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SOME TRANSCENDENTAL ASPECTS OF ALGEBRAIC GEOMETRY

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0. INTRODUCTORY REMARKS

The purpose of these seven lectures is to discuss some transcendental aspects of algebraic geometry. Historically, a great deal of the subject was initially developed by analytical and topological methods. This was probably due to the origins of much of algebraic geometry as a branch of complex function theory (Gauss, Abel, Jacobi, Riemann, Weierstrass,

AMS (MOS) subject classifications (1970). Primary 14C30, 14D05, 22E15, 30A42, 32C10, 32C25, 32C30, 32C35, 32G05, 32G20, 32H20, 32H25, 32J25, 32L10, 32M10, 32M15, 32G13, 35D05, 35D10, 35N15, 53B35, 53C30, 53C55, 53C65, 58C10, 58G05, 58G10; Secondary 14F05, 14F10, 14F25, 14H15, 14M15, 30A70, 14F35.

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Poincaré, Picard, etc.). Another possible reason is that the intimate relationship between algebraic geometry and topology is more visible in the complex case (Lefschetz, and more recently the Hirzebruch-Riemann-Roch formula). Finally, the beautiful local and global methods of differential geometry are available in the complex case (Hodge, de Rham, Chern, Kodaira, etc.).

During the last 100 years, and especially in the last 40 years, much of the theory which was initially discovered by analytical methods has, quite properly, been put in a purely algebraic setting (the foundations, Torelli for curves, Riemann-Roch, etc.). Moreover, longstanding problems have been proved by algebraic methods (resolution of singularities). Finally, due primarily to the input from arithmetic, new and striking results over the complex numbers have been suggested and sometimes proved by algebra (Deligne's theory of mixed Hodge structures, the Tate conjectures, etc.). However, because of the fundamental nature of the complex numbers, there remains a transcendental aspect of algebraic geometry which is both essential to an understanding of the subject and quite beautiful for its own sake. In these lectures, we hope to focus on some facets of this theory.

Specifically, we shall concentrate on the following topics as illustrating transcendental algebraic geometry:

a) The Hirzebruch-Riemann-Roch formula for compact, complex manifolds;

b) Hodge theory for a single compact Kähler manifold, and the related vanishing theorems for cohomology, theory of mixed Hodge structures, and homotopy type of Kähler manifolds;

c) variation of Hodge structure culminating in the recent work of Schmid; and

d) the global theory of transcendental holomorphic mappings (Nevanlinna theory) viewed as non-compact algebraic geometry.

The Hirzebruch-Riemann-Roch formula is of course well-known and has a purely algebraic proof in much stronger form. However, it is a basic result first proved by transcendental methods and has inspired a great deal of mathematics over the last 20 years. Moreover, there has recently been given an "elementary" proof by Toledo and Tong, one in which the local and

global properties of the $\bar{\partial}$-operator are brought nicely into focus, and so this proof will be discussed in the second lecture. The complete argument will be presented in the analysis seminar.

Hodge theory for a compact Kähler manifold is again a subject which has been around for quite some time. However, we have deemed it worthwhile to sketch the theory in some detail, emphasizing Chern's conceptual explanation of the plethora of Kähler identities. As applications, we have given Le Potier's recent extension of the Kodaira-Nakano vanishing theorem to vector bundles, and a brief account of Deligne's theory of mixed Hodge structures and the homotopy type of Kähler varieties. Finally, in the belief that many, if not most, algebraic geometers are aware of the formal aspect of Hodge theory but have not been through the grubby analysis, we have (in the appendix to lecture one) given an account of the underlying analysis of the Laplacian which hopefully may appeal to tastes of the algebraists.

The main thrust of these lectures will be to discuss variation of Hodge structure leading up to the recent work of Schmid. Here the methods of complex analysis, Hermitian differential geometry, and Lie group theory blend together to maximally illustrate the flavor of transcendental algebraic geometry. The complete proofs of most of the main results will be covered between the lectures and the accompanying analysis seminar.

We shall also discuss some "non-compact" algebraic geometry. For example, Picard's theorem and its subsequent refinement, the beautiful value distribution theory of R. Nevanlinna, appear naturally as transcendental analogues of the fundamental theorem of algebra. The extension of this theory to transcendental holomorphic curves in $\mathbb{P}^n$ is based on the non-compact Plücker formulae of H. and J. Weyl. Although it is not yet established, one may adopt the philosophy that a global result concerning complex algebraic varieties is not properly understood unless one has analogous results for non-compact manifolds, and this is to some extent the viewpoint we shall take. It is on non-compact varieties that the full richness of the larger class of generally transcendental analytic functions and holomorphic mappings can be perhaps best exploited. An example of this is the consequence of a theorem of Grauert that all of the rational, even dimensional homology on a smooth, affine algebraic variety is representable by analytic subvarieties, although very little of it is so by algebraic cycles. Conversely, the existing theory in the compact case frequently points the way to profound analytical results in the non-compact case, as illustrated by the L^2-methods for studying the $\bar{\partial}$-operator on open manifolds which has developed so fruitfully in the last decade, and whose basic estimate is just the identity Kodaira initially used in his vanishing theorem for a compact Kähler manifold.

GENERAL BIBLIOGRAPHICAL REFERENCES

We shall give a few references to articles and books which were turning points in the development of the subject.

N. Abel, Démonstration d'une propriété générale d'une certaine classe de fonctions transcendentes, J. reine angew. Math., vol. 4 (1829), 200-201.

B. Riemann, Theorie der Abelschen Funktionen, J. reine angew. Math., vol. 54 (1857), 115-155.

E. Picard and G. Simart, Théorie des fonctions algébriques de deux variables independantes (tomes I et II), Gauthier-Villars, Paris, 1892-1906.

H. Weyl, Die Idee der Riemannschen Fläche, Teubner, Berlin (1913); third edition Teubner, Stuttgart (1955).

S. Lefschetz, L'analysis situs et la géométrie algébrique, Gauthier-Villars, Paris (1924).

W. V. D. Hodge, The theory and applications of harmonic integrals, Cambridge Univ. Press, New York (second edition 1952).

S. S. Chern, Characteristic classes of hermitian manifolds, Annals of Math., vol. 47 (1946), 85-121.

K. Kodaira, On Kähler varieties of restricted type, Annals of Math., vol. 60 (1954), 43-76.

F. Hirzebruch, Topological methods in algebraic geometry, Springer-Verlag, New York (1966).

1. GENERALITIES ON COMPLEX MANIFOLDS

A complex manifold M is a manifold M provided with a distinguished open covering $\{U, V, \dots\}$ and coordinate charts $f_U : U \to \mathbb{C}^n$, $f_V : V \to \mathbb{C}^n$, ... such that $f_{UV} = f_U \circ f_V^{-1}$ is biholomorphic where defined. Such manifolds are even-dimensional, oriented, and we shall assume them to be connected. Some standard examples are:

a) $\mathbb{C}^n$ with global coordinates $z = (z_1, \dots, z_n)$;

b) $\mathbb{P}^n$ with homogeneous coordinates $Z = [z_0, \dots, z_n]$,

c) the Grassmannian $G(k, n)$ of k-planes through the origin in $\mathbb{C}^n$,

d) complex tori $T_\Lambda = \mathbb{C}^n/\Lambda$ where Λ is a lattice in $\mathbb{C}^n$;

e) among the spheres, only $S^2 = \mathbb{P}^1$ and possibly S^6 have complex structures; however, the product $S^{2p+1} \times S^{2q+1}$ of odd spheres has a continuous family of complex structures (Calabi-Eckmann);

f) let Q be the skew-form with matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and define $\mathcal{H}_n \subset G(n, 2n)$ to be all subspaces Λ satisfying the Riemann bilinear relations

$$Q(e, e') = 0 \qquad (e, e' \in \Lambda)$$
$$\sqrt{-1}\, Q(e, \bar{e}) > 0 \qquad (0 \neq e \in \Lambda)$$

then $\mathcal{H}_n$ is a complex manifold biholomorphic to the Siegel-upper-half-plane of all $n \times n$ matrices $Z = X + \sqrt{-1}\, Y$ satisfying $Z = {}^tZ$, $Y > 0$;

g) a one-dimensional complex manifold is a Riemann surface; and

h) the most important examples are the projective algebraic manifolds, which are the complex submanifolds of $\mathbb{P}^n$ given as the zeroes of homogeneous polynomials. This class includes the Grassmannians, compact Riemann surfaces, some but not all complex tori, any quotient D/Γ of a bounded domain D in $\mathbb{C}^n$ by a discrete group of $\mathrm{Aut}(D)$ acting without fixed points and with compact quotient; except for $p = q = 0$, the product of odd spheres is not projective.

An important tool for linearizing the study of complex manifolds are the holomorphic vector bundles, defined as in the real case but with holomorphic transition functions. We denote such bundles by $E \to M$; the fibres $E_p = \pi^{-1}(p)$ $(p \in M)$ are (non-canonically) isomorphic to $\mathbb{C}^r$ where r is the rank. Aside from the trivial bundle $M \times \mathbb{C}^r$, some examples are:

a) the holomorphic tangent bundle $T(M) \to M$, whose local holomorphic sections are the holomorphic vector fields $\sum_i \theta_i(z) \frac{\partial}{\partial z_i}$ on M;

b) the normal bundle to a complex submanifold;

c) the line bundle $(r = 1)$ $[D] \to M$ determined by a divisor D on

M ; here D has local defining equations $f_U = 0$ $(f_U \in \mathcal{O}(U))$, and the transition functions of $[D]$ are $f_{UV} = f_U/f_V \in \mathcal{O}^*(U \cap V)$;

d) the universal sub-bundle S and quotient bundle Q over the Grassmannian $G(k, n)$; for a k-plane $\Lambda \in G(k, n)$

$$S_\Lambda = \Lambda , \qquad Q_\Lambda = \mathbb{C}^n/\Lambda ;$$

e) over a projective variety, there are generally "very few" holomorphic (= algebraic) vector bundles, since the Chern classes (defined below) of such a bundle must have Hodge type (p, p) . On the other hand, it is a theorem of Grauert that on an affine algebraic variety (= algebraic subvariety of $\mathbb{C}^N$), every topological bundle has a unique analytic structure. A nice problem is to determine the "growth" of such bundles.

The standard constructions of linear algebra $(\otimes, \oplus, \Lambda^p$, Hom, duality, ...) apply fibre-wise to vector bundles. We follow the usual notational conventions: $\mathcal{O}(E)$ is the sheaf of holomorphic sections of $E \to M$; E^* is the dual of E , $\Omega^p = \mathcal{O}(\Lambda^p T(M)^*)$, etc. Noteworthy are the universal exact sequence

$$0 \to S \to G(k, n) \times \mathbb{C}^n \to Q \to 0 ,$$

and natural identification

$$T(G(k, n)) \cong \text{Hom}(S, Q) . \tag{1.1}$$

(PROOF: A variation Λ_t $(|t| < \varepsilon)$ of a k-plane Λ_0 is measured infinitesimally by choosing $e_t \in \Lambda_t$ and then projecting $\left.\frac{de_t}{dt}\right|_{t=0}$ into $\mathbb{C}^n/\Lambda_0$.)

We also denote by $A^{p,q}(M, E)$ the C^∞ , E-valued (p, q) forms on M . The operator

$$\bar\partial : A^{p,q}(M, E) \to A^{p,q+1}(M, E)$$

is well-defined since the transition functions of E are holomorphic.

In some sense, transcendental algebraic geometry is distinguished by the use of metrics. A holomorphic vector bundle with Hermitian metric in the fibres will be called a Hermitian vector bundle. Given such, the fundamental invariant is the curvature matrix $\Theta \in A^{1,1}(M, \text{Hom}(E, E))$.

The curvature is of twofold importance: First, it gives rise to Chern forms and Chern classes, as in the real case. Secondly, peculiar to the complex case are the notions of positivity and negativity, which tie in with the analytic concept of pluri-subharmonicity.

Given a Hermitian vector bundle $E \to M$, there exists a canonical metric connection

$$D : A^0(M, E) \to A^1(M, E)$$

uniquely characterized by the conditions

$$\begin{aligned} d(\xi, \eta) &= (D\xi, \eta) + (\xi, D\eta) \qquad (\xi, \eta \in A^0(M, E)) \\ D'' &= \bar\partial \end{aligned} \tag{1.2}$$

where $D = D' + D''$ is the type decomposition of D . A unitary frame field $\xi_1, \ldots, \xi_r$ is given by smooth sections ξ_ν of E over an open set $U \subset M$ which give a unitary basis in each fibre E_p $(p \in U)$. For such a frame field, the connection matrix θ and curvature Θ are defined by

$$\begin{aligned} D\xi_\nu &= \sum_\mu \theta_{\nu\mu}\xi_\mu \\ d\theta + \theta \wedge \theta &= \Theta . \end{aligned} \tag{1.3}$$

The conditions (1.2) characterizing D imply that

$$\begin{aligned} &\Theta + {}^t\bar\Theta = 0 \\ &\Theta \text{ is a matrix of } (1, 1) \text{ forms.} \end{aligned} \tag{1.4}$$

As examples of Hermitian vector bundles, observe that the inclusions

$$S \subset \mathbb{C}^n$$

induce a Hermitian metric on the universal sub-bundle. Similarly, for the universal sub-bundle $S|\mathcal{H}_n$ restricted to the Siegel-upper-half-plane, we may set

$$(\xi, \eta) = \sqrt{-1}\, Q(\xi, \bar\eta)$$

to obtain a canonical Hermitian metric of the sort encountered in variation of Hodge structure.

From the curvature, one constructs the Chern forms $c_q(\Theta)$ by taking elementary symmetric functions:

$$\sum_{q=0}^{r} c_q(\Theta)t^{r-q} = \det\left(\frac{\sqrt{-1}\,\Theta}{2\pi} + tI\right) .$$

Using (1.4), the Chern forms are global real forms of type (q, q), and the Bianchi identity $D\Theta = 0$ implies that

$$dc_q(\Theta) = 0 .$$

Thus, the Chern forms define classes

$$c_q \in H^{2q}_{DR}(M) ,$$

the de Rham cohomology of M. A basic result is that these Chern classes c_q are independent of the metric. In fact, this is true for real manifolds. In the complex case Bott and Chern proved more; namely that the Chern forms associated to two metrics $(\ ,\ )$ and $(\ ,\ )'$ satisfy

$$c_q(\Theta) - c_q(\Theta') = dd^c\eta_{q-1}$$

where $d^c = \sqrt{-1}\,(\bar\partial - \partial)$.

To define positivity, we use the curvature form

(1.5) $$\Theta(\eta) = \frac{\sqrt{-1}}{2\pi}(\Theta\eta, \eta) = \frac{\sqrt{-1}}{2\pi}\sum_{\mu,\nu}\Theta_{\mu\nu}\eta^{\mu}\bar\eta^{\nu}$$

where $\eta = \sum_\mu \eta^\mu \xi_\mu$ is a vector. Each $\Theta(\eta)$ is a real $(1, 1)$ form,

$$\Theta(\eta) = \frac{\sqrt{-1}}{2\pi}\sum_{i,j} a_{ij}\,dz_i \wedge d\bar z_j \qquad (a_{ij} = \bar a_{ji}) ,$$

and the Hermitian bundle is positive in case $\Theta(\eta) > 0$ for every non-zero η, in the sense that $(a_{ij}) > 0$. For line bundles, we note that

$$\Theta(\eta) = c_1(\Theta)\cdot(\eta, \eta) .$$

Here is an illustration of the use of the curvature form.

(1.6) Suppose that M is compact and that $E \to M$ is a Hermitian vector bundle whose curvature form has everywhere at least one negative eigenvalue. Then $H^0(M, \mathcal{O}(E)) = 0$.

PROOF: Suppose that $0 \neq \xi \in H^0(M, \mathcal{O}(E))$, and let $p_0 \in M$ be a point where the length $|\xi(p)|^2$ has a maximum. Then, at p_0,

(1.7) $$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial|\xi(p)|^2 \leq 0 .$$

On the other hand, using (1.2), $\bar\partial\xi = 0$, and $D^2 = \Theta$, we obtain

$$\begin{aligned}\partial\bar\partial(\xi, \xi) &= \partial(\xi, D'\xi)\\ &= (D'\xi, D'\xi) + (\xi, D''D'\xi)\\ &= (D'\xi, D'\xi) + (\xi, \Theta\xi) ;\quad \text{i.e.}\end{aligned}$$

(1.8) $$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial(\xi, \xi) &= \frac{\sqrt{-1}}{2\pi}(D'\xi, D'\xi) - \frac{\sqrt{-1}}{2\pi}(\Theta\xi, \xi)\\ &\geq -\Theta(\xi) ,\end{aligned}$$

which contradicts (1.7). Q.E.D.

The relationship between positivity and Chern forms is not yet sufficiently understood. For example, aside from vector bundles of rank ≤ 2, it is not known if the Chern forms of a positive bundle are positive. In general, the relationship between positive (differential-geometric), ample (algebro-geometric), and numerically positive (topological and algebro-geometric) vector bundles has not been explained.

