A simple proof of the projectivity of Kontsevich’s space of maps

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Sunto: Si dà una semplice dimostrazione della proiettività della compattificazione dello spazio delle mappe da curve algebriche a spazi proiettivi recentemente introdotta da Kontsevich.

The stacks of stable maps from curves to projective space have been introduced by Kontsevich [5][6]. It has been observed by several people that the underlying algebraic spaces are in fact projective. A proof can be found in [7]. Here we wish to present a simple proof based on the methods of [1]. We work over \( \mathbb{C} \) throughout.

Consider a complete, connected, reduced curve \( C \) whose singularities are at worst nodes, \( n \) smooth numbered points \( x_1, \ldots, x_n \) on \( C \), and a morphism \( \mu : C \to \mathbb{P}^r \). According to Kontsevich, one says that the datum of \( C, x_1, \ldots, x_n, \mu \) is a stable map if the following condition is satisfied. Let \( E \) be a smooth component of \( C \) such that \( \mu(E) \) is a point; if the genus of \( E \) is zero (resp., one) then \( E \) contains at least three (resp., one) points which are among the \( x_i \) or are singular in \( C \) but not in \( E \). A family of stable maps is a flat proper morphism \( f : C \to S \) together with \( n \) sections \( \sigma_i : S \to C, i = 1, \ldots, n \) and a morphism \( \mu : C \to \mathbb{P}^r \) such that, for every \( s \in S \), \( (f^{-1}(s), \sigma_1(s), \ldots, \sigma_n(s), \mu_{|f^{-1}(s)}) \) is a stable map. One has obvious notions of pullback and of isomorphism between families of stable maps.

Let \( F = (C, x_1, \ldots, x_n, \mu) \) be a stable map of degree \( d \). If \( Q \) is a sufficiently general member of \( |\mathcal{O}_{\mathbb{P}^r}(3)| \), then \( \mu^*(Q) = \sum p_i \) is a divisor consisting of \( 3d \) smooth points of \( C \), each occurring with multiplicity one. Furthermore, \( \Gamma = (C, x_1, \ldots, x_n, p_1, \ldots, p_{3d}) \) is a stable \( (n + 3d) \)-pointed curve. We then have an exact sequence of groups

\[
1 \to G \to \text{Aut}(F) \to G',
\]

where \( G = \text{Aut}(\Gamma) \cap \text{Aut}(F) \) and \( G' \) is the group of permutations of \( p_1, \ldots, p_{3d} \). This shows that there is an upper bound for the order of \( \text{Aut}(F) \) which depends only on \( d, n, \) and the genus \( g \) of \( C \). The fact that \( \Gamma \) is stable also implies that the number of singular points of \( C \) is bounded by \( 3g - 3 + n + 3d \).

Fix non-negative integers \( g, n, r, d \). Then the functor

\[
\mathcal{F}(S) = \left\{ \text{families of stable maps of degree } d \text{ from } n \text{-pointed genus } g \text{ curves to } \mathbb{P}^r \right\} / \text{isomorphisms}
\]

is coarsely represented by a complete separated algebraic space \( \overline{M}_{g,n}(r,d) \) (cf. [5][7]). Clearly, \( \overline{M}_{g,n}(r,d) \) is non-empty if and only if \( 2g - 2 + n + 3d > 0 \), and \( d = 0 \) for

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$r = 0$. We wish to show that $\overline{M}_{g,n}(r,d)$ is projective. This is clear if $d = 0$. In fact, $\overline{M}_{g,n}(r,0) = \overline{M}_{g,n} \times \mathbb{P}^r$, where $\overline{M}_{g,n}$ is the usual moduli space of stable $n$-pointed genus $g$ curves, and we know that $\overline{M}_{g,n}$ is projective.

For $d > 0$ we argue as follows. For any family

$$C \xrightarrow{\mu} \mathbb{P}^r$$

$$f \bigg|\bigg| \sigma_i, i = 1, \ldots, n$$

$S$

of stable maps of degree $d$ from $n$-pointed curves of genus $g$ to $\mathbb{P}^r$, which we denote by $F$, set

$$L_F = \omega_f(\sum D_i) \otimes \mu^*\mathcal{O}(3),$$

where $\omega_f = \omega_{C/S}$ is the relative dualizing sheaf and $D_i = \sigma_i(S)$. We also set

$$\mathcal{L}_F = \langle L_F, L_F \rangle,$$

where $\langle , \rangle$ is Deligne’s bilinear symbol (cf. [2][3]); $\mathcal{L}_F$ is a line bundle on $S$ which behaves nicely under base change. Therefore this construction defines a line bundle $\mathcal{L}$ on the moduli stack of stable maps of degree $d$ from $n$-pointed curves of genus $g$ to $\mathbb{P}^r$. Since, as we observed, the orders of the automorphisms groups of such maps are bounded, $\mathcal{L}$ can be viewed as a fractional line bundle on $\overline{M}_{g,n}(r,d)$. We shall prove the following result.

