatorname{grad}\vec{\varphi}\, da \\
&+\langle \mathbf{A}(t), \mathbf{R}^T(t)\operatorname{grad}\vec{\varphi} - (\operatorname{grad}\vec{\varphi})^T\mathbf{R}(t)\rangle + \int_{\mathcal{D}_a} \operatorname{grad}\Delta\Phi(t) : \operatorname{grad}\Delta\vec{\varphi}\, da \\
&= \int_{\mathcal{D}_a} \vec{f}(t)\cdot\vec{\varphi}\, da \quad \textit{for all } \vec{\varphi}\in\hat{\mathcal{V}}, \ \ \textit{for a.e. } t\in(0,\hat{T}).
\end{aligned} \tag{36}
$$

9. What Occurs When Time $T < \hat{T}$. A Mechanical Remark

After time T, there may be at some time t, a null measure set, for instance a curve, where

$$\left|\mathbf{R}\frac{d\hat{\Psi}}{d\mathbf{B}}(\mathbf{W})(a,t)\right| = \infty, \quad \hat{\Psi}(\mathbf{W})(a,t) = \infty,$$

together with

$$\int_{\mathcal{D}_a} \hat{\Psi}(\mathbf{W})\, da \le c.$$

On this curve, it happens that

$$\mathbf{W}(a,t) \in \partial C_\alpha.$$

and the velocity

$$\dot{\mathbf{W}}(a,t)$$

cannot be continuous with respect to time because $\mathbf{W}(a,t)$ has to remain in C_α. Thus there is a time discontinuity of the velocities $\dot{\Phi}$ and Ω satisfying

$$\mathbf{R}^T\operatorname{grad}\left[\dot{\Phi}\right] = \left[\dot{\mathbf{W}}\right] + \mathbf{R}^T\left[\boldsymbol{\Omega}\right]\mathbf{F},$$

owing to relationship (8) (the brackets denote the time discontinuities). A collision occurs. The predictive theory we have chosen does not take into account collisions. Thus, it is a mechanical phenomenon which makes impossible to have a solution, i.e., a smooth motion, after time T.

Let us stress that before time T the deformations may be large up to $-(1-\gamma)$. This means negative elongation up to $-99,999..\%$, which is enough for many practical problems. We may say that if the exterior forces are not extreme, we can predict the motion of a solid with large deformations. In any case, we predict the motion when it begins and before a possible flattening or crushing, for instance when you crush fresh pasta between two fingers.

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