Before discussing harmonic forms, we want to give a device for computing curvatures. Suppose given an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of holomorphic vector bundles. A Hermitian metric in E induces one in E' and E'', and the differential geometry is analogous to that of submanifolds in $\mathbb{R}^n$. More precisely, the metric connection D in $E \to M$ induces a map

$$\sigma : E' \to E'' \otimes T(M)^*$$

in the obvious way (σ is of type $(1, 0)$ since $D'' = \bar\partial$ and E' is a holomorphic sub-bundle). We call σ the 2nd fundamental form of E' in E. As an example, in the universal exact sequence over the Grassmannian, $\sigma \in \mathrm{Hom}(T(M), \mathrm{Hom}(S, Q))$ gives the isomorphism (1.1). The curvature form of E' is given by

(1.9) $$\Theta_{E'}(\xi) = \Theta_E(\xi) - (\sigma(\xi), \sigma(\xi)) \qquad (\xi \in E') .$$

In particular,

(1.10) $$\Theta_{E'} \le \Theta_E .$$

The principle that <u>curvatures decrease in sub-bundles</u> is of fundamental importance in Hermitian differential geometry.

A second major use of Hermitian metrics is the study of cohomology by using harmonic forms. Given a holomorphic vector bundle $E \to M$ over a compact complex manifold M, we consider the Dolbeault complex

$$\cdots \to A^{p,q-1}(M, E) \xrightarrow{\bar\partial} A^{p,q}(M, E) \to \cdots .$$

The holomorphic analogue of de Rham's theorem is the isomorphism

(1.11) $$H^q(M, \Omega^p(E)) = H^{p,q}_{\bar\partial}(M, E)$$

between the Čech cohomology of the sheaf $\Omega^p(E) = \Omega^p \otimes \mathcal{O}(E)$ and the cohomology of the Dolbeault complex. If we introduce metrics in E and in the tangent bundle, then the Dolbeault cohomology $H^{p,q}_{\bar\partial}(M, E)$ is represented by harmonic forms as follows: The spaces $A^{p,q}(M, E)$ are pre-Hilbert spaces using the L^2 inner product

$$(\phi, \psi) = \int_M \phi \wedge *\psi$$

where

(1.12) $$* : A^{p,q}(M, E) \to A^{n-p,n-q}(M, E^*)$$

is the pointwise duality operator. Next, the adjoint operator

$$\bar\partial^* : A^{p,q}(M, E) \to A^{p,q-1}(M, E)$$

and Laplacian

$$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

are defined as usual, as is the harmonic space

$$\begin{aligned} \mathcal{H}^{p,q}(M, E) &= \{\phi \in A^{p,q}(M, E) : \Box\phi = 0\} \\ &= \{\phi \in A^{p,q}(M, E) : \bar\partial\phi = 0 = \bar\partial^*\phi\} . \end{aligned}$$

What is by now standard P.D.E. gives, among other things, the isomorphism

(1.13) $$H^{p,q}_{\bar\partial}(M, E) \cong \mathcal{H}^{p,q}(M, E) .$$

There are three immediate applications of this representation of cohomology by harmonic forms:

a) The cohomology is <u>finite dimensional</u>;

$$\dim H^q(M, \Omega^p(E)) < \infty .$$

b) Next, we consider two compact, complex manifolds M, N over which we are given holomorphic vector bundles E, F. Denoting by $E \otimes F \to M \times N$ the bundle with fibres $(E \otimes F)_{(x \times y)} = E_x \otimes F_y$ $(x \in M, y \in N)$, if we choose the obvious product metrics throughout and make straightforward considerations of the various Laplacians, then

$$\mathcal{H}^{p,q}(M \times N, E \otimes F) = \sum_{r,s} \mathcal{H}^{r,s}(M, E) \otimes \mathcal{H}^{p-r,q-s}(N, F) .$$

This leads to the Künneth formula

(1.14) $$H^*(M \times N, \Omega^*_{M \times N}(E \otimes F)) \cong H^*(M, \Omega^*_M(E)) \otimes H^*(N, \Omega^*_N(F)) .$$

c) Since $\bar\partial^* = \pm *\bar\partial*$ where $*$ is the operator (1.12), it follows that $*\Box = \Box*$ and

$$* : \mathcal{H}^{p,q}(M, E) \to \mathcal{H}^{m-p,m-q}(M, E^*)$$

is an isometry. This implies that the pairing

(1.15) $$H^q(M, \Omega^p(E)) \otimes H^{n-q}(M, \Omega^{m-p}(E^*)) \to \mathbb{C}$$

given by using (1.11) and

$$\phi \otimes \psi \to \int_M \phi \wedge \psi$$

is a pairing of dual vector spaces (<u>Kodaira-Serre duality</u>).

The relationship between harmonic forms and positivity will be discussed in lecture 3 when we talk about vanishing theorems.

REFERENCES FOR CHAPTER ONE

As general references for complex manifolds and Hermitian differential geometry, we suggest:

S. S. Chern, Complex manifolds without potential theory, Van Nostrand, New York (1967).

R. O. Wells, Differential analysis on complex manifolds, Prentice-Hall, Englewood Cliffs (1973).

J. Morrow and K. Kodaira, Complex manifolds, Holt, Rinehart and Winston, New York (1971).

The curvature form, 2nd fundamental form of a holomorphic sub-bundle, etc. are discussed in

P. Griffiths, Hermitian differential geometry, Chern classes, and positive vector bundles, in Global Analysis, Princeton University Press, Princeton, N. J. (1969).

APPENDIX TO LECTURE 1: PROOF OF THE HODGE THEOREM*

Let M be a compact Hermitian manifold. We shall prove the Hodge theorem for the Laplace-Beltrami operator $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting on the Dolbeault complex $A^{0,*}(M)$.

a) FOURIER SERIES. Let $T = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be a standard torus with coordinate $x = (x_1,\dots,x_n)$. We consider the space F of formal Fourier series

$$u \sim \sum_{\xi\in\mathbb{Z}^n} u_\xi e^{\sqrt{-1}<\xi,x>}$$

with complex coefficients. For any integer s, the <u>Sobolev</u> s-<u>norm</u> is

$$\|u\|_s^2 = \sum_\xi (1+\|\xi\|^2)^s |u_\xi|^2 ,$$

and we denote by H_s the Hilbert space of all $u \in F$ with finite s-norm. Note that

$$H_s \subset H_r \qquad (s \geq r) ,$$

and we set

$$H_\infty = \bigcap_s H_s$$
$$H_{-\infty} = \bigcup_s H_s .$$

* We shall assume only elementary Hilbert space theory up to the spectral theorem for compact operators.

The mapping

$$u_\xi \to (1+\|\xi\|^2)^s u_\xi$$

is an isometry from H_s to H_{-s}, so that we may identify H_{-s} with the dual of H_s by the pairing

$$(u, v) \to \sum_\xi u_\xi \bar v_\xi .$$

A basic tool is the

RELLICH LEMMA: For $s > r$, the inclusion $H_s \subset H_r$ is a compact operator.

PROOF: We must show that a bounded sequence $\{u_k\}$ in H_s has a convergent subsequence in H_r. Since $\|u_k\|_s^2 \leq C < \infty$, the sequences

$$(1+\|\xi\|^2)^{r/2} u_{k,\xi}$$

are bounded. By diagonalization and passing to a subsquence of the u_k, we may assume that

$$(1+\|\xi\|^2)^{r/2} u_{k,\xi}$$

is Cauchy for each fixed ξ. Given $\varepsilon > 0$, we may then choose R and m such that

$$\frac{4C}{(1+\|\xi\|^2)^{s-r}} < \varepsilon/2 \qquad \text{for } \|\xi\| > R$$

$$\sum_{\|\xi\|\leq R} (1+\|\xi\|^2)^r |u_{k,\xi} - u_{\ell,\xi}|^2 < \varepsilon/2 \qquad \text{for } k, \ell \geq m .$$

Then, for $k, \ell \geq m$,

$$\|u_k - u_\ell\|_r^2 = \sum_{\|\xi\|\leq R} (1+\|\xi\|^2)^r |u_{k,\xi} - u_{\ell,\xi}|^2$$
$$+ \sum_{\|\xi\|>R} (1+\|\xi\|^2)^r |u_{k,\xi} - u_{\ell,\xi}|^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Denote by $C^s(T)$ the functions of differentiability class s on the torus. Each continuous function $\phi \in C^0(T)$ generates a formal Fourier series $\sum_\xi \phi_\xi e^{\sqrt{-1}<\xi,x>}$ whose <u>Fourier coefficients</u> are given by

$$\phi_\xi = \int_T \phi(x) e^{-\sqrt{-1}<\xi,x>} .$$

Set $D_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$ and use the standard multi-index notations:

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \qquad \alpha = (\alpha_1, \ldots, \alpha_n)$$
$$[\alpha] = \alpha_1 + \cdots + \alpha_n$$
$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} .$$

Then the formulae

$$\text{(A.1)} \qquad \begin{aligned} D^\alpha e^{\sqrt{-1}<\xi,\alpha>} &= \xi^\alpha e^{\sqrt{-1}<\xi,\alpha>} \\ \int_T (D^\alpha\phi)(\bar{\psi}) &= \int_T (\phi)(\overline{D^\alpha\psi}) \qquad (\phi, \psi \in C^\infty(T)) \end{aligned}$$

are valid. If $\phi \in C^s(T)$ and $[\alpha] \leq s$, the Fourier coefficients of $D^\alpha\phi$ are

$$(D^\alpha\phi)_\xi = \xi^\alpha \phi_\xi .$$

Parseval's identity

$$\int_T |\phi|^2 = \sum_\xi |\phi_\xi|^2$$

shows that the Fourier series mapping $C^0(T) \to H_0$ is injective, and that $C^s(T)$ maps to H_s. A partial converse is given by our second basic tool, the important

> SOBOLEV LEMMA: $H_{s+[n/2]+1} \subset C^s(T)$, in the sense that each $u \in H_{s+[n/2]+1}$ is the Fourier series of a unique function $u \in C^s(T)$, and moreover this Fourier series converges uniformly to the function.

PROOF: In case $s = 0$, we let $\sum_\xi u_\xi e^{\sqrt{-1}<\xi,x>}$ satisfy $\sum_\xi (1 + \|\xi\|^2)^{[n/2]+1} |u_\xi|^2 < \infty$. The partial sums $S_R = \sum_{\|\xi\| \leq R} u_\xi e^{\sqrt{-1}<\xi,x>}$ are continuous functions, and for $R \leq R'$

$$\begin{aligned} |S_R(x) - S_{R'}(x)| &\leq \sum_{\|\xi\| \geq R} |u_\xi| \\ &= \sum_{\|\xi\| \geq R} \frac{((1 + \|\xi\|^2)^{[n/2]+1} |u_\xi|^2)^{1/2}}{((1 + \|\xi\|^2)^{[n/2]+1})^{1/2}} \\ &\leq C \left(\sum_{\|\xi\| \geq R} (1 + \|\xi\|^2)^{[n/2]+1} |u_\xi|^2 \right)^{1/2} \end{aligned}$$

by the Schwarz inequality and $\sum_\xi \frac{1}{(1 + \|\xi\|^2)^{[n/2]+1}} \leq C < \infty$. Consequently, the $S_R(x)$ converge uniformly to a continuous function $u(x)$, whose Fourier coefficients are in turn just the u_ξ by virtue of the orthogonality relations

$$\int_T e^{\sqrt{-1}<\xi,x>} e^{-\sqrt{-1}<\xi',x>} = \begin{cases} 1 & \xi = \xi' \\ 0 & \xi \neq \xi' \end{cases}$$

We shall do the case $s = n = 1$ to illustrate the general situation for $s > 0$. Thus, let $u(x) = \sum_{\xi \in \mathbb{Z}} u_\xi e^{\sqrt{-1}<\xi,x>}$ be a continuous function where $\sum_\xi |\xi|^4 |u_\xi|^2 < \infty$. Using the previous case, the partial sums of $v(x) = \sum_\xi \sqrt{-1}\, \xi u_\xi e^{\sqrt{-1}<\xi,x>}$ converge uniformly to a continuous function. Integrating term by term gives

$$\int_0^x v(t)dt = \sum_{\xi \neq 0} u_\xi e^{\sqrt{-1}<\xi,x>} = u(x) - u_0 .$$

It follows that $u(x)$ is of class C^1 and $u'(x) = v(x)$.

In summary, we have proved:

The Fourier series mapping $C^0(T) \to F$ leads to inclusions

$$C^s(T) \subset H_s$$
$$H_{s+[n/2]+1} \subset C^s(T)$$

In particular, we shall make the identification

$$C^\infty(T) = H_\infty .$$

We remark that the Fourier series of a function $\phi \in C^\infty(T)$ may be differentiated term by term. Indeed, $D^\alpha\phi$ is given by a Fourier series whose Fourier coefficients are $(D^\alpha\phi)_\xi = \xi^\alpha\phi$ by (A.1).

Another useful comment is that the proof of the Sobolev lemma gives an estimate

$$\text{(A.2)} \qquad \sup_{x \in T} |\phi(x)| \leq C\|\phi\|_{[n/2]+1} .$$

The norm $\| \|_0$ is just the usual L^2 norm on $C^\infty(T)$. Similarly, the Sobolev s-norm is equivalent to the norm

(A.3) $$|||\phi|||_s^2 = \sum_{[\alpha]\leq s} \|D^\alpha\phi\|_0^2 = \sum_{\substack{[\alpha]\leq s\\ \xi}} |\xi^{2\alpha}||u_\xi|^2 .$$

This follows from the inequalities

$$\sum_{[\alpha]\leq s} |\xi^{2\alpha}| \leq (1+\|\xi\|^2)^s \leq C_s \sum_{[\alpha]\leq 2s} |\xi^{2\alpha}| .$$

Thus, H_s <u>is the L^2-completion of $C^\infty(T)$ with respect to the norm (A.3).</u> Combining these remarks with (A.2) gives

(A.4) $$\sup_{x\in T} |D^\alpha\phi(x)| \leq C_\alpha\|\phi\|_{[\alpha]+[n/2]+1} .$$

We shall conclude this discussion of Fourier series with some remarks concerning the <u>distributions</u>, defined as the linear functionals

$$\lambda : C^\infty(T) \to C$$

which are continuous in the sense that

(A.5) $$|\lambda(\phi)| \leq C \sup_{\substack{[\alpha]\leq k\\ x\in T}} |D^\alpha\phi(x)| .$$

Each distribution generates a formal Fourier series $\sum_\xi \lambda_\xi e^{\sqrt{-1}<\xi,x>}$ where

$$\lambda_\xi = \lambda(e^{-\sqrt{-1}<\xi,x>}) .$$

It follows from (A.5) that λ is continuous in the norm $\| \|_k$, so that $\lambda \in H_{-k}$ by previous remarks. Indeed, the formal Fourier series of λ in H_{-k} is just $\sum_\xi \lambda_\xi e^{\sqrt{-1}<\xi,x>}$. On the other hand, by (A.4) any $\lambda \in H_{-k}$ gives a distribution by setting

$$\lambda(\phi) = \sum_\xi \lambda_\xi\phi_\xi .$$

Thus, we may identify the distributions with $H_{-\infty}$.

The derivatives of any distribution λ are defined by

$$(D^\alpha\lambda)\phi = \lambda(D^\alpha\phi) .$$

A distribution λ is said to <u>be in</u> L^2 in case $\lambda \in H_0 \subset H_{-\infty}$. Putting our remarks together, we may give the following description of the Sobolev spaces H_s for $s \geq 0$:

H_s consists of the distributions λ such that all derivatives $D^\alpha\lambda$ are in L^2 for $[\alpha] \leq s$.

Referring to lecture 2, the delta-function distribution

$$\delta(\phi) = \phi(0)$$

has formal Fourier series $\sum_\xi e^{\sqrt{-1}<\xi,x>}$. Letting $dV = dx_1 \wedge \cdots \wedge dx_n$, the equation of currents

$$dk = \delta\cdot dV - dV$$

has the solution $k = \sum_{j=1}^n (-1)^j k_j dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_n$ where $k_j = \sum_\xi \sqrt{-1}\,\frac{C_j(\xi)}{\xi_j}\, e^{\sqrt{-1}<\xi,x>}$ and $C_j(\xi)$ is the number of non-zero ξ_j in $\xi = (\xi_1, \ldots, \xi_n)$.

b) GLOBALIZATION TO MANIFOLDS; THE HODGE DECOMPOSITION. Any compactly supported function on $\mathbb{R}^n$ may be regarded as a function on the torus. Using a partition of unity, we may thus globalize the above discussion to a compact Hermitian manifold. Let ∇ be the metric connection on the complexified tangent bundle $T_C = T' \oplus T''$. ∇ induces a connection on all associated tensor bundles. Thus

$$\nabla : C^\infty(\Lambda^q T''^*) \to C^\infty(\Lambda^q T''^* \otimes T_C{}^*) ,$$

$$\nabla : C^\infty(\Lambda^q T''^* \otimes T_C{}^*) \to C^\infty(\Lambda^q T''^* \otimes T_C{}^* \otimes T_C{}^*) ,$$

etc. are defined. If $\phi \in C^\infty(\Lambda^q T''^*) = A^{0,q}(M)$, we set $\nabla^k = \underbrace{\nabla(\nabla(\cdots(\nabla\phi)\cdots)}_{k\text{-times}}$. In any local coordinate system,

$$\sum_{[\alpha]\leq s} |D^\alpha\phi|^2 \leq C \sum_{k\leq s} (\nabla^k\phi, \nabla^k\phi) \leq C' \sum_{[\alpha]\leq s} |D^\alpha\phi|^2$$

since, for any tensor τ, in local coordinates

$$\nabla\tau = \sum_{i=1}^m \frac{\partial\tau}{\partial z_i} dz_i + \sum_{i=1}^m \frac{\partial\tau}{\partial\bar{z}_i} d\bar{z}_i + \chi\cdot\tau$$

where χ is an algebraic operator. Thus, if we let $H_s^{0,q}(M)$ be the completion of $A^{0,q}(M)$ in the norm

$$\|\phi\|_s^2 = \sum_{k\leq s} \int_M (\nabla^k\phi, \nabla^k\phi)\,dV ,$$

then $H_s^{0,q}(M)$ localizes to the Sobolev spaces H_s for C^∞ forms compactly supported in a fixed coordinate patch. In particular,

$$\bigcap_s H_s^{0,q}(M) = A^{0,q}(M) ;$$
$$\bigcup_s H_s^{0,q}(M) = D^{0,q}(M) \text{ are the currents of type } (0, q) ;^*$$
$$H_0^{0,q}(M) = L_2^{0,q}(M) \text{ are the } L^2\text{-forms; and}$$

the inclusion

$$H_s^{0,q}(M) \to H_r^{0,q}(M)$$

is compact for $s > r$.

In addition to the local Fourier analysis, our basic tool in the study of harmonic theory is

GÅRDING'S INEQUALITY: For $\phi \in A^{0,q}(M)$,

(A.6) $$\|\phi\|_1^2 \leq C\mathcal{D}(\phi, \phi)$$

where $\mathcal{D}(\phi, \phi)$ is the Dirichlet norm

(A.7) $$\mathcal{D}(\phi, \phi) = (\phi, \phi) + (\bar\partial\phi, \bar\partial\phi) + (\bar\partial^*\phi, \bar\partial^*\phi) = (\phi, (I + \Box)\phi) .$$

Assuming (A.6) we shall prove the basic results in Hodge theory, and then derive (A.6) in the next section. The Gårding inequality says that the norm $\mathcal{D}(\phi, \phi)^{1/2}$ is equivalent to $\|\phi\|_1$ on $H_1^{0,q}$. Moreover, for $\psi \in H_1^{0,q}(M)$ and $\eta \in A^{0,q}(M)$,

(A.8) $$\mathcal{D}(\psi, \eta) = (\psi, (I + \Box)\eta) .$$

(A.9) LEMMA: Given $\phi \in L_2^{0,q}$, there exists a unique $\psi \in H_1^{0,q}$ such that

(A.10) $$(\phi, \eta) = \mathcal{D}(\psi, \eta) \qquad (\eta \in A^{0,q}(M)) .$$

The map $T(\phi) = \psi$ is a compact self-adjoint operator on $L_2^{0,q}(M)$ whose range is contained in $H_1^{0,q}(M)$. As a mapping of $L_2^{0,q}(M)$ into $H_1^{0,q}(M)$, T is continuous.

* Actually, we have only defined the $H_s^{0,q}(M)$ for $s \geq 0$, but this may be extended to all s.

PROOF: The linear functional

$$\eta \to (\phi, \eta) \qquad (\eta \in A^{0,q}(M))$$

extends to a bounded linear form on $H_1^{0,q}(M)$, by virtue of $|(\phi, \eta)| \leq \|\phi\|_0\|\eta\|_0 \leq \|\phi\|_0\mathcal{D}(\eta)$. Thus the equation (A.10) has a unique solution $\psi \in H_1^{0,q}(M)$. The mapping $T(\phi) = \psi$ is characterized by

$$(\phi, \eta) = \mathcal{D}(T\phi, \eta) = (T\phi, (I + \Box)\eta) \qquad (\eta \in A^{0,q}(M)) .$$

T is self-adjoint since

$$(\phi, T\gamma) = \mathcal{D}(T\phi, T\gamma) = \overline{\mathcal{D}(T\gamma, T\phi)} = \overline{(\gamma, T\phi)}$$

for smooth forms ϕ, γ. From

$$\|T\phi\|_1^2 \leq \mathcal{D}(T\phi, T\phi) = (\phi, T\phi) \leq \|\phi\|_0 \cdot \|T\phi\|_0$$

and inequalities of the form

$$2\alpha\beta \leq \epsilon\alpha^2 + \frac{1}{\epsilon}\beta^2 ,$$

it follows that T is bounded, and Rellich's theorem implies that T is compact. Q.E.D.

We may now prove the regularity theorem. Given $\phi \in H_s^{0,q}(M)$ and $\psi \in L_2^{0,q}(M)$, we say ψ _is a weak solution of the equation_

(A.11) $$\Box\psi = \phi$$

in case

$$(\psi, \Box\eta) = (\phi, \eta) \qquad (\eta \in A^{0,q}(M)) .^*$$

REGULARITY THEOREM: If ψ is a weak solution of (A.11) where $\phi \in H_s^{0,q}(M)$, then $\psi \in H_{s+2}^{0,q}(M)$.

PROOF: We write

$$\Box = P^2$$
$$P = \bar\partial + \bar\partial^*$$

* A _strong solution_ is a solution in the usual sense.

and notice that in proving the regularity theorem we may assume that ϕ, ψ are both supported in a fixed coordinate patch. Regularity then follows from

(A.12) PROPOSITION: Let P be a first order differential operator on $\mathbb{R}^n$:

$$Pu = Qu + Ru$$
$$(Qu)_i = \sum a^h_{ij} \frac{\partial u_j}{\partial x_h}$$
$$(Ru)_i = \sum b_{ij} u_j$$

where a^h_{ij}, b_{ij} are smooth functions. Suppose that the Gårding inequality

(A.13) $$\alpha\|Pu\|_0 + \beta\|u\|_0 \geq \|u\|_1$$

holds for any compactly supported smooth u. If $u, v \in H_s$ and

$$Pu = v$$

in the weak sense (as distributions), then $u \in H_{s+1}$.

PROOF: We will assume that $s \geq 0$, although the proof works for any s. Before actually proving (A.12) we recall some standard facts about mollifiers. Let χ be a positive, compactly supported smooth function on $\mathbb{R}^n$ such that

$$\chi(-x) = \chi(x)$$
$$\int \chi(x)dx = 1 .$$

We define

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right)$$
$$(v * u)(x) = \int v(y)u(x-y)dy = \int v(x-y)u(y)dy$$

and notice that, no matter what the differentiability properties of u are, $\chi_\varepsilon * u$ is C^∞ and

$$\frac{\partial}{\partial x_h}(\chi_\varepsilon * u) = \frac{\partial}{\partial x_h}\chi_\varepsilon * u = \chi_\varepsilon * \frac{\partial u}{\partial x_h} .$$

Moreover it is a standard and easily proved fact that, if u is compactly

supported and belongs to L_2 (resp. H_s), then $\chi_\varepsilon * u$ converges to u in L_2-norm (H_s-norm) as $\varepsilon \to 0$. To prove (A.12) it is sufficient to uniformly bound the H_{s+1}-norm of $\chi_\varepsilon * u$, since then a sequence $\chi_{\varepsilon_n} * u$, $\varepsilon_n \to 0$ converges weakly to an element of H_{s+1} which can only be u.

It is a consequence of (A.13) that we can bound the H_{s+1}-norm of $\chi_\varepsilon * u$ in terms of the H_s-norms of $Q(\chi_\varepsilon * u)$ and $\chi_\varepsilon * u$. The latter, in turn, is bounded by a constant times the H_s-norm of u. We know how to bound the H_s-norm of $\chi_\varepsilon * Qu$, so we must bound the H_s-norm of the difference

(A.14) $$\chi_\varepsilon * Qu - Q(\chi_\varepsilon * u) .$$

For simplicity we will do this when $s = 0$, the argument being the same in general. The i-th component of (A.14) is

$$\left[\frac{\partial}{\partial x_h}\chi_\varepsilon * \left(\sum_{j,h} a^h_{ij}u_j\right) - \sum_{j,h} a^h_{ij}\frac{\partial}{\partial x_h}\chi_\varepsilon * u_j\right] - \chi_\varepsilon * \left(\sum_{j,h} u_j \frac{\partial}{\partial x_h} a^h_{ij}\right) .$$

The second term is bounded by a constant times the L_2-norm of u. The other term is

$$\frac{1}{\varepsilon^{n+1}}\sum_{j,h}\int \frac{\partial}{\partial x_h}\chi(y/\varepsilon)\,[a^h_{ij}(x-y) - a^h_{ij}(x)]u_j(x-y)dy$$

and the Minkowski inequality implies that its L_2-norm is less than

$$\frac{1}{\varepsilon^{n+1}}\int_{|y|\leq\varepsilon} \left|\frac{\partial}{\partial x_h}\chi\left(\frac{y}{\varepsilon}\right)\right| K|y|\,\|u_j\|_0\,dy \leq K'\|u\|_0$$

for suitable constants K, K'. Q.E.D.

As an application, we call any weak solution of the equation

$$\Box\phi = \lambda\phi \qquad (\lambda \in C)$$

an <u>eigenfunction</u> for the Laplacian.

COROLLARY: Any eigenfunction of $\Box$ is smooth.

PROOF: From $\Box\phi = \lambda\phi$, it follows inductively on s that $\phi \in \bigcap_s H^{0,q}_s(M)$, which is just $A^{0,q}(M)$ by the Sobolev lemma. Q.E.D.

The eigenspace $A^{0,q}_\lambda(M) = \{\phi \in L^{0,q}_2(M) : \Box\phi = \lambda\phi\}$ is just the eigenspace with eigenvalue $1/(1+\lambda)$ for T. Since T is compact and self-adjoint these eigenspaces are all finite dimensional. In particular, the harmonic space $H^{0,q}(M)$ corresponding to the eigenvalue $\lambda = 0$ is finite dimensional. Moreover, the spectral theorem for such operators gives the discrete decomposition

$$L_2^{0,q}(M) = \bigoplus_{\lambda_m} A_{\lambda_m}^{0,q}(M)$$

where $A_{\lambda_m}^{0,q}$ is the eigenspace for $1/(1+\lambda_m)$ for T. In particular, $0 \leqq \lambda_0 < \lambda_1$ and $\lambda_m \to \infty$ as $m \to \infty$. On the orthogonal complement $H^{0,q}(M)^{\perp} = \bigoplus_{m \geqq 1} A_{\lambda_m}^{0,q}(M)$ of the harmonic space, the estimate

$$\|\square\eta\|_0 \geqq \lambda_1\|\eta\|_0 \qquad (\lambda_1 > 0)$$

is valid for all smooth forms. This is the essential estimate needed to prove the

HODGE DECOMPOSITION: The equation

$$\square\psi = \phi \qquad (\phi \in A^{0,q}(M))$$

has a unique solution $\psi \in A^{0,q}(M) \cap H^{0,q}(M)^{\perp}$ if and only if $\phi \in H^{0,q}(M)^{\perp}$. The map $\phi \to \psi$ is a compact self-adjoint operator G (the Green's operator) which commutes with $\bar\partial$ and $\bar\partial^*$. For any $\phi \in A^{0,q}(M)$, one has the Hodge decomposition

$$\phi = H(\phi) + \square G(\phi) \qquad \text{(A.15)}$$

where $H(\phi)$ is the projection of ϕ in $H^{0,q}(M)$ and G has been extended to all of $L_2^{0,q}(M)$ by setting $G = 0$ on $H^{0,q}(M)$.

PROOF: The necessity condition $\phi \in H^{0,q}(M)$ and the uniqueness statement are obvious. An explicit formula for G is as follows. Write

$$\phi = \sum \phi_i$$

where $\phi_i \in A_{\lambda_i}^{0,q}$, $\lambda_i \neq 0$. Then

$$G\phi = \sum \frac{1}{\lambda_i}\phi_i .$$

It follows from the regularity theorem that $\psi = G\phi$ is smooth and is a solution of $\square\psi = \phi$ in the usual sense. Moreover the Hodge decomposition (A.15) is valid, and the operator properties of G follow from those of T. The commutation relations $[G, \bar\partial] = 0 = [G, \bar\partial^*]$ result from $[\square, \bar\partial] = 0 = [\square, \bar\partial^*]$. Q.E.D.

In particular, any $\phi \in A^{0,q}(M)$ may be written in the chain homotopy form

$$\phi = H(\phi) + \bar\partial(G\bar\partial^*)\phi + (G\bar\partial^*)\bar\partial\phi , \qquad \text{(A.16)}$$

and this, together with what has been previously said, proves the results on cohomology which were stated in lecture 1.

The proof of Hodge theory we have given depends only on the Gårding inequality (A.6), which is the basic ellipticity estimate for $\square$. It works equally well for Riemannian manifolds and vector bundle cohomology $H_{\bar\partial}^{*}(M, E)$. The basic defect is that the more subtle properties of the Green's operator as a kernel on $M \times M$ with certain precise singularities along the diagonal are not readily visible by the Hilbert space method. We remark that the total operator

$$G\bar\partial^* : A^{0,*}(M) \to A^{0,*-1}(M)$$

comes from a very beautiful kernel $k(x, y)$ on $M \times M$, which in particular solves the equation of currents

$$\bar\partial k = T_{0,\Delta} - s$$

with $s(x, y)$ being the smooth form $\Psi(x, y)$ in lecture 2 corresponding to taking the ψ_μ^q and ψ_μ^{*q} to be an orthonormal basis for the harmonic forms.

c) THE GÅRDING INEQUALITY. For any differential operator D of order k on a compact manifold M, one trivially has estimates

$$\|D\phi\|_s^2 \leq C_s\|\phi\|_{k+s}^2 .$$

Very roughly speaking, ellipticity is the converse. For compactly supported functions $\phi(x)$ on $\mathbb{R}^n$, the Euclidean Laplacian $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$

satisfies

$$\int_{\mathbb{R}^n} (\Delta\phi\cdot\phi)\,dV = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left|\frac{\partial\phi}{x_i}\right|^2 dV$$

by an obvious integration by parts. Since

$$\Delta\Big(\sum_I \phi_I\, dx^I\Big) = \sum_I (\Delta\phi_I)\, dx^I ,$$

the same is true for compactly supported p-forms. Thus, one has the equality

$$\|\phi\|_1^2 = \mathcal{D}(\phi, \phi)$$

in the flat Euclidean case. The same is true for the Laplacian $\square$ on $\mathbb{C}^m$, since $\square = \frac{1}{2}\Delta$ by the Kähler property (Lecture 3).

In the general case, one proves (2.6) by writing out the formula for $\square$ (Weitzenböck identity) and doing an integration by parts (Stokes' theorem). The principal part of $\square$ has the same form as on $\mathbb{C}^m$, but there are first order terms coming from the torsion and zero order terms coming from the curvature. These may be estimated out by repeatedly using the inequality

$$2\alpha\beta \le \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2 .$$

Here is the argument in more detail. Write locally $ds^2 = \sum_{i=1}^m \omega_i\bar\omega_i$ where $\omega_1, \ldots, \omega_m$ is an orthonormal coframe for the (1, 0) forms. On any tensor τ, $\nabla'\tau = \sum_{i=1}^m \nabla_i\tau\otimes\omega_i$ where ∇_i is covariant differentiation along the (1, 0) vector field dual to ω_i. A (0, q) form is written as $\phi = \sum_I \phi_I\bar\omega_I$ where $\bar\omega_I = \bar\omega_{i_1}\wedge\cdots\wedge\bar\omega_{i_q}$, etc. The Weitzenböck identity (in crude form) is

$$(\square\phi)_I = -\sum_j \nabla_j\nabla_{\bar j}\phi_I + (\delta\phi)_I \tag{A.17}$$

where $\delta\phi$ is a first order operator. In the Kähler case, $\delta\phi$ is the zero order operator given by

$$(\delta\phi)_I = \sum_j R_{i_p\bar j}\,\phi_{\bar i_1\cdots\underset{p}{\bar j}\cdots\bar i_q}$$

where $R_{i\bar j} = c_1(K_M)$ is the Ricci tensor of the metric. To prove (A.17),

one first derives the formulae

$$\text{(A.18)}\quad \begin{aligned} (\bar\partial\phi)_{\bar i_1\cdots\bar i_{q+1}} &= \sum_{k=1}^{q+1} (-1)^k \nabla_{\bar i_k}\phi_{\bar i_1\cdots\hat{\bar i}_k\cdots\bar i_{q+1}} + (\tau\phi)_{\bar i_1\cdots\bar i_{q+1}} \\ (\bar\partial^*\phi)_{\bar i_1\cdots\bar i_{q-1}} &= \sum_{k,i} (-1)^k \nabla_i\phi_{\bar i_1\cdots\underset{k}{\bar i}\cdots\bar i_{q-1}} + (\tau^*\phi)_{\bar i_1\cdots\bar i_{q-1}} \end{aligned}$$

where τ, τ^* are algebraic operators involving the torsion. These equations follow in turn from the basic structure equations

$$d\omega_i = \sum_j \theta_{ij}\wedge\omega_j + \tau_i$$

$$\theta_{ij} + \bar\theta_{ji} = 0 , \quad \tau_i \text{ has type } (2, 0)$$

for the Hermitian connection in $T'(M)$, θ being the connection matrix.* Now then the Weitzenböck formula (A.17) results from (A.18) and $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ together with $[\nabla_i, \nabla_{\bar j}] = \chi_{ij}$, an operator involving the curvature and torsion.

A little reflection should convince the reader that whatever computation works for $\mathbb{C}^m$ will carry over to a general manifold in the crude form (A.17).

Assuming (A.17), we proceed as follows: The $(m-1, m)$ form

$$\Psi = c_1\sum_j (-1)^j \nabla_{\bar j}\phi_I\bar\phi_I\, \omega_1\wedge\cdots\wedge\hat\omega_j\wedge\cdots\wedge\omega_m\wedge\bar\omega_1\wedge\cdots\wedge\bar\omega_m$$

is intrinsically defined on M. Choosing the constant c_1 properly,

$$d\Psi = \sum_{I,j} |\nabla_{\bar j}\phi_I|^2 dV + c_2\sum_{j,I} (\nabla_j\nabla_{\bar j}\phi_I\bar\phi_I)dV$$

for some constant c_2. Applying Stokes' theorem and keeping track of constants (including signs) gives

$$\int_M \sum_{I,j} |\nabla_{\bar j}\phi_I|^2 dV = (\square\phi, \phi) + (\delta_1\phi, \phi)$$

where δ_1 is first order. Similarly,

$$\int_M \sum_{I,j} |\nabla_j\phi_I|^2 = (\square\phi, \phi) + (\delta_2\phi, \phi) .$$

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* A proof of (A.18) in the Kähler case $\tau \equiv 0$ runs as follows: Both sides are intrinsically defined first order operators. Hence, using osculation of the metric to $\mathbb{C}^m$ to second order, it will suffice to verify (A.18) in $\mathbb{C}^m$, which is easy.

Adding $\|\phi\|_0^2$ to these equations gives

$$\|\phi\|_1^2 \leq C(\mathcal{D}(\phi, \phi) + 2(\delta_3\phi, \phi)) .$$

Then, by the Schwarz inequality,

$$2|(\delta_3\phi, \phi)| \leq \varepsilon\|\delta_3\phi\|_0^2 + \frac{1}{\varepsilon}\|\phi\|_0^2 ,$$

and taking $\varepsilon\|\delta_3\phi\|_0^2 \leq C'\varepsilon\|\phi\|_1^2$ to the left gives

$$\|\phi\|_1^2 \leq C\mathcal{D}(\phi, \phi)$$

for a suitable large constant C . Q.E.D.

REMARK: In the Kähler case, the precise Weitzenböck identity plus integration by parts as above gives the Kodaira identity

$$(\Box\phi, \phi) = \|\overline{\nabla}\phi\|_0^2 + (R\phi, \phi) .$$

In particular, if the canonical bundle is negative (as on $\mathbf{P}^m$), then $H_{\bar\partial}^{0,q}(M) = 0$ for $q > 0$. Applying a similar method to the groups $H_{\bar\partial}^{0,q}(M, E)$ gives Kodaira's original proof of his vanishing theorem.

REFERENCES FOR THE APPENDIX TO LECTURE ONE

A proof of the Hodge theorem along the same general lines as that above was given by J. J. Kohn in a course at Princeton University in 1961-62. A different argument, based on Fourier transforms and pseudo-differential operators as opposed to Fourier series, is in the book by R. O. Wells listed in the references to lecture one.

2. RIEMANN-ROCH AND FIXED POINT FORMULAE

a) FORMALISM. Let $E \to M$ be a holomorphic vector bundle over a compact, complex manifold. Since the cohomology $H^*(M, \mathcal{O}(E)) = \sum_{q=0}^m H^q(M, \mathcal{O}(E))$ is finite dimensional, the Euler characteristic

$$\chi(M, E) = \sum_{q=0}^{m} (-1)^q \dim H^q(M, \mathcal{O}(E))$$

is defined. The Hirzebruch-Riemann-Roch formula

$$\chi(M, E) = \int_M T(c_1, \ldots, c_m; d_1, \ldots, d_r) \tag{2.1}$$

expresses the Euler characteristic as a universal polynomial (the Todd polynomial) in the Chern classes c_i of M and d_μ of E evaluated on M . We shall outline Toledo and Tong's proof of (2.1), giving as an application the Atiyah-Bott fixed point formula. For notational simplicity, we shall take E to be the trivial line bundle, and shall write $\chi(M, \mathcal{O})$ for the Euler characteristic in this case.

The basic formal step is to use Kodaira-Serre duality and the Künneth formula to express the Euler characteristic as the value of a cohomology class on $M \times M$ on the diagonal Δ . This is the holomorphic analogue of the usual formula for the topological Euler characteristic as the self-intersection number of the diagonal, and goes as follows: Let $\psi_\mu^q \in A^{0,q}(M)$ be $\bar\partial$-closed forms giving a basis for $H_{\bar\partial}^{0,q}(M)$, and $\psi_\mu^{*q} \in A^{m,m-q}(M)$ forms yielding a dual basis for $H_{\bar\partial}^{m,m-q}(M)$. Using (x, y) to denote points on $M \times M$ and $A^{(p,q)(r,s)}(M \times M)$ to denote forms on the product which are of type (p, q) in x and (r, s) in y (bi-type for short), we set

$$\Psi_q(x, y) = (-1)^q \sum_\mu \psi_\mu^q(x) \wedge \psi_\mu^{*q}(y)$$

$$\Psi(x, y) = \sum_{q=0}^{m} \Psi_q(x, y) .$$

Then clearly

$$\chi(M, \mathcal{O}) = \int_\Delta \Psi(x, x) .$$

In general, any smooth form $s_q(x, y) \in A^{(0,q)(m,m-q)}(M \times M)$ gives an operator

$$S_q : A^{0,q}(M) \to A^{0,q}(M)$$

defined by

$$(S_q\phi)(x) = \int_M s_q(x, y) \wedge \phi(y) ,$$

which we call a smoothing operator with kernel $s_q(x, y)$. Suppose that $S = \sum_{q=0}^m S_q$ is any smoothing operator with kernel $s(x, y) = \sum_{q=0}^m s_q(x, y)$ having the properties

$s_q(x, y) \in A^{(0,q)(m,m-q)}(M \times M)$;

S_q commutes with $\bar\partial$; and

S_q induces the identity on $H^{0,q}_{\bar\partial}(M)$.

Then, from the Künneth formula on $M \times M$, $s(x, y)$ is $\bar\partial$-cohomologous to $\Psi(x, y)$, and consequently

$$\chi(M, \mathcal{O}) = \int_\Delta s(x, x) . \tag{2.2}$$

Toledo and Tong give a universal procedure for finding such a smoothing operator.

To explain in other terms what (2.2) means, it is convenient to use the language of currents. On a (possibly non-compact) complex manifold N , the <u>currents of type</u> (p, q) , denoted $D^{p,q}(N)$, are the continuous linear functionals

$$T : A_c^{n-p,n-q}(N) \to \mathbb{C}$$

on the compactly supported forms $(n = \dim N)$. The usual formula for distributional derivatives

$$\bar\partial T(\phi) = (-1)^{p+q+1} T(\bar\partial\phi)$$

defines $\bar\partial : D^{p,q}(N) \to D^{p,q+1}(N)$ with $\bar\partial^2 = 0$.

Currents are introduced to have a formalism including both subvarieties and smooth forms in one large complex. For example, a codimension-k analytic subvariety $Z \subset N$ defines a current $T_Z \in D^{q,q}(N)$ by integration over the regular points of Z :

$$T_Z(\phi) = \int_{Z_{reg}} \phi .$$

It can be proved that $\bar\partial T_Z = 0$ (intuitively, this is because the boundary $\bar Z - Z$ has real codimension <u>two</u> in $\bar Z$). As another example, any form $\eta \in A^{p,q}(N)$ defines a current $T_\eta \in D^{p,q}(N)$ by the formula

$$T_\eta(\phi) = \int_M \eta \wedge \phi .$$

Stokes' theorem implies that $\bar\partial T_\eta = T_{\bar\partial\eta}$.

A basic result (<u>smoothing of cohomology</u>) is that the inclusion

$$A^{p,q}(N) \to D^{p,q}(N)$$

induces an isomorphism on cohomology. In particular, given a subvariety $Z \subset N$ as above, the equation of currents

$$\bar\partial k = T_Z - \phi \tag{2.3}$$

can be solved for a smooth (k, k)-form ϕ . We may take k to be a locally integrable $(k, k-1)$-form which is smooth outside Z and has "residue type" singularities along Z , i.e. singularities generalizing the Cauchy formula

$$\bar\partial \left(\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z}\right) = T_{\{0\}}$$

in the complex plane.

Returning to $M \times M$, the current T_Δ defined by the diagonal has an expansion into bi-type

$$T_\Delta = \sum_{p=0}^{m} T_{p,\Delta} \quad \text{where}$$
$$T_{p,\Delta} = \sum_{q=0}^{m} (T_\Delta)^{(p,q)(m-p,m-q)} .$$

The equation $\bar\partial T_\Delta = 0$ implies that $\bar\partial T_{p,\Delta} = 0$ for each p . In particular, $\bar\partial T_{0,\Delta} = 0$, and as in (2.3) we may solve the equation of currents

$$\bar\partial k(x, y) = T_{0,\Delta} - s(x, y) \tag{2.4}$$

where $s(x, y) \in \sum_{q=0}^{m} A^{(0,q)(m,m-q)}(M \times M)$ is a smooth form and $k(x, y)$ is a residue form. $s(x, y)$ is the kernel of a smoothing operator with the properties listed above (2.2), and by that equation we may write the Euler characteristic

$$\chi(M, \mathcal{O}) = T_{0,\Delta} \cdot T_{m,\Delta} \tag{2.5}$$

as the <u>holomorphic self-intersection</u> number of the diagonal (note that $T_{0,\Delta} \cdot T_{m,\Delta} = T_{0,\Delta} \cdot T_\Delta$ by type considerations).

A final remark concerning (2.4) is that, if we have $k_0(x, y)$ defined only in a neighborhood of the diagonal in $M \times M$ and satisfying

$$\bar\partial k_0 = T_{0,\Delta} - s_0$$

there, then we may find a global k satisfying (2.4) and with $s_0(x, x) = s(x, x)$. Indeed, let ρ be a bump function which is one near Δ and set $k = \rho k_0$. Then (2.4) is satisfied with $s = \rho s_0 + \bar\partial\rho \wedge k_0$.

b) THE ČECH PARAMETRIX. Let M be a complex manifold with coordinate covering $\mathcal{U} = \{U_\alpha; \phi_\alpha : U_\alpha \to \mathbb{C}^m ; \phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ where defined$\}$. We may consider $\mathcal{U}$ as the raw data of the manifold. The cohomology groups $H^*(M, \mathcal{O})$ are given in terms of this raw data by the Čech method (for a suitable such covering). The Dolbeault cohomology $H^{0,*}_{\bar\partial}(M)$ is also defined intrinsically by the $\bar\partial$-operator, but the explicit formulae in the isomorphism $H^*(M, \mathcal{O}) \cong H^{0,*}_{\bar\partial}(M)$ depends on a choice of partition of unity (or something similar). On the other side of the Hirzebruch-Riemann-Roch formula, the Chern classes have been defined using a metric on M. However, there is a procedure due to Atiyah and reviewed below for defining the Chern classes in terms of the raw data. Thus, both sides of (2.1) are defined purely in terms of $\mathcal{U}$, and it makes formal sense to look for a means of evaluating the holomorphic intersection number $T_{0,\Delta}\cdot T_\Delta$ without introducing the extraneous data of a partition of unity and metric. In other words, we should try and solve the Čech analogue of (2.4) purely in terms of the raw data of M. Furthermore, since currents are an essential part of the problem, we should use a Čech complex containing currents as well as holomorphic material. Finally, by the remark at the end of (a) it will suffice to solve the Čech analogue of (2.4) in a neighborhood of the diagonal.

Putting all of this together leads to the following formulation of the problem: On a complex manifold N, we denote by $\mathcal{D}^{p,q}$ and $A^{p,q}$ the respective sheaves of currents and smooth forms of type (p, q). Given a coordinate covering $\mathcal{V} = \{V_\mu\}$ of N, we denote by

$$C^*(\mathcal{V}, \mathcal{D}^{0,*})$$

the Čech bi-complex where $C^p(\mathcal{V}, \mathcal{D}^{0,q})$ are the p-cochains with coefficients in $\mathcal{D}^{0,q}$. The total differential $D = \delta \pm \bar\partial$ satisfies $D^2 = 0$, and thus the Čech hypercohomology groups

$$\mathbb{H}^*(\mathcal{V}, \mathcal{D}^{0,*})$$

are defined. Assuming that the covering $\mathcal{V}$ is $\bar\partial$-acyclic, the inclusion $\mathcal{O} \to \mathcal{D}^{0,*}$ induces an isomorphism on cohomology

$$H^*(\mathcal{V}, \mathcal{O}) \cong \mathbb{H}^*(\mathcal{V}, \mathcal{D}^{0,*}) . \tag{2.6}$$

In fact, there are two spectral sequences E' and E" abutting to $\mathbb{H}^*(\mathcal{V}, \mathcal{D}^{0,*})$ and with

$$E_2' = H^*_\delta(\mathcal{V}, H^*_{\bar\partial}(\mathcal{D}^{0,*}))$$
$$E_2'' = H^*_{\bar\partial}(H^*_\delta(\mathcal{V}, \mathcal{D}^{0,*})) .$$

By the $\bar\partial$-acyclicity of $\mathcal{V}$, $H^0_{\bar\partial}(\mathcal{D}^{0,*}) = \mathcal{O}$ and $H^q_{\bar\partial}(\mathcal{D}^{0,*}) = 0$ for $q > 0$. Thus, the first spectral sequence degenerates and gives (2.6). Using a partition of unity, $H^q_\delta(\mathcal{V}, \mathcal{D}^{0,*}) = 0$ for $q > 0$ while $H^0_\delta(\mathcal{V}, \mathcal{D}^{0,*}) = \mathcal{D}^{0,*}(N)$. Consequently the second spectral sequence degenerates also and gives

$$H^*_{\bar\partial}(N) \cong \mathbb{H}^*(\mathcal{V}, \mathcal{D}^{0,*}) .$$

The composition of this isomorphism and (2.6) gives the Dolbeault isomorphism, whose explicit formulae depend on the partition of unity.

On $M \times M$ we consider the covering $\mathcal{U} \times \mathcal{U} = \{U_\alpha \times U_\alpha, \phi_\alpha \times \phi_\alpha, \ldots\}$ of a neighborhood of the diagonal. The current $T_{0,\Delta} \in C^0(\mathcal{U} \times \mathcal{U}, \mathcal{D}^{(0,*)(m,m-*)})$ satisfies

$$\bar\partial T_{0,\Delta} = 0 = \delta T_{0,\Delta} ,$$

and the Čech analogue of (2.4) is the relation

$$Dk = T_{0,\Delta} - s \tag{2.7}$$

where s is to be smooth. When written out, what this amounts to are the equations

$$k = k_0 + \cdots + k_{m-1} ;$$
$$\bar\partial k_0 = T_{0,\Delta}$$
$$\bar\partial k_j = \delta k_{j-1} \qquad (j = 1, \ldots, m-1)$$
$$\delta k_{m-1} = s ;$$

and

$$k_j \in C^j(\mathcal{U} \times \mathcal{U}, \mathcal{D}^{(0,*)(m,m-j-1-*)})$$
$$s \in C^m(\mathcal{U} \times \mathcal{U}, \mathcal{D}^{(0,0)(m,0)}) .$$

It follows that

$$\bar{\partial} s = \pm \delta^2 k_{m-2} = 0 , \qquad \delta s = 0$$

so that, by Hartog's theorem for $m > 1$ and by elementary reasons in case $m = 1$,

$$s \in H^m(\mathcal{U} \times \mathcal{U}, \Omega^m_{M \times M}) .$$

A solution of (2.7) is called a Čech parametrix, and for any such we have by (2.2)

(2.8) $$\chi(M, \mathcal{O}) = s[\Delta] .$$

The main result of Toledo and Tong may be informally stated as follows:

> (2.9) Given a complex manifold M and raw data $\{U_\alpha, \phi_\alpha, \phi_{\alpha\beta}\}$, there exists a Čech parametrix $s = s\{U_\alpha, \phi_\alpha, \phi_{\alpha\beta}\}$ which is functorial in the raw data. In particular, the formulae for the Čech cocycle $s \in H^m(\mathcal{U}, \Omega^m_M)$ are given by universal expressions in terms of the $\phi_\alpha, \phi_{\alpha\beta}$ and finitely many of their derivatives.

c) ČECH CHERN CLASSES. To explain how (2.9) gives a Riemann-Roch formula, we recall the following construction of Atiyah:

> Given a complex manifold M, an open covering $\{U_\alpha\}$, and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, C)$ for a holomorphic vector bundle $E \to M$, the Atiyah curvature cocycle $\Theta \in Z^1(\{U_\alpha\}, \Omega^1_M(\mathrm{Hom}(E, E)))$ is defined by
>
> $$\Theta_{\alpha\beta} = g^{-1}_{\alpha\beta} \cdot dg_{\alpha\beta} .$$

Note that Θ is functorial in the raw data $\{U_\alpha, g_{\alpha\beta}\}$. For any invariant polynomial $P(X)$ $(X \in \mathfrak{gl}(r, C))$ with corresponding multilinear form $P(X_1, \ldots, X_q)$ the Atiyah Chern polynomial is defined by

$$P(\Theta) \in Z^q(\{U_\alpha\}, \Omega^q_M)$$

where $q = \deg P$ and where

$$P(\Theta)_{\alpha_0 \ldots \alpha_q} = P(\Theta_{\alpha_0\alpha_1}, \mathrm{Ad}g_{\alpha_0\alpha_1}\Theta_{\alpha_1\alpha_2}, \ldots, \mathrm{Ad}g_{\alpha_{q-2}\alpha_{q-1}}\Theta_{\alpha_{q-1}\alpha_q}) .$$

The Atiyah Chern polynomials are functorial in the raw data, and a fairly straightforward argument using invariant theory shows that:

> (2.10) Any method of functorially assigning to the raw data $\{U_\alpha, g_{\alpha\beta}\}$ a cocycle $c = c(\{U_\alpha\}, g_{\alpha\beta}) \in Z^m(\{U_\alpha\}, \Omega^m_M(\mathrm{Hom}(E, E))$ is necessarily an Atiyah Chern polynomial.

Combining (2.8), (2.9), and (2.10) gives a formula

$$\chi(M, \mathcal{O}) = \int_M T(c_1, \ldots, c_m)$$

for some universal polynomial T. We shall give the explicit formula for T after discussing the first step in the construction of a Čech parametrix and using this to deduce the Atiyah-Bott fixed point formula.

d) THE FIXED POINT FORMULA. We shall give the first step in the construction of a universal Čech parametrix, and shall use this to deduce the Atiyah-Bott-Lefschetz-Woods Hole fixed point formula (this conference being on the tenth anniversary of Woods Hole).

For this we use the Bochner-Martinelli kernel on $C^m \times C^m$ defined by

(2.11) $$k_0(z, \zeta) = C_m \frac{\sum_{i=1}^m (-1)^i \overline{(z_i - \zeta_i)\phi_i(z-\zeta)} \wedge \Phi(\zeta)}{\|z - \zeta\|^{2m}}$$ where

$$\Phi(w) = dw_1 \wedge \cdots \wedge dw_m \quad \text{and} \quad \phi_i(w) = dw_1 \wedge \cdots \wedge \widehat{dw_i} \wedge \cdots \wedge dw_m .$$

Note that $k_0(z, \zeta)$ has bi-type $(0, *)(m, m-1-*)$, and is formally $\bar{\partial}$-closed. As distributions,

$$\bar{\partial} k_0(z, \zeta) = T_{0,\Delta}$$

for a suitable normalizing constant C_m.

Using the coordinate charts $\phi_\alpha \times \phi_\alpha : U_\alpha \times U_\alpha \to C^m \times C^m$ to pull back the Bochner-Martinelli kernels gives $k_0 \in C^0(\mathcal{U} \times \mathcal{U}, \mathcal{D}^{(0,*)(m,m-1-*)})$ satisfying

$$\bar{\partial} k_0 = T_{0,\Delta} .$$

It is usually not the case that $k_{0,\alpha} = k_{0,\beta}$ in $(U_\alpha \times U_\alpha) \cap (U_\beta \times U_\beta)$. However, from general principles, the equation

$$\bar{\partial} k_{1,\alpha\beta} = k_{0,\alpha} - k_{0,\beta}$$

has a solution, as does the succeeding equation

$$\bar{\partial} k_{2,\alpha\beta\gamma} = k_{1,\alpha\beta} - k_{1,\beta\gamma} + k_{1,\alpha\gamma} ,$$

and so forth. The thrust of (2.9) is that all of these equations may be solved in a suitable universal manner.

Now to the fixed point formula! Let $f : M \to M$ be a biholomorphic automorphism having isolated transversal fixed points. There are induced maps on cohomology

$$f^{*,q} : H^q(M, \mathcal{O}) \to H^q(M, \mathcal{O}) ,$$

and the holomorphic Lefschetz number is defined by

$$L(f, \mathcal{O}) = \sum_{q=0}^{m} (-1)^q \operatorname{Trace} f^{*,q} .$$

What we wish to find is a formula for $L(f, \mathcal{O})$ in terms of the eigenvalues of the induced linear map

$$f_{*,p} : T_p(M) \to T_p(M)$$

at the fixed points of f, and the result is:

$$L(f, \mathcal{O}) = \sum_{f(p)=p} \frac{1}{\det(I - f_{*,p})} . \tag{2.12}$$

For the proof, we denote by $G_f = \{(p, f(p)\}$ the graph of f in $M \times M$. As in the formal part of the Riemann-Roch, one may prove the intersection relation

$$L(f, \mathcal{O}) = T_{0,\Delta} \cdot G_f . \tag{2.13}$$

Choose a coordinate covering $\{U_\alpha\}$ of M such that each fixed point p is in exactly one open set U_α, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to this covering and such that $\rho_\alpha \equiv 1$ near any p. Let $k_{0,\alpha}$ be the Bochner-Martinelli kernel in $U_\alpha \times U_\alpha$ and consider the form

$$k = \sum_\alpha \rho_\alpha k_{0,\alpha}$$

on $M \times M$. The equation of currents

$$\bar{\partial} k = T_{0,\Delta} - s_0 , \tag{2.14}$$

where s_0 is the integrable form $\sum_\alpha \bar{\partial}\rho_\alpha \wedge k_{0,\alpha}$, is valid. This form s_0 is not smooth on $M \times M$, but it is smooth on the graph of f, and consequently

$$T_{0,\Delta} \cdot G_f = \int_{G_f} s_0 .$$

Letting $B_\varepsilon(p)$ be a ball of radius ε around p, Stokes' theorem together with (2.13) give

$$L(f, \mathcal{O}) = \sum_{f(p)=p} \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p)} k . \tag{2.15}$$

To evaluate the integrals, we choose local holomorphic coordinates $z = (z_1, \ldots, z_m)$ in a neighborhood U_α around p such that $f(z) = A \cdot z + (\cdot\cdot)$ where A is a non-singular matrix and $(\cdot\cdot)$ denotes higher order terms. These latter will disappear in the limit, and so we shall assume for simplicity that $f(z) = Az$. Letting (z, ζ) denote coordinates in $U_\alpha \times U_\alpha$, the form

$$\sigma = C_m \frac{\sum_{i=1}^{m} (-1)^i \overline{(z_i - \zeta_i)\phi_i(z - \zeta)} \wedge \phi(z - \zeta)}{\|z - \zeta\|^{2m}}$$

gives the standard d-closed volume form on the normal spheres to the diagonal having constant value one on any such sphere. On the graph $\zeta = f(z)$,

$$k_0 = \frac{\sigma}{\det(I - A)}$$

so that $\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p)} k_0 = \frac{1}{\det(I - A)}$.

We shall conclude by explaining how (2.12) leads to the expression for the Todd polynomials in the Riemann-Roch formula. For this, we first need to discuss Bott's residue theorem. In simplest form, this result states that, if M has a nowhere vanishing holomorphic vector field v and if $P(c_1, \ldots, c_m)$ is any polynomial of degree m in the Chern classes, then

$$\int_M P(c_1, \ldots, c_m) = 0 . \tag{2.16}$$

To prove (2.16), we shall choose a special connection for the tangent bundle such that the Chern polynomial $P(c_1(\Theta), \ldots, c_m(\Theta))$ is identically zero. For this we take a coordinate covering $\{U_\alpha\}$ of M with coordinates $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^m)$ in U_α such that $v = \partial/\partial z_\alpha^1$. Let D_α be the flat Euclidean connection in U_α and $D = \sum_\alpha \rho_\alpha D_\alpha$. The curvature Θ of D satisfies the contraction relation

$$< \Theta, v > \equiv 0 ;$$

i.e., in U_α the curvature does not involve dz_α^1. To see this, we need only observe that in $U_\alpha \cap U_\beta$ the connection matrix θ_β for D_β <u>written in terms of the coordinates</u> z_α satisfies $< \theta_\beta, v > \equiv 0$. Now it is clear that $P(c_1(\Theta), \ldots, c_m(\Theta)) \equiv 0$ for this curvature.

Bott's general formula deals with a holomorphic vector field v having isolated non-degenerate zeroes. Near such a zero p, we choose holomorphic coordinates $(z_1, \ldots, z_m)$ such that

$$v = \sum_{i,j} a_{ij} z_i \frac{\partial}{\partial z_j} + \text{(higher order terms)} .$$

Denote by $\lambda_1, \ldots, \lambda_m$ the eigenvalues of the matrix (a_{ij}) and by σ_q the q^{th} elementary symmetric function of the λ_i's. Bott's formula is the relation

$$\int_M P(c_1, \ldots, c_m) = \sum_{v(p)=0} \frac{P(\sigma_1, \ldots, \sigma_m)}{\lambda_1 \cdots \lambda_m} . \tag{2.17}$$

To prove (2.17), we choose a connection D_1 on M which is flat near each zero of v, and a connection D_2 on $M - \{\text{zeroes of } v\}$ satisfying $< \Theta_2, v > \equiv 0$ as before. Now then the proof that the de Rham Chern classes are independent of the choice of connection gives a relation

$$P(c_1(\Theta_1), \ldots, c_m(\Theta_1)) - P(c_1(\Theta_2), \ldots, c_m(\Theta_2)) = d\eta$$

on $M - \{\text{zeroes of } v\}$ for an explicit η (note that $P(c_1(\Theta_2), \ldots, c_m(\Theta_2)) \equiv 0$). Stokes' theorem then gives

$$\int_M P(c_1(\Theta_1), \ldots, c_m(\Theta_1)) = \sum_{v(p)=0} \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p)} \eta ,$$

which when computed out yields (2.17).

Finally, the explicit form of the Todd genus arises as follows: Suppose that v is a holomorphic vector field with isolated zeroes and set $f_t = \exp(tv)$. From the homotopy formula $\mathrm{Lie}_v = v \lrcorner \bar\partial + \bar\partial v \lrcorner$ on the Dolbeault complex, it follows that $f_t^{*,q}$ is the identity on $H^q(M, \mathcal{O})$. Thus, for all t,

$$\chi(M, \mathcal{O}) = L(f_t, \mathcal{O}) = \sum_{v(p)=0} \prod_{i=1}^m \frac{1}{1 - e^{t\lambda_i}} .$$

Taking $t = -1$ gives

$$\chi(M, \mathcal{O}) = \sum_{v(p)=0} \frac{1}{\lambda_1 \cdots \lambda_m} \prod_{i=1}^m \frac{\lambda_i}{1 - e^{-\lambda_i}} .$$

Combining this with (2.1) and (2.17) yields

$$\chi(M, \mathcal{O}) = \int_M T(c_1, \ldots, c_m)$$

where the Todd polynomial T is obtained by writing formally $1 + c_1 + \cdots + c_m = \prod_{i=1}^m (1 + \gamma_i)$ and then setting

$$T(c_1, \ldots, c_m) = \left\{ \prod_{i=1}^m \frac{\gamma_i}{1 - e^{-\gamma_i}} \right\}_m$$

where $\{\cdots\}_m$ is the component of degree m.

REFERENCES FOR LECTURE 2

The Hirzebruch-Riemann-Roch formula first appeared in the book by Hirzebruch listed as a general reference. Since then the subject has mushroomed into index theory, for which the most recent account is

M. Atiyah, R. Bott, and V. K. Patodi, On the heat equation and the index theorem, Inventiones Math., vol. 19 (1973), 279-330.

The proof of the Hirzebruch formula we are following is from

D. Toledo and Y. Tong, A parametrix for $\bar\partial$ and Riemann-Roch in Čech theory, to appear.

The general theory of currents and smoothing of cohomology is given in

G. de Rham, Variétés differentiables, Hermann, Paris (1955).

A discussion of the currents which appear in complex analysis appears in

J. King, The currents defined by analytic varieties, Acta Math., vol. 127 (1971), 185-220.

The Čech Chern class construction is given in

M. F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc., 85 (1957), 181-207.

The various integral formulae which interpolate between the Bochner-Martinelli kernel and Cauchy kernel are discussed in

R. Harvey, Integral formulae connected by Dolbeault's isomorphism, Rice Univ. Studies, vol. 56 (1970), 77-97.

The Atiyah-Bott fixed point formula is given in

M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes (II), Annals of Math., vol. 88 (1968), 451-491.

Proofs of Bott's residue theorem may be found in

R. Bott, Vector fields and characteristic numbers, Michigan Math. J., vol. 14 (1967), 231-244;

S. S. Chern, Geometry of characteristic classes, Proc. 13th Biennial Seminar (1972), 1-40.

3. KÄHLER MANIFOLDS; HODGE THEORY

Among complex manifolds some are singled out by the fact that their holomorphic tangent bundle can be endowed with a special kind of hermitian metric, a Kähler metric. The presence of such a metric has important global implications and we shall discuss some of them.

There are several possible definitions of what a Kähler metric is. To a hermitian metric

$$h = \sum_{i,j} h_{ij}\, dz_i d\bar{z}_j$$

on a complex manifold M there is naturally attached a distinguished connection D_h on the holomorphic tangent bundle $T_h(M)$, the metric connection. $D = D_h \oplus \bar{D}_h$ is a connection on the complexification $T_h \oplus \bar{T}_h$ of the real tangent bundle of M.

i) The hermitian metric h is said to be Kähler if D is the riemannian connection on M. In other terms, set

$$\tau(X, Y) = D_X Y - D_Y X - [X, Y]$$

for any couple X, Y of vector fields on M. τ is easily seen to be a tensor, the so-called torsion tensor of the metric h. Saying that h is Kähler means that τ vanishes identically.

ii) To give a second, equivalent definition of what a Kähler metric is, let

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{ij}\, dz_i \wedge d\bar{z}_j$$

be the exterior form associated to h. A necessary and sufficient condition for h to be Kähler is that ω be d-closed.

iii) A third definition is the following: a hermitian metric h is said to be Kähler if it can be written locally as

$$\sum_i dz_i d\bar{z}_i + [2]$$

where $[2]$ stands for terms which vanish of order at least two at $z = 0$, relative to suitable holomorphic coordinate systems.

Each of the above definitions has some advantage over the others. Definition ii), for example, makes it clear that the property of having a Kähler metric is inherited by subvarieties and that a Kähler metric gives a distinguished cohomology class in $H^2(M, \mathbb{R})$. Definition iii) is most useful in computation as it allows us to prove the basic Kähler identities by verifying them for the flat metric on $\mathbb{C}^m$.

As for the proof of the equivalence of i), ii), iii), ii) follows from i) by noticing that for the metric connection D_h

$$\partial h_{ij} = (D_h' \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$$

holds and that i) means that:

$$D_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = D_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_i} .$$

The converse is done in the same way. iii) clearly implies ii), and one obtains iii) from i) by choosing the coordinate systems to be geodesic coordinate systems up to second order terms.

We now want to discuss the basic Kähler identities satisfied by the operators acting on the algebra

$$A^*(M) = \sum A^{p,q}(M) .$$

We define

$$L : A^{p,q}(M) \rightarrow A^{p+1,q+1}(M)$$

to be multiplication by ω, and Λ to be the adjoint of L relative to the given metric. Also we denote by $\pi_{p,q}$ the projection onto $A^{p,q}(M)$ and set $\pi_k = \sum_{p+q=k} \pi_{p,q}$. It is a consequence of the theory of unitary invariants that on a "general" Kähler manifold the operators L, Λ, $\pi_{p,q}$ generate the algebra of all invariant algebraic operators on $A^*(M)$.

Probably the neatest and most compact way of presenting the identities satisfied by L, Λ and $\pi_{p,q}$ is the following: Write

$$H = \sum_k (m-k)\pi_k$$

where m is the complex dimension of M. An easy computation then gives the commutation relations:

$$(3.1) \qquad \begin{aligned} [H, \Lambda] &= 2\Lambda \\ [H, L] &= -2L \\ [\Lambda, L] &= H . \end{aligned}$$

This means that the assignments:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \Lambda$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow H$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow L$$

give a representation

$$\rho : s\ell_2 \rightarrow \mathrm{End}(A^*(M))$$

which can also be viewed as a continuously varying family of finite dimensional representations. The standard representation theory for $s\ell_2$ then implies that:

$$(3.2) \qquad L^k : A^{m-k}(M) \rightarrow A^{m+k}(M)$$

is an isomorphism and that

$$(3.3) \qquad A^k(M) = \sum L^h P^{k-2h} A(M) \qquad \text{(direct sum)}$$

where $PA^\ell(M)$ (the primitive part of A^ℓ) stands for the kernel of $L^{m-\ell+1} \mid A^\ell(M)$. It is also clear that ρ is compatible with decomposition into (p, q)-type, hence $PA^\ell(M)$ decomposes into a direct sum:

$$PA^\ell(M) = \sum_{p+q=\ell} PA^{p,q}(M) .$$

What we have done so far does not require that the metric be Kähler. The crucial fact, however, is that, when the metric is Kähler, (3.2) and (3.3), together with the decomposition into type, pass over to cohomology.

Proving this involves exploring the commutation relation between ρ and exterior differentiation. The basic such relation, from which all others can be deduced, is:

$$(3.4) \qquad [\Lambda, d] = (d^c)^*$$

where $(d^c)^*$ is the adjoint of

$$d^c = C^{-1}dC = \sqrt{-1}(\bar\partial - \partial)$$

and $C = \sum \sqrt{-1}^{\,p-q}\pi_{p,q}$ is the Weil operator. (3.4) is actually equivalent to the metric being Kähler. The two (easy) consequences of (3.4) that will be important for us are:

$$(3.5) \qquad [\Delta, \rho] = 0$$

where $\Delta = dd^* + d^*d$ is the Laplace operator, and

$$(3.6) \qquad \Delta = 2\Box$$

where $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is the complex Laplacian. (3.6) in turn implies that

$$(3.7) \qquad [\Delta, \pi_{p,q}] = 0 .$$

since $\square$ is of pure (0, 0) type.

As for the proof of (3.4), it suffices to do the case of the flat metric on C^m, which is straightforward.

The main consequence of (3.5) is that, on a compact Kähler manifold, ρ passes over to cohomology, and therefore, by standard representation theory,

$$L^k : H^{m-k}(M, \mathbb{R}) \to H^{m+k}(M, \mathbb{R}) \tag{3.8}$$

is an isomorphism (Hard Lefschetz Theorem) and one has the Lefschetz decomposition

$$H^k(M, \mathbb{R}) = \bigoplus_h L^h P^{k-2h}(M, \mathbb{R}) \tag{3.9}$$

where $P^\ell(M, \mathbb{R})$ (primitive cohomology) is the kernel of $L^{m-\ell+1}|H^\ell(M, \mathbb{R})$.

On the other hand, (3.7) implies that $H^k(M, C)$ has a direct sum decomposition (Hodge decomposition)

$$H^k(M, C) = \sum_{p+q=k} H^{p,q}(M) \tag{3.10}$$

such that

$$H^{p,q}(M) = \overline{H^{q,p}(M)} .$$

Here $H^{p,q}(M)$ stands for the subspace of $H^k(M, C)$ generated by d-closed (p, q)-forms. Moreover the Hodge decomposition is obviously compatible with the Lefschetz decomposition. It should also be noticed that the equality (3.6) gives an isomorphism (which does not depend on the metric) between $H^{p,q}(M)$ and $H^q(M, \Omega^p)$.

As a final ingredient, if we define a bilinear form

$$Q_k(\xi, \eta) = \int_M \omega^{m-k} \wedge \xi \wedge \eta$$

on $P^k(M)$, then the Hodge-Riemann bilinear relations hold:

$$(3.11) \quad \begin{array}{ll} \text{I)} \ Q_k(P^{p,q}, P^{q',p'}) = 0 & \text{unless } p = p', \ q = q' \\ \text{II)} \ Q_k(C\xi, \bar{\xi}) > 0 & \xi \in P^k(M), \ \xi \neq 0 \end{array}$$

In case the cohomology class of ω is rational — which, by a theorem of Kodaira, is equivalent to saying that an integral multiple of the class

of ω is induced by the generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ via an embedding into $\mathbb{P}^n$ — the representation ρ, and hence the Lefschetz decomposition, are defined over Q.

To conclude, we may remark that, while the Lefschetz decomposition is of a topological nature, the Hodge decomposition depends on the complex structure of M, and in fact is a very significant invariant for this.

As another application of the Kähler identities we will prove the following result, due to Kodaira, Nakano and Le Potier.

(3.12) Let $E \to M$ be a rank r positive vector bundle on a compact complex manifold M of dimension m. Then

$$H^q(M, \Omega^p(E)) = 0$$

if $p + q \geq m + r$.

PROOF: The proof is done by reducing to the rank 1 case. We shall deal with this first. It follows from the hypotheses that M is Kähler, and we shall choose (the hermitian form associated to) $\sqrt{-1}\,\Theta$ as a metric on M, where Θ is the curvature form of E. We shall show that, for any E-valued harmonic (p, q)-form ϕ, the Nakano inequalities

$$(3.13) \quad \begin{array}{l} (\Lambda L\phi, \phi) \geq 0 \\ (L\Lambda\phi, \phi) \leq 0 \end{array}$$

hold. Combining them gives

$$0 \leq ([\Lambda, L]\phi, \phi) = (m - p - q)(\phi, \phi)$$

which implies that ϕ is zero if $p + q > m$, as desired. Now to the proof of (3.13)! A slight generalization of (3.4) gives the commutation relation:

$$[\Lambda, D''] = -\sqrt{-1}\, D'^*$$

where $D = D' + D''$, $D'' = \bar{\partial}$, is the metric connection of E. Therefore:

$$\begin{aligned} (\Lambda L\phi, \phi) &= \sqrt{-1}\,(\Lambda\Theta\phi, \phi) = \\ &= \sqrt{-1}\,(\Lambda\bar{\partial}D'\phi, \phi) = \\ &= \sqrt{-1}\,(\bar{\partial}\Lambda D'\phi, \phi) + (D'^*D'\phi, \phi) = \end{aligned}$$

$$= (D'\phi, D'\phi) \geq 0$$

since ϕ , being harmonic, is $\bar\partial$-closed and $\bar\partial^*$-closed. This is the first of the Nakano inequalities and the proof of the other one is similar.

When the rank of E is larger than one, we may argue as follows. Let $\mathbb{P}(E^*)$ be the complex manifold whose points are the hyperplanes lying in the fibres of E . $\mathbb{P}(E^*)$ has dimension $m + r - 1$ and is obviously a bundle

$$\mathbb{P}(E^*) \xrightarrow{\pi} M$$

with projective $(r-1)$-spaces as fibres. Let H be the standard tautological line bundle over $\mathbb{P}(E^*)$. Direct computation shows that:

H is positive

$$R^i\pi_*\Omega^j(H) = \begin{cases} 0 & \text{if } i > 0 \\ \Omega^j(E) & \text{if } i = 0 . \end{cases}$$

In particular $\pi_*\mathcal{O}(H) = \mathcal{O}(E)$ and the E_2-term of the Leray spectral sequence abutting to $H^*(\mathbb{P}(E^*), \Omega^p(H))$ is

$$E_2^{ij} = H^j(M, R^i\pi_*\Omega^p(H)) = \begin{cases} 0 & \text{if } i > 0 \\ H^j(M, \Omega^p(E)) & \text{if } i = 0 . \end{cases}$$

Therefore the above spectral sequence degenerates at the E_2-term and

$$H^q(M, \Omega^p(E)) = H^q(\mathbb{P}(E^*), \Omega^p(H)) = 0 \quad \text{if } p + q \geq m + r$$

by applying to H the line bundle case of the theorem.

A classical application of the Kodaira-Nakano vanishing theorem is an analytic proof of the Lefschetz theorem on hyperplane sections with $\mathbb{R}$-coefficients. Let D be a smooth, ample divisor on M (this is the same as saying that the line bundle $[D]$ is positive). $[D]|_D$ can be identified with the normal bundle to D in M . It follows that there are exact sequences

$$0 \to \Omega^p_M[D]^* \to \Omega^p_M \to \Omega^p_{M|D} \to 0$$

$$0 \to \Omega^{p-1}_D[D]^* \to \Omega^p_{M|D} \to \Omega^p_D \to 0 .$$

Applying the vanishing theorem and Serre duality to $H^q(M, \Omega^p_M[D]^*)$ and $H^q(D, \Omega^p_D[D]^*)$ gives that the restriction map

$$H^{p,q}(M) \to H^{p,q}(D)$$

is an isomorphism if $p + q \leq m - 2$ and is injective if $p + q = m - 1$. It follows from (3.10) that this is just the Lefschetz theorem on hyperplane sections. Conversely, Mumford and Ramanujan have proved that the topological Lefschetz theorem implies the Kodaira-Nakano vanishing theorem.

It is perhaps worth noticing that the integral formulae underlying the vanishing theorems are exactly the same as those used to establish the basic estimates used in the study of the $\bar\partial$-operator on non-compact manifolds by L^2-methods. This is an illustration of our contention that the compact case has been properly understood only after there are results for general possibly non-compact complex manifolds which specialize to the existing ones in the compact case.

Another implication of the fundamental Kähler identity (3.4) and the subsequent equality (3.6) among the various Laplacians is the

PRINCIPLE OF TWO TYPES: Given a (p, q) form ϕ which is also exact, $\phi = d\eta$, then we may write either

$$\text{(3.14)} \quad \begin{array}{ll} \phi = d\eta' , & \eta' \text{ has type } (p-1, q) \text{ , or} \\ \phi = d\eta'' , & \eta'' \text{ has type } (p, q-1) . \end{array}$$

An application of this is that:

(3.15) All Massey products on a compact Kähler manifold are zero.

Recall that, given closed differential forms α, β, γ of degrees p, q, and r ,

$$\alpha \wedge \beta = d\rho$$
$$\beta \wedge \gamma = d\sigma$$

the Massey triple product is the closed $p + q + r - 1$ form

$$\alpha \wedge \sigma + (-1)^{(\cdots)}\gamma \wedge \rho .$$

Its cohomology class $[\alpha, \beta, \gamma]$ is well-defined in

$$\text{(3.16)} \quad H^{p+q+r-1}(M)/H^p(M) \cup H^{q+r-1}(M) + H^r(M) \cup H^{p+q-1}(M) .$$

To see that $[\alpha, \beta, \gamma] = 0$, we decompose α, β, γ under the Hodge decomposition and consider the case where they all three have pure Hodge type. Using (3.14), we may write

$$\alpha \wedge \beta = d\rho' \quad \text{or} \quad \alpha \wedge \beta = d\rho''$$
$$\beta \wedge \gamma = d\sigma' \quad \text{or} \quad \beta \wedge \gamma = d\sigma''$$

where ρ', ρ'' and σ', σ'' have different Hodge types. But then $[\alpha, \beta, \gamma]$ has the two representatives $\alpha \wedge \sigma' + (-1)^{(\cdots)}\gamma \wedge \rho'$ and $\alpha \wedge \sigma'' + (-1)^{(\cdots)}\gamma \wedge \rho''$ of different Hodge types, and so must be zero since the quotient space (3.16) has a direct sum Hodge decomposition.

Now it is well-known to topologists that Massey triple products are homotopy invariants of a space which are not contained in its cohomology. What is suggested by the above argument, together with the vanishing of the higher Massey products which is proved similarly, is that: among all spaces with a given cohomology ring, the Kähler manifolds (if there are any) have the "simplest" homotopy type. In particular, for simply-connected M the rational homotopy groups and rational Whitehead products should be determined from the cohomology alone. Likewise, the rational nilpotent completion of $\pi_1(M)$ should be determined by $H^1(M, Q)$ together with the cup product $H^1(M, Q) \otimes H^1(M, Q) \to H^2(M, Q)$. This can all be easily proved using (3.15) together with Sullivan's recent de Rham homotopy theory.

The philosophy here, as well as in Deligne's degeneration of spectral sequence arguments used in proving the existence of mixed Hodge structures on general varieties (to be discussed in the next talk), is that any sort of naturally defined higher cohomology operation on a compact Kähler manifold must, because of the principle of two types, be zero.

REFERENCES FOR LECTURE 3

The basic Kähler identities are proved in

A. Weil, Introduction à l'étude des variétés kähleriennes, Hermann, Paris (1958).

The conceptual explanation of the Kähler identities via $s\ell_2$ appears in

S. S. Chern, On a generalization of Kähler geometry, in A Symposium in Honor of S. Lefschetz, Princeton Univ. Press, Princeton, N. J. (1957), 103-121.

The theorem of Kodaira referred to just below (3.11) is given in his paper listed in the general references.

The vanishing theorem (3.12) appears in

S. Nakano, On complex analytic vector bundles, J. Math. Soc. Japan, vol. 7 (1955), 1-12; and

J. Le Potier, Annulation de la cohomologie à valeurs dans certains fibrés vectoriels holomorphes et applications, Thèse présentée à l'Université de Poitiers, (1974).

The equivalence of the vanishing theorem and Lefschetz theorem on hyperplane sections is given in

C. P. Ramanujam, Remarks on the Kodaira vanishing theorem, Jour. Indian Math. Soc. Vol. 36 (1972), 41-50.

The result on the homotopy type of Kähler manifolds will appear in

P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, On the homotopy type of compact Kähler manifolds, to appear.

4. GENERALIZATIONS OF HODGE STRUCTURES

a) The first topic we shall discuss today is Hodge structure and mixed Hodge structure. The definition of a Hodge structure is obtained by extracting the essential features of the Hodge decomposition on the primitive part of the cohomology of a projective variety. Formally, a Hodge structure of weight M consists of a real vector space $H_{\mathbb{R}}$, a lattice $H_{\mathbb{Z}}$ and a decreasing filtration (the Hodge filtration) on the complexification H_C of $H_{\mathbb{R}}$

$$0 \subset F^m \subset \cdots \subset F^0 = H_C$$

such that, for any p,

$$H_C = F^p \oplus \overline{F^{m-p+1}} . \tag{4.1}$$

An equivalent way of defining a Hodge structure is saying that H_C should have a direct sum decomposition

$$(4.2)\qquad H_C = \sum_{p+q=m} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}} .$$

From the filtration one obtains (4.2) by setting

$$H^{p,m-p} = F^p \cap \overline{F^{m-p}}$$

and conversely, given (4.2) one can recover the Hodge filtration by setting

$$F^p = \sum_{i\geq p} H^{i,m-i} .$$

The <u>Weil</u> <u>operator</u> of the Hodge structure H is defined, as usual, to be

$$C \sum \phi^{pq} = \sum \sqrt{-1}^{\,p-q}\, \phi^{p,q} .$$

A <u>polarization</u> on H is a rational (non-degenerate) bilinear form Q, symmetric if m is even, skew if m is odd, such that the Hodge-Riemann bilinear relation

$$(4.3)\qquad Q(F^p, F^{m-p+1}) = 0, \qquad Q(C\xi, \bar{\xi}) > 0 \quad \text{if } \xi \neq 0$$

hold. In lecture 3 we have sketched a proof of the fact that the primitive cohomology

$$P^m(M)$$

of a projective variety M carries a natural polarized Hodge structure of weight m.

A morphism of type (r, r) between two Hodge structures H and H' is a rationally defined homomorphism

$$\phi : H_C \to H'_C$$

such that

$$\phi(F^p) \subset F'^{p+r}$$

or, equivalently:

$$\phi(H^{p,q}) \subset H^{p+r,q+r} .$$

Most linear algebra constructions, like taking Hom's, tensor products, etc. of Hodge structures of arbitrary weights, or direct sums of Hodge structures of the same weight, can be performed within the category of Hodge structures.

The structure of the cohomology of general open or singular algebraic varieties is more complex than a plain Hodge structure. However it is a fundamental theorem of Deligne that:

The cohomology groups of a general algebraic variety carry natural, functorial mixed Hodge structures.

We shall now define what such objects are and prove a very special case of Deligne's theorem.

A <u>mixed</u> <u>Hodge</u> <u>structure</u> consists of a real vector space $H_{\mathbb{R}}$, a lattice $H_{\mathbb{Z}}$ and two finite filtrations of $H_C = H_{\mathbb{R}} \otimes C$:

$$0 \subset \cdots \subset W_m \subset W_{m+1} \subset \cdots \subset H_C \qquad \text{(weight filtration)}$$
$$0 \subset \cdots \subset F^p \subset F^{p-1} \subset \cdots \subset H_C \qquad \text{(Hodge filtration)}$$

such that:

i) $\{W_m\}$ is rationally defined

ii) $\{F^p\}$ induces a Hodge structure of weight m on each of the quotients W_m/W_{m-1}.

"Regular" Hodge structures of weight m can be viewed as mixed Hodge structures with a trivial weight filtration:

$$0 = W_{m-1} \subset W_m = H_C .$$

A <u>morphism</u> <u>of</u> <u>type</u> (r, r) of mixed Hodge structures is a rationally defined homomorphism

$$\phi : H_C \to H'_C$$

such that

$$\phi(W_m) \subset W'_{m+2r}$$

$$\phi(F^p) \subset F'^{p+r} .$$

Such morphisms turn out to be <u>strict</u> relative to both the weight and Hodge filtration, i.e.

$$\operatorname{Im}\phi \cap W'_{m+2r} = \phi(W_m)$$

$$\operatorname{Im}\phi \cap F'^{p+r} = \phi(F^p) .$$

All the standard linear algebra constructions can be performed within the category of mixed Hodge structure, which is, moreover, <u>abelian</u>.

We shall now prove, using differential forms, Deligne's theorem for varieties of the very special form

$$X = \bigcup_i X_i$$

where the X_i's are smooth, compact Kähler subvarieties of X of the same dimension meeting <u>transversally</u>.

To do this we first have to give an analogue of de Rham's theorem for X. For every multiindex $I = (i_0, \ldots, i_q)$ we set:

$$|I| = q+1$$

$$X_I = X_{i_0} \cap \cdots \cap X_{i_q} .$$

We also set:

$$X^{[q]} = \coprod_{|I|=q+1} X_I$$

$$A^{r,s}(X) = r\text{-forms on } X^{[s]} .$$

The differentials d (exterior differentiation) and

$$\delta : A^{r,s-1}(X) \to A^{r,s}(X)$$

defined by the formula:

$$\delta\phi(i_0 \cdots i_s) = \sum (-1)^j \phi(i_0 \cdots \hat{i}_j \cdots i_s)|_{X_{(i_0 \cdots i_s)}}$$

make

$$A^{*,*}(X) = \bigoplus A^{r,s}(X)$$

into a double complex.

(4.4) LEMMA: The (total) cohomology of $A^{**}(X)$ is canonically isomorphic to the cohomology of X.

PROOF: We may define <u>sheaves</u> $A^{r,s}(X)$ whose sections over an open set U are $A^{r,s}(U)$. These sheaves are obviously acyclic and their direct sum is made into a double complex by d and δ. All we have to do is then to show that $A^{**}(X)$ is a resolution of the constant sheaf $\mathbb{C}$ (or $\mathbb{R}$, depending on the coefficients we are using). It is clear that a section ϕ of $A^{0,0}(X)$ such that

$$d\phi = \delta\phi = 0$$

must be constant. It remains to prove the Poincaré lemma for $A^{**}(X)$. The E_2-term of one of the two spectral sequences of the double complex $A^{**}(X)$ is:

$$E_2^{pq} = H^q_\delta H^p_d(A^{**}(X)) .$$

By the usual Poincaré lemma this is zero whenever $p > 0$, therefore $E_2 = E_\infty$ and

$$H^q(A^{**}(X)) = E_2^{0q} .$$

On the other hand E_2^{0q} is the q-th cohomology sheaf of the complex of sheaves

$$(4.5) \qquad \cdots \to \mathbb{C}_{X^{[q]}} \xrightarrow{\delta} \mathbb{C}_{X^{[q+1]}} \to \cdots$$

where $\mathbb{C}_Y$ stands for the constant sheaf $\mathbb{C}$ on Y. Now the formula for δ is the same as the one for the coboundary operator on a simplex, therefore

$$E_2^{0q} = \begin{cases} 0 & \text{if } q > 0 \\ \mathbb{C}_X & \text{if } q = 0 \end{cases}$$

which proves the Lemma. Q.E.D.

We now define the weight and Hodge filtrations on $A^{**}(X)$ as follows:

$$W_m A^{**}(X) = \sum_{r \leq m} A^{r,*}(X)$$

$$F^p A^{**}(X) = \sum_{r,s} F^p A^{r,s}(X) .$$

These two filtrations induce filtration $\{W_m\}$ and $\{F^p\}$ on the cohomology of X.

We are going to show that:

$\{W_m\}$ and $\{F^p\}$ give a mixed Hodge structure on the cohomology of X.

PROOF: The first step is to replace the filtration $\{W_m\}$ with a decreasing filtration $\{\tilde{W}^h\}$ which will induce the same filtration on each of the cohomology groups of X (up to indices, of course) and will allow us to use a spectral sequence argument. We set:

$$\tilde{W}^h A^{**} = \sum_{s \geq h} A^{*s} .$$

We shall show that the spectral sequence associated with the filtration $\{\tilde{W}^h\}$ degenerates at the E_2-term and that E_2^{pq} has a Hodge structure of pure weight p. When coupled with the remark that

$$\tilde{W}^h H^m(X, C) = W_{m-h} H^m(X, C)$$

this will prove our contention.

We notice that the E_1-term of the above spectral sequence is

$$E_1^{pq} = H^p(X^{[q]}, C)$$

and that

$$d_1 : H^p(X^{[q]}, C) \to H^p(X^{[q+1]}, C)$$

is obviously a morphism of Hodge structures, therefore E_2^{pq} has a Hodge structure of pure weight p. We shall show that $d_2 = 0$, the proof that $d_3 = d_4 = \cdots = 0$ being similar. We choose a differential form α representing a class ξ in E_2^{pq}. We may assume that α has pure type (r, s). Then

$$d\alpha = 0$$
$$\delta\alpha = d\beta$$

and $\delta\beta$ is a representative of d_2. By the principle of two types we may choose β to have type $(r, s-1)$ or $(r-1, s)$, therefore $d_2\xi$ has two different types and must be zero. Q.E.D.

A by-product of the above proof is that the weight filtration on $H^m(X, C)$ is of the form:

$$0 \subset W_0 \subset \cdots \subset W_m = H^m(X, C) .$$

The same limitations hold for the mixed Hodge structure on the cohomology of a general complete variety.

The main reason for discussing mixed Hodge structures in these lectures is that, given a one-parameter family $\{V_t\}_{0<|t|<1}$ of projective varieties degenerating into a singular one, V_0, the Hodge structure on the cohomology of V_t tends in a precise manner to a mixed Hodge structure which is related to the mixed Hodge structure on the cohomology on V_0. The comparison of these two structures yields important geometric information.

b) VARIATION OF HODGE STRUCTURE. As mentioned in the preceding lecture, the Hodge decomposition

$$H^*(M, C) = \sum_{p,q} H^{pq}(M, C) \tag{4.6}$$

of the cohomology of a compact Kähler manifold reflects the particular complex structure on M and it is therefore natural to study how (4.6) behaves as the complex structure on M varies.

Geometrically, the basic situation is a family

$$f : X \to S \tag{4.7}$$

of compact Kähler manifolds. Here X and S are generally non-compact complex manifolds, X Kähler, and f is smooth and proper. The fibres $X_s = f^{-1}(s)$ constitute an analytic family of Kähler manifolds. The most important case is when everything is algebraic, X and S are quasi-projective and S is a curve. Thus S is obtained from a compact Riemann surface $\bar{S}$ by deleting finitely many points and, by resolution of singularities, we may embed (4.7) in a smooth compactification

(4.8)
$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ f \downarrow & & \downarrow \bar{f} \\ S & \hookrightarrow & \bar{S} \end{array}$$

where $\bar{X}$, $\bar{S}$ are smooth, but where $\bar{f}$ may fail to be smooth on $\bar{f}^{-1}(\bar{S} - S)$. Moreover it may be assumed that the fibre of $\bar{f}$ over each point of $\bar{S} - S$ is a divisor with normal crossings.

Returning to the situation (4.7), we will denote by E^n the holomorphic vector bundle on S whose sheaf of sections is

$$R^n f_*(C_X) \otimes \mathcal{O}_S .$$

E^n comes to us naturally equipped with a distinguished sheaf of locally constant sections, namely $R^n f_*(C_X)$, and hence with a flat holomorphic connection ∇' (the so-called Gauss-Manin connection). The solutions of

$$\nabla' s = 0$$

are precisely the sections of $R^n f_*(C)$. In the following we will denote by $\nabla = \nabla' + \nabla''$ the flat connection on E^n whose $(0, 1)$-part ∇'' is the $\bar{\partial}$ operator.

The Hodge numbers $h^{pq}(s) = \dim H^{pq}(X_s, C)$, $p + q = n$ are upper semi-continuous functions of s, as follows from general elliptic principles; on the other hand they add up to $\dim H^n(X_s, C)$, which is locally constant, hence they are locally constant, too. It follows, again from general principles, that the groups $H^{pq}(X_s, C)$ fit together to give a smooth vector subbundle E^{pq} of E^n. E^{pq} has a natural holomorphic structure, which comes to it from being the vector bundle associated to the holomorphic sheaf $R^q f_* \Omega^p_{X/S}$, where $\Omega^p_{X/S}$ stands for holomorphic p-forms along the fibres of f. However, *in general* E^{pq} *is not a holomorphic subbundle of* E^n. We also set:

$$F^p = \sum_{r \geq p} E^{p,n-p} .$$

To see how the $E^{p,q}$ behave relative to the complex structure of E^n, we need an explicit formula for the connection ∇. The argument we shall give is due, in an algebraic setting, to Katz-Oda.

Let $\{\phi_s\}$ be a smooth family of closed n-forms along the fibres of (4.7) giving a smooth section e of E^n. Let u be a vector field on S, v a C^∞ lifting of u to X. Then it makes sense to consider the Lie derivative of $\{\phi_s\}$ along v, $\mathrm{Lie}_v(\{\phi_s\})$. Since the Lie derivative commutes with exterior differentiation along the fibres, this yields a family of closed n-forms whose cohomology classes depend only on the cohomology classes of the ϕ_s. Therefore $\mathrm{Lie}_v(\{\phi_s\})$ determines a section of E^n which we may denote by $\mathrm{Lie}_v(e)$. We want to show that:

(4.9)
$$\nabla_u e = \mathrm{Lie}_v\, e .$$

It is quite obvious from the definition of Lie derivative that if e is flat $\mathrm{Lie}_v(e)$ is zero. One thing that has to be proved is that $\mathrm{Lie}_v(e)$ does not depend on the particular lifting v of u. Granting this, Lie_v behaves like a connection except for the fact that it might not depend linearly, over the C^∞ functions, on u.

Now to take the Lie derivative of $\{\phi_s\}$ we may proceed as follows: we lift $\{\phi_s\}$ to a form ϕ on X, take the Lie derivative of ϕ and restrict the result to the fibres of (4.7). But then we may use the *homotopy formula*:

(4.10)
$$\mathrm{Lie}_v\, \phi = v \lrcorner d\phi + d(v \lrcorner \phi)$$

where $\lrcorner$ stands for contraction. $d(v \lrcorner \phi)$ restricts to a family of exact form, therefore Lie_v has the required linearity properties. It is also clear from (4.10) that, when v projects to zero on S, $\mathrm{Lie}_v(e)$ is zero: in fact, in this case, $v \lrcorner d\phi$ vanishes. This proves (4.9).

It follows immediately from (4.9) and (4.10) that, if u is a vector field of type $(1, 0)$, then

(4.11)
$$\nabla_u C^\infty(F^p) \subset C^\infty(F^{p-1})$$

and that if v has type $(0, 1)$, then

(4.12)
$$\nabla_u C^\infty(F^p) \subset C^\infty(F^p) .$$

Formula (4.12) says that F^p *is a holomorphic subbundle* of E^n, whereas (4.11) is the *horizontality property* of variations of Hodge structure which is a crucial ingredient in all applications. One should notice that there is a natural isomorphism of holomorphic vector bundles:

$$F^p/F^{p+1} \xrightarrow{\sim} E^{p,n-p} .$$

To apply methods of differential geometry and analysis to study the variation of the Hodge structures of the X_s, one needs to have metrics in the various vector bundles involved. Such metrics are given, quite naturally, by the Hodge-Riemann bilinear relations, provided of course that we pass to the primitive cohomology. This is permissible, since the Lefschetz decomposition is of a topological nature and hence is locally constant in $E \to S$.

The resulting data may be codified in the following, somewhat lengthy

DEFINITION: A variation of Hodge structure of weight n, $V = \{S, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$ is given by a holomorphic vector bundle $E \to S$ over a complex manifold S having the following structure:

i) E has a flat holomorphic connection ∇ and contains a flat bundle of lattices $E_{\mathbb{Z}}$ (integral cohomology, in the geometric case).

ii) The F^p, $p = 0, \ldots, n$, are a decreasing filtration of E by holomorphic subbundles which satisfy the horizontality condition

$$\nabla \mathcal{O}(F^p) \subset \Omega^1_S(F^{p-1}) .$$

iii) $Q : E_{\mathbb{Z}} \otimes E_{\mathbb{Z}} \to \mathbb{Z}$ is a non degenerate, flat bilinear form such that

$$Q(e, e') = (-1)^n Q(e', e) .$$

iv) For each $s \in S$ the filtration $\{F^p_s\}$ and Q_s give a polarized Hodge structure on E_s. Thus if we set:

$$E^{p,n-p} = F^p \cap \overline{F^{n-p}}$$

then $E = \sum_{p+q=n} E^{pq}$ and the Hodge-Riemann bilinear relations (4.3) are satisfied.

All we have done here is to abstract the data arising from the variation of the Hodge decomposition in a family of compact Kähler manifolds whose metric form is rational. It is, however, important to note that V is not assumed to arise from geometry in this way. Moreover, even when the base space S is algebraic, one should not assume that an algebraic structure is given on V.

We will discuss some foundational results on variation of Hodge structure. To state the first of these we assume for simplicity that S is an algebraic curve, i.e. that $S = \bar{S} - \{s_1, \ldots, s_N\}$ is a compact Riemann surface minus N points, and we set

$$S[r] = \{s \in \bar{S} \mid |s - s_\mu| \geq \frac{1}{r} \text{ for all } \mu\}.$$

The result is the following:

(4.13) Let $V = \{S, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$ be a variation of Hodge structure. Then E has an intrinsic algebraic structure where the algebraic sections of E are those holomorphic sections ξ which satisfy the growth estimate

$$\max_{s \in S[r]} \log (Q(C_s \xi(s), \overline{\xi(s)})) = O(\log r) \tag{4.14}$$

where C_s is the Weil operator. Moreover, the F^p are algebraic subbundles of E and ∇ is algebraic.

Intuitively, (4.14) means that ξ has at most "poles at infinity" using the intrinsic Hodge norm to measure size.

Actually the proof of the above result will yield more, namely that the Gauss-Manin connection ∇ has regular singular points; this means that E has an algebraic extension $\bar{E}$ to $\bar{S}$ such that for every holomorphic section e of $\bar{E}$, ∇e has at most simple poles. Another, equivalent way of defining regular singular points is that the length of a (multivalued) flat section of E, measured using a C^∞ metric on $\bar{E}$, should grow at most like a polynomial when approaching a point in $\bar{S} - S$.

When V comes from algebraic geometry there is another, possibly different, algebraic structure on E, arising from Grothendieck's algebraic de Rham theorem in relative form:

$$R^q f_* C_X \otimes \mathcal{O}_S \cong \mathbb{R}^q f_* (\Omega^*_{X/S})$$

where $\Omega^*_{X/S}$ is the complex of algebraic forms along the fibres of $X \to S$. One can prove, however, that this algebraic structure is the same as the one given by (4.13), thus proving the usual regularity theorem in algebraic geometry.

The next results center around the local monodromy group. Given a variation of Hodge structure $V = \{S, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$, there is a monodromy representation

$$\rho : \pi_1(S, s_0) \to \mathrm{Aut}(E_{s_0})$$

gotten by displacing elements of E_{s_0} around closed paths by parallel translation. $\Gamma = \rho\pi_1(S, s_0)$ is called the monodromy group of V. When S is a punctured disk Δ^*, the image of the generator of $\pi_1(\Delta^*)$ will be denoted by T and called the Picard-Lefschetz transformation. This is suggested by looking at the geometric situation $f : X \to S$, where $S = \bar{S} - \{s_1, \ldots, s_N\}$ is a curve and then localizing around one of the s_i. The basic fact in this localized situation is the

MONODROMY THEOREM: The Picard-Lefschetz transformation T is quasi-unipotent of index of unipotency n, where n is the weight of the variation of Hodge structure V. In other terms,

$$(T^{\mu} - I)^{n+1} = 0$$

for some positive integer μ.

The simplest situation occurs when T is of finite order. Then one may go to a finite covering and prove the

REMOVABLE SINGULARITY THEOREM: If the Picard-Lefschetz transformation T is the identity, the variation of Hodge structure V extends across the puncture of Δ^*.

In the general case, it was conjectured by Deligne and proved by Schmid that, as $\zeta \in \Delta^*$ tends to zero, the Hodge decomposition of E_ζ tends to a mixed Hodge structure whose weight filtration is constructed from T.

The final results we wish to mention center around the global monodromy representation

$$\rho : \pi_1(S, s_0) \to \mathrm{Aut}(E_{s_0})$$

when the base space S is an algebraic variety. One of these results is the

RIGIDITY THEOREM: Let V, V' be two variations of Hodge structure of weight n and assume that there is an isomorphism σ between them at one point s_0. If σ is equivariant with respect to ρ, ρ', then it extends to a global isomorphism between V and V'.

Finally, there is Deligne's:

SEMI-SIMPLICITY THEOREM: The global monodromy representation ρ is completely reducible and the variation of Hodge structure V decomposes accordingly.

In classical terms, one has known for a long time that the position of the singular points and global monodromy determine a wide class of ordinary differential equations (e.g. the hypergeometric equations) on $\mathbb{P}^1$ having regular singular points. The above results assert the overwhelming influence of monodromy in variation of Hodge structures.

REFERENCES FOR LECTURE 4

Deligne's theory of mixed Hodge structures is given in

P. Deligne, Théorie de Hodge II, Publ. Math. I.H.E.S., vol. 40 (1972), 5-57.

An alternate more analytic account of his results may be found in

P. Griffiths and W. Schmid, Variation of Hodge structure (a discussion of recent results and methods of proof), to appear in Proc. Tata Institute Conf. on Discrete Groups and Moduli.

This paper also contains an exposition of the theory of variation of Hodge structure, together with additional references on the subject.

The original proof of the monodromy theorem has finally appeared in

A. Landman, On the Picard-Lefschetz transformations, Trans. Amer. Math. Soc. vol. 181 (1973), 89-126.

5. HERMITIAN DIFFERENTIAL GEOMETRY

In this talk we shall use the E. Cartan method of moving frames to discuss the theory of holomorphic curves, including local versions of the Wirtinger theorem and Plücker formulae as illustrations of non-compact algebraic geometry, and classifying spaces for variations of Hodge structure.

We begin with a homogeneous space G/H of a Lie group G by a closed subgroup H. In practice, G may frequently be identified with a set of "frames" on G/H, and when this is done the left-invariant Maurer-Cartan forms on G appear in the structure equations of a moving frame. Furthermore, when mapping a manifold M into G/H, there will frequently appear natural "Frênet frames" or liftings of the map to G. Restricting the Maurer-Cartan forms on G to these natural frames gives a complete set of invariants for the map, by virtue of the following general principle:

(5.1) Let M be a connected manifold and G a Lie group with basis $\{\omega_i\}$ for the Maurer-Cartan forms. Two maps $f, \tilde{f} : M \to G$ differ by a left translation in G if, and only if, $f^*\omega_i = \tilde{f}^*\omega_i$ for all i.