**Theorem 1.** For any choice of non-negative integers $g$, $n$, $r$, and $d$ such that

$$2g - 2 + n + 3d > 0,$$

$$d > 0 \text{ if } r > 0,$$

$\mathcal{L}$ is ample on $\overline{M}_{g,n}(r,d)$.

Notice that, for $r = d = 0$, the theorem reduces to the well-known statement that Mumford’s class $\kappa_1$ is ample on $\overline{M}_{g,n}$ (cf. [1], for instance). The first step in the proof is to observe that there is a family $G$ of stable maps of degree $d$ from $n$-pointed curves of genus $g$ to $\mathbb{P}^r$. Since, as we observed, the orders of the automorphisms groups of such maps are bounded, $\mathcal{L}$ can be viewed as a fractional line bundle on $\overline{M}_{g,n}(r,d)$. We shall prove the following result.

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$$\nu : Z \rightarrow \overline{M}_{g,n}(r,d)$$

is finite. A proof of this is sketched for instance in [7], based on a modification of a construction of Kollár [4]. To show that $\mathcal{L}$ is ample it suffices to show that $\nu^*(\mathcal{L}) = \mathcal{L}_G$ is ample on $Z$. In order to prove this we shall use Seshadri’s criterion. In other terms, we shall show that there is a positive constant $\alpha$ such that, for any integral complete curve $\Gamma$ in $Z$, one has

$$(\mathcal{L}_G \cdot \Gamma) \geq \alpha m(\Gamma),$$

where $m(\Gamma)$ stands for the maximum multiplicity of points of $\Gamma$. Since the intersection number $(\mathcal{L}_G \cdot \Gamma)$ is the degree of $\mathcal{L}_G$, where $G'$ is the pullback of $G$ via the inclusion $\Gamma \subset Z$, we will be done if we can show that there is a positive constant $\alpha$ such that $\deg \mathcal{L}_F \geq \alpha m(S)$ for any non-isotrivial family $F$ of stable maps of degree $d$ from $n$-pointed genus $g$ curves to $\mathbb{P}^r$ parametrized by an integral complete curve $S$. Here non-isotrivial means that the moduli map $S \rightarrow \overline{M}_{g,n}(r,d)$ does not send $S$ to a point. Taking into account the definition of $\mathcal{L}_F$, what needs to be proved is
**Lemma 2.** If \(2g - 2 + n + 3d > 0\) and \(d > 0\) or \(r = d = 0\), there is a positive constant \(\alpha = \alpha(g, n, r, d)\) such that, for any non-isotrivial family \(F\) of degree \(d\) stable maps from \(n\)-pointed genus \(g\) curves to \(\mathbb{P}^r\) over an integral complete curve \(S\),

\[
(L_F \cdot L_F) \geq \alpha m(S).
\]

The proof is essentially by reduction to the known case \(r = d = 0\). From now on we assume that \(d > 0\). Let the family \(F\) be given by maps \(f : C \to S\), \(\mu : C \to \mathbb{P}^r\) and sections \(\sigma_i : S \to C\), \(i = 1, \ldots, n\). We begin by reducing to the case when the general fiber of \(f\) is smooth. Denote by \(\Sigma(F)\) the union of all one-dimensional components of the locus of nodes in the fibers of \(f\), and by \(\pi_F : N(F) \to C\) the normalization of \(C\) along \(\Sigma(F)\). Let \(\psi : S' \to S\) be a finite unramified base change, and let

\[
\begin{array}{ccc}
C' & \xrightarrow{\mu'} & \mathbb{P}^r \\
\downarrow f' & & \downarrow \sigma'_i, i = 1, \ldots, n \\
S' & &
\end{array}
\]

be the pullback family, which we call \(F'\). We can choose \(\psi\) in such a way that \(\pi^{-1}_F(\Sigma(F'))\) is a disjoint union of sections of \(N(F') \to S'\). Moreover, since the number of singular points in the fibers of \(f\) is bounded independently of \(F\), the degree of \(\psi\) can also be chosen to be bounded. Thus, in proving Lemma 2, we may assume that \(\pi^{-1}_F(\Sigma(F))\) is a disjoint union of sections of \(N(F) \to S\). Let \(C_1, \ldots, C_h\) be the connected components of \(N(F)\), set \(\pi_i = \pi_F|_{C_i}\), \(f_i = f\pi_i\), \(\mu_i = \mu\pi_i\). Let \(\sigma_{i,1}, \ldots, \sigma_{i,n_i}\) be the sections of \(f_i\) that come from components of \(\pi^{-1}_F(\Sigma(F))\) lying on \(C_i\) or from sections \(\sigma_j\) such that \(\sigma_j(S)\) lies on \(\pi_F(C_i)\). Then the datum of \(f_i : C_i \to S\), \(\mu_i : C_i \to \mathbb{P}^r\), and \(\sigma_{i,1}, \ldots, \sigma_{i,n_i}\) is a family of stable maps of degree \(d_i\) with the property that the general fiber of \(f_i\) is smooth of genus \(g_i\). It is clear from the definitions that

\[
L_{F_i} = \pi^*_i(L_F),
\]

so that

\[
(L_F \cdot L_F) = \sum (L_{F_i} \cdot L_{F_i}).
\]

Moreover the invariants \(g_i, n_i, d_i\) satisfy the inequalities

\[
g_i \leq g, \quad d_i \leq d, \quad n_i \leq n + 2(3g - 3 + n + 3d).
\]

This shows that it suffices to prove Lemma 2 for families whose general fiber is smooth; in fact, the possible objection that some of the families \(F_i\) might be such that \(d_i = 0\), so that Lemma 2 is false for them if \(r \neq 0\), may be countered as follows. Suppose all the \(F_i\) with \(d_i \neq 0\) are isotrivial, but \(F_j\) is not. Then \(\mu_j(C_j)\) is a single point, so \(f_j : C_j \to S\) is non-isotrivial as a family of stable curves, and one can apply to it Lemma 2 with \(r = d = 0\).

From now on we assume that the general fiber of \(f : C \to S\) is smooth. We set \(D_i = \sigma_i(S)\) for \(i = 1, \ldots, n\). A simple dimension count shows that, if \(\mathcal{H}\) is a sufficiently general hyperplane, then

i) \(\mu^{-1}(\mathcal{H})\) does not contain components of fibers of \(f\);
ii) \(\mu^{-1}(\mathcal{H})\) does not contain singular points of fibers of \(f\);
iii) $\mu^{-1}(\mathcal{H})$ does not contain points of intersection between one of the $D_i$, $i = 1, \ldots, n$, and the fibers of $f$ which are singular or lie above singular points of $S$.

iv) $\mu^{-1}(\mathcal{H})$ does not contain $D_i$ for $i = 1, \ldots, n$;

v) $\mu^{-1}(\mathcal{H})$ cuts transversely all the fibers of $f$ which are singular or lie over singular points of $S$.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ be distinct hyperplanes satisfying i), ii), iii), iv) and v). Possibly after a finite base change of bounded degree, $\mu^{-1}(\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3)$ becomes a sum of distinct sections. Moreover, since $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ satisfy v), we may choose a base change that does not affect $m(S)$. We may thus assume that

$$
\mu^{-1}(\mathcal{H}_1) = D_{n+1} + \cdots + D_{n+d},
$$

$$
\mu^{-1}(\mathcal{H}_2) = D_{n+d+1} + \cdots + D_{n+2d},
$$

$$
\mu^{-1}(\mathcal{H}_3) = D_{n+2d+1} + \cdots + D_{n+3d},
$$

where $D_{n+1}, \ldots, D_{n+3d}$ are distinct sections, different from $D_1, \ldots, D_n$. The family of curves $f : C \to S$, together with the sections $D_1, \ldots, D_{n+3d}$, has all the characters of a family of stable $(n+3d)$-pointed curves, except for the fact that some of the $D_i$ may meet; however, by properties ii) and iii), this may occur only on smooth fibers of $f$ not lying above singular points of $S$. To obtain a family $(f'' : C'' \to S, D''_1, \ldots, D''_{n+3d})$ of semi-stable $(n+3d)$-pointed curves it is necessary to blow up, perhaps repeatedly, the points of intersection of two or more of the $D_i$ and possibly, in genus zero, blow down some exceptional curves of the first kind. At each blow-up, the selfintersection of $\omega_f(\sum_{i=1}^{n+3d} D_i)$ decreases. If $g = 0$ and, at any stage of the process, the (proper transforms of the) $D_i$ all meet at a point $p$ of a smooth fiber $\Gamma$, the proper transform of $\Gamma$ under the blow-up at $p$ is an exceptional curve of the first kind not meeting sections, which needs to be blown down. The blow-down increases the selfintersection of $\omega_f(\sum D_i)$ exactly by one. Thus, in any case

$$
(\omega_f(\sum D_i) \cdot \omega_f(\sum D_i)) \geq (\omega_f''(\sum D''_i) \cdot \omega_f''(\sum D''_i)).
$$