(PROOF IN CASE G IS A MATRIX GROUP: In this case the ω_i are the matrix entries in $\omega = g^{-1}dg$. Writing $f(m) = g(m)\cdot\tilde{f}(m)$ $(m \in M)$, $f(m)^{-1}df(m) = \tilde{f}(m)^{-1}d\tilde{f}(m) + \tilde{f}(m)^{-1}[g(m)^{-1}dg(m)]\tilde{f}(m)$. Thus $f^*\omega = \tilde{f}^*\omega \iff dg(m) = 0$. Q.E.D.)

Here are some examples.

a) COMPLEX PROJECTIVE SPACE. Points in $\mathbb{P}^n$ will be written as homogeneous coordinate vectors $Z = [z_0, \ldots, z_n]$. The frame manifold $F(\mathbb{P}^n)$ consists of all unitary bases $\mathbb{F} = \{Z_0, Z_1, \ldots, Z_n\}$ for $\mathbb{C}^{n+1}$.

Choosing a reference frame $\mathbb{F}_0$, any frame $\mathbb{F}$ is uniquely of the form

$$\mathbb{F} = T\cdot\mathbb{F}_0$$

for some unitary transformation $T \in U_{n+1}$. The correspondence $\mathbb{F} \longleftrightarrow T$ gives an identification $F(\mathbb{P}^n) \cong U_{n+1}$.

The vectors Z_i may be considered as smooth maps $Z_i : F(\mathbb{P}^n) \to \mathbb{C}^{n+1}$. Expanding the differential $dZ_i(\mathbb{F})$ in terms of the basis vectors in the frames $\mathbb{F}$ leads to the structure equations of a moving frame:

$$(5.2)\qquad \begin{aligned} dZ_i &= \sum_j \theta_{ij}Z_j \\ \theta_{ij} + \bar{\theta}_{ji} &= 0, \end{aligned}$$

which should be read: "Under infinitesimal displacement, the frame $\mathbb{F}$ undergoes an infinitesimal unitary transformation with coefficient matrix θ_{ij}." Since $Z_i(T\cdot\mathbb{F}) = TZ_i(\mathbb{F})$ for any fixed T, the θ_{ij} give a basis for the left-invariant Maurer-Cartan forms on U_{n+1}. The Maurer-Cartan equations

$$(5.3)\qquad d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj}$$

follow from $d(dZ_i) = 0$ in (5.2).

A holomorphic curve is a holomorphic mapping $Z : S \to \mathbb{P}^n$ from a Riemann surface into $\mathbb{P}^n$. In case S is compact, $Z(S)$ is an algebraic curve and hence has a degree, satisfies various Plücker formulae, etc. We shall eventually discuss certain non-compact analogues of these. In terms of a local coordinate ζ on S, Z is given by $Z(\zeta) = [z_0(\zeta), \ldots, z_n(\zeta)]$ where the $z_i(\zeta)$ are holomorphic. A frame field is given by a C^∞ lifting of Z to $F(\mathbb{P}^n)$; i.e. by a C^∞ frame $\mathbb{F}(\zeta) = \{Z_0(\zeta), \ldots, Z_n(\zeta)\}$ where $Z_0(\zeta) \wedge Z(\zeta) \equiv 0$. For such a frame field, the Maurer-Cartan forms $\theta_{ij} = \theta'_{ij} + \theta''_{ij}$ are linear combinations of $d\zeta$ and $d\bar{\zeta}$, and we claim that

$$(5.4)\qquad \theta''_{0\alpha} = 0 \qquad (\alpha = 1, \ldots, n).$$

(PROOF: $0 = \bar{\partial}(Z_0(\zeta) \wedge Z(\zeta)) = \bar{\partial}Z_0(\zeta) \wedge Z(\zeta) = \sum_{\alpha=1}^{n} \theta''_{0\alpha}Z_\alpha(\zeta) \wedge Z(\zeta)$.)

Similarly, one may prove that

(5.5) $$\Omega_0 = \frac{\sqrt{-1}}{2\pi}\left\{\sum_{\alpha=1}^{n} \theta_{0\alpha} \wedge \bar{\theta}_{0\alpha}\right\}$$

is independent of the frame field, and is the pull-back to S of the standard Kähler form on $\mathbb{P}^n$ (Fubini-Study metric).

b) CLASSIFYING SPACES FOR VARIATION OF HODGE STRUCTURE. Let E be a complex vector space with integral lattice $E_{\mathbb{Z}}$ and non-degenerate bilinear form

$$Q : E_{\mathbb{Z}} \otimes E_{\mathbb{Z}} \to \mathbb{Z}$$

$$Q(e, e') = (-1)^n Q(e', e) .$$

Given a set of Hodge numbers $h^{p,q}$ with $\sum_{p+q=n} h^{p,q} = \dim E$, $h^{p,q} = h^{q,p}$, the set of all polarized Hodge structures

(5.6) $$E = \bigoplus_{p+q=n} E^{p,q}$$

with $\dim E^{p,q} = h^{p,q}$ * forms a classifying space D for polarized Hodge structures of weight n.

A Hodge frame associated to a Hodge decomposition (5.6) is a collection $\mathbb{F} = \{\underline{f}_n, \underline{f}_{n-1}, \ldots, \underline{f}_0\}^{\dagger}$ where each $\underline{f}_p$ is a set $\{f_{p_1}, \ldots, f_{p_k}\}$ $(k = h^{p,q})$ of vectors giving an orthonormal basis for $E^{p,q}$, and where $\underline{f}_{n-p} = \bar{\underline{f}}_p$. Upon choosing a reference frame $\mathbb{F}_0$, the relation

$$\mathbb{F} = T \cdot \mathbb{F}_0$$

gives an identification $F(D) \cong G_{\mathbb{R}}$ of the manifold $F(D)$ of all Hodge frames with the Lie group $G_{\mathbb{R}}$ of real automorphisms of E which preserve Q. In particular, the classifying space is a homogeneous manifold

$$D = G_{\mathbb{R}}/H$$

with compact isotropy group H.

A $G_{\mathbb{R}}$-invariant complex structure on D may be given by the requirement that a C^∞ curve $\{E^{p,q}(\zeta)\}$ $(\zeta \in U \subset \mathbb{C})$ in D varies holomorphically if, and only if,

(5.7) $$\frac{\partial F^p(\zeta)}{\partial \bar{\zeta}} \subseteq F^p(\zeta)$$

where $F^p(\zeta) = E^{n,0}(\zeta) + \cdots + E^{p,n-p}(\zeta)$ is the associated Hodge filtration.

A variation of Hodge structure $V = \{S, \mathbb{E}, \mathbb{F}^p, \nabla, Q\}$, as defined in the third lecture, gives rise to:

a) A classifying space D as above, where $E = \mathbb{E}_{s_0}$, $Q = Q_{s_0}$, $h^{p,q} = \dim \mathbb{E}^{p,q}_{s_0}$, etc.;

b) the monodromy group Γ, which is the subgroup of the arithmetic group $G_{\mathbb{Z}}$ of all linear automorphisms of $E_{\mathbb{Z}}$ which preserve Q obtained by displacing flat frames around closed paths $\gamma \in \pi_1(S, s_0)$; and

c) a holomorphic period mapping

(5.8) $$\Phi : S \to \Gamma\backslash D$$

satisfying the infinitesimal period relation

(5.9) $$\frac{\partial F^p(\zeta)}{\partial \zeta} \subseteq F^{p-1}(\zeta) ,$$

where by definition $\Phi(s)$ is the Hodge decomposition $\mathbb{E}_s = \bigoplus_{p+q=n} \mathbb{E}^{p,q}(s)$ combined with an isomorphism $\mathbb{E}_s \cong \mathbb{E}_{s_0}$ depending on a homotopy class of paths from s to s_0, and $F^p(\zeta)$ is the image of $\mathbb{E}^{n,0}(s) + \cdots + \mathbb{E}^{p,n-p}(s)$ with ζ being the coordinate of s.

Conversely, such a period mapping Φ gives rise to a variation of Hodge structure by pulling back the universal family over $\Gamma\backslash D$. Henceforth, a variation of Hodge structure shall mean either the bundle data $V = \{S, \mathbb{E}, \mathbb{F}^p, \nabla, Q\}$, or a period mapping (5.8) satisfying (5.9).

Suppose now that we are either working locally or on a universal covering so that Γ may be taken to be trivial. Then we have $\Phi : S \to D$ satisfying (5.9). In terms of a local coordinate ζ on S, a Hodge frame field $\mathbb{F}(\zeta) = \{\underline{f}_n(\zeta), \underline{f}_{n-1}(\zeta), \ldots, \underline{f}_0(\zeta)\}$ is defined to be a smooth lifting of $\Phi(\zeta)$ to $F(D)$. Given such a frame field, we set

* This means that $E^{p,q} = \bar{E}^{q,p}$, and that the Hodge-Riemann bilinear relations (I) and (II) from lecture 3 are satisfied.

† Throughout our discussions of Hodge theory, indices will appear in decreasing order.

$$d\underline{f}_p = \sum_{q=n}^{0} \omega_{p,q}\underline{f}_q$$

where the $\omega_{p,q}$ are matrices of 1-forms on S which are the pull-backs of the Maurer-Cartan forms on $F(D) \cong G_{\mathbb{R}}$. The symmetry relations

$$\begin{aligned} &\omega_{p,q} + (-1)^{q-p}\, {}^t\bar{\omega}_{q,p} = 0, \quad \omega_{p,q} = \bar{\omega}_{n-p,n-q}, \\ (5.10) \qquad &\omega''_{p,q} = 0 \quad \text{for} \quad q < p \\ &\omega_{p,q} = 0 \quad \text{for} \quad |p-q| \geq 2 \end{aligned}$$

result from the orthonormal symmetry relations on $\mathbb{F}(\zeta)$, the Cauchy-Riemann equations (5.7) (as in the case of holomorphic curves in $\mathbb{P}^n$, cf. (5.4)), and the infinitesimal period relation (5.9). Setting $\psi_p = \omega_{p,p-1}$* and $\phi_p = \omega_{p,p}$, the Maurer-Cartan matrix for a variation of Hodge structure has the form

$$(5.11) \qquad \omega = \begin{pmatrix} \phi_n & \psi_n & & 0 & \cdots & 0 \\ {}^t\bar{\psi}_n & \phi_{n-1} & & & & \vdots \\ 0 & & \ddots & & & 0 \\ \vdots & & & & \phi_1 & \psi_1 \\ 0 & \cdots & 0 & & {}^t\bar{\psi}_1 & \phi_0 \end{pmatrix}$$

and satisfies the Maurer-Cartan equation

$$(5.12) \qquad d\omega = \omega \wedge \omega$$

as in the case of curves in $\mathbb{P}^n$.

The idea of how one proves the global results on variation of Hodge structure is: (a) To apply curvature arguments as in lecture one to the Hodge bundles $\mathbb{E}^p \longrightarrow S$ in the case when S is compact, the ϕ_p in (5.11) are the connection matrices in the Hodge bundles, and the curvature is computed by (5.12); and (b) in case S is non-compact, the Ahlfors lemma (lecture 6) gives an estimate on ψ_p, and judiciously choosing our frame field then allows one to estimate ϕ_p using (5.12), the upshot being that the arguments in the compact case carry over to the general situation. This will all be explained in more detail in lecture 7 and in the seminar.

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* The ψ_p are (1, 0) forms with values in $\mathrm{Hom}(\mathbb{E}^p, \mathbb{E}^{p-1})$ which measure the variation of Hodge structure, and may be identified with the Kodaira-Spencer class (cf. lecture 7 below).

For the remainder of this lecture, we shall return to the study of holomorphic curves in projective space.

A holomorphic curve $Z: S \to \mathbb{P}^n$ is <u>non-degenerate</u> in case the image does not lie in a proper linear subspace. Analytically, this is expressed by the Wronskian condition

$$W(\zeta) = Z(\zeta) \wedge Z'(\zeta) \wedge \cdots \wedge Z^{(n)}(\zeta) \neq 0 .$$

Near a <u>regular point</u> ζ_0 where $W(\zeta_0) \neq 0$, we define <u>Frênet frames</u> by the conditions:

$$\mathrm{Span}\,\{Z_0, \ldots, Z_k\} = \mathrm{Span}\,\{Z, Z', \ldots, Z^{(k)}\} = P^k(\zeta),$$

where $P^k(\zeta)$ is the $k^{\underline{th}}$ <u>osculating space</u>. For a Frênet frame field, dZ_k is clearly a linear combination of $Z_0, \ldots, Z_{k+1}$ and $\bar{\partial}Z_k$ is a linear combination of $Z_0, \ldots, Z_k$. This implies that $\theta_{k,\ell} = 0$ for $|k-\ell| \geq 2$ and $\theta''_{k,k+1} = 0$, and thus the structure equations (5.2) for a Frênet frame field reduce to the Frênet equations

$$\begin{aligned} &dZ_k = \theta_{k,k-1}Z_{k-1} + \theta_{k,k}Z_k + \theta_{k,k+1}Z_{k+1} \\ (5.13) \qquad &\theta_{k,k} + \bar{\theta}_{k,k} = 0, \quad \theta''_{k,k+1} = 0, \quad \theta_{k,k-1} + \bar{\theta}_{k-1,k} = 0.^* \end{aligned}$$

The $k^{\underline{th}}$ <u>associated curve</u> is the locus of the osculating spaces $P^k(\zeta)$. Analytically, this curve is given by the holomorphic mapping

$$\Lambda_k(\zeta) = Z(\zeta) \wedge Z'(\zeta) \wedge \cdots \wedge Z^{(k)}(\zeta)$$

from S into the Grassmannian $\mathbb{P}G(k,n)$ of projective k-planes in $\mathbb{P}^n$. Here, we are tacitly using the Plücker coordinates on $\mathbb{P}G(k,n)$. Since first $\Lambda_k(\zeta) = \lambda_k Z_0(\zeta) \wedge \cdots \wedge Z_k(\zeta)$ is a multiple of $Z_0 \wedge \cdots \wedge Z_k$ as vectors in $\Lambda^{k+1}C^{n+1}$, and secondly by the Frênet equations (5.13)

$$d(Z_0 \wedge \cdots \wedge Z_k) = \mu_k Z_0 \wedge \cdots \wedge Z_k \pm \theta_{k,k+1} Z_0 \wedge \cdots \wedge Z_{k-1} \wedge Z_{k+1},$$

it follows from (5.5) that

$$(5.14) \qquad \Omega_k = \frac{\sqrt{-1}}{2\pi}\,\theta_{k,k+1} \wedge \bar{\theta}_{k,k+1}$$

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* We note the similarity between the Maurer-Cartan matrices for a variation of Hodge structure and for holomorphic curves.

is the pull-back under Λ_k of the standard Kähler form on $\mathbb{P}G(k, n)$. The quantities $\Omega_0, \Omega_1, \Omega_2, \ldots$ are the complex analogues of arclength, curvature, torsion, ... for ordinary curves in $\mathbb{R}^n$. As in that situation, one may show:

(5.15) Two holomorphic curves $Z, \tilde{Z} : S \to \mathbb{P}^n$ differ by a rigid unitary motion if, and only if, $\Omega_k = \tilde{\Omega}_k$ for $k = 0, \ldots, n-1$.

(PROOF: The result is local, and we choose Frênet frame fields $\{Z_i\}$ and $\{\tilde{Z}_i\}$ for Z and $\tilde{Z}$. Fixing the $\tilde{Z}_i$, we seek to rotate the Z_i by

(5.16) $$Z_i \to e^{\sqrt{-1}\,\psi_i} Z_i \qquad (i = 0, \ldots, n)$$

such that the Maurer-Cartan matrices agree on the new frame fields. The result then follows from (5.1). Writing $\theta_{k,k+1} = h_k d\zeta$ and $\tilde{\theta}_{k,k+1} = \tilde{h}_k d\zeta$, the assumption $\Omega_k = \tilde{\Omega}_k$ gives $|h_k| = |\tilde{h}_k|$ or

$$h_k = e^{\sqrt{-1}\,\gamma_k}\,\tilde{h}_k .$$

Under a rotation (5.16),

$$\theta_{k,k+1} \to e^{\sqrt{-1}\,(\psi_k - \psi_{k+1})}\,\theta_{k,k+1} ,$$

so that choosing $\psi_k - \psi_{k+1} = \gamma_k$ for $k = 0, \ldots, n-1$ gives

(5.17) $$\theta_{k,k+1} = \tilde{\theta}_{k,k+1}, \quad \text{and by (5.13)}$$
$$\theta_{k,k-1} = \tilde{\theta}_{k,k-1} .$$

We are still free to rotate all Z_i through the same angle ψ.

By the structure equations (5.3) and (5.13),

(5.18) $$d\theta_{k,k+1} = (\theta_{k,k} - \theta_{k+1,k+1}) \wedge \theta_{k,k+1}$$
$$d\theta_{k,k} = \theta_{k,k-1} \wedge \theta_{k-1,k} + \theta_{k,k+1} \wedge \theta_{k+1,k} .$$

Using (5.17) in the second equation gives $d(\theta_{k,k} - \tilde{\theta}_{k,k}) = 0$. By the Poincaré lemma, we may rotate all Z_i through angle ψ where $\theta_{n,n} - \tilde{\theta}_{n,n} = \sqrt{-1}\,d\psi$ to have $\theta_{n,n} = \tilde{\theta}_{n,n}$. Now, using (5.17) and the characterization of the Hermitian connection given in lecture 3 in the first equation in (5.18) gives $\theta_{k,k} - \theta_{k+1,k+1} = \tilde{\theta}_{k,k} - \