Now, if

$$
F' = (f' : C' \to S, D'_1, \ldots, D'_{n+3d})
$$

is the stable model of $(f'' : C'' \to S, D''_1, \ldots, D''_{n+3d})$, we have that

$$
(\omega_{f'}(\sum D'_i) \cdot \omega_{f'}(\sum D'_i)) = (\omega_{f''}(\sum D''_i) \cdot \omega_{f''}(\sum D''_i)),
$$

so we conclude that

$$
(L_{F'} \cdot L_{F'}) \geq (L_{F'} \cdot L_{F'}).
$$

If $F'$ is not isotrivial, we are done, since $\kappa_1$ is ample on $\overline{M}_{g,n+3d}$. From now on, we assume that $F'$ is isotrivial. In particular, this implies that all the fibers of $f : C \to S$ are smooth. When $g > 0$, $C'$ dominates $C$, so $C' = C$ and the $D_i$ do not meet. Another consequence is that $\mu(C)$ is a surface. To see it, just combine the non-isotriviality of $F$ with the following result.

**Lemma 3.** Let $X$ and $Y$ be smooth curves, denote by $p$ the genus of $Y$, let $U$ be a disk, and let $y_1, \ldots, y_h$ be distinct points of $Y$. Suppose $2p - 2 + h > 0$. Let $\Psi : X \times U \to Y$ be a morphism such that the divisor $\Psi^{-1}(\sum y_i)$ is a sum $\sum \{x_j\} \times U$, where the $x_j$ are $k$ distinct points of $X$. Then, for any $x \in X$, $\Psi(x, u)$ is independent of $u$. 

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To proof uses elementary deformation theory. Let $\psi : X \to Y$ be a morphism such that $\psi^{-1}(\sum y_i) = \sum x_j$. The first order deformations of $\psi$ as a map from the $k$-pointed curve $(X, x_1, \ldots, x_k)$ to $Y$ sending the $x_j$ to the $y_i$ are classified by $H^0(X, F)$, where $F$ stands for $\psi^*(T_Y(-\sum y_i))/T_X(-\sum x_j)$, and those such that the moduli of $(X, x_1, \ldots, x_k)$ do not vary by the image of $H^0(X, \psi^*(T_Y(-\sum y_i)))$ in $H^0(X, F)$. The conclusion follows from the fact that the degree of $\psi^*(T_Y(-\sum y_i))$ is a multiple of $2 - 2p - h$, and hence negative.

For $g > 0$ we reach a contradiction establishing Lemma 2 by noticing that, since $\mu(C)$ is a surface, $H_1 \cap H_2 \cap \mu(C)$ is non-empty, so the $D_i$ cannot be disjoint, contrary to what we established earlier. When $g = 0$, we argue somewhat differently. Since $\mu(C)$ is a surface, by choosing $H_1, H_2$, and $H_3$ to be sufficiently general, we may assume that $\mu(D_i)$ is not a point for $i > n$. Thus, if $i > n$ and $p$ is any point of $D_i$, there is a hyperplane passing through $\mu(p)$ but not containing $\mu(D_i)$. It follows that

$$(\mu^* O(1) \cdot D_i) \geq m(S) \quad \text{for any } i > n.$$ 

Now set

$$\eta_h = \omega_f \left( \sum_{i \leq h} D_i \right).$$

We wish to show that, for any $h \geq 2$ and any section $D$ of $f$,

$$(\eta_h \cdot \eta_h) \geq 0, \quad (\eta_h \cdot D) \geq 0.$$ 

In fact $\eta_2 = O(\sum a_i \Gamma_i)$, where the $\Gamma_i$ are fibers of $f$, so $\sum a_i = (\eta_2 \cdot D_1) = (D_2 \cdot D_1) \geq 0$ and

$$(\eta_2 \cdot \eta_2) = 0, \quad (\eta_2 \cdot D) = \sum a_i \geq 0.$$ 

In general

$$(\eta_h \cdot \eta_h) = (\eta_2 \cdot \eta_2) + \sum_{2<i\leq h} (D_i \cdot \eta_2) + \sum_{2<j\leq h} (D_j \cdot \omega_f(D_j)) + \sum_{i \leq h, 2<j \leq h, i \neq j} (D_i \cdot D_j) \geq 0,$$

while

$$(\eta_h \cdot D) = (\omega_f(D_j) \cdot D_j) + \sum_{0<i\leq h, i \neq j} (D_i \cdot D_j) \geq 0,$$

if $D = D_j$ for some $j \leq h$, and

$$(\eta_h \cdot D) = (\eta_2 \cdot D) + \sum_{2<i\leq h} (D_i \cdot D) \geq 0$$

otherwise. Thus

$$(L_F \cdot L_F) = (\eta_{n+2d} \cdot \eta_{n+2d}) + 2 \sum_{i>n+2d} (D_i \cdot \eta_{n+2d}) + \sum_{i>n+2d} (D_i \cdot \mu^* O(1)) \geq d m(S).$$

This finishes the proof of Lemma 2, and hence of Theorem 1.
Bibliography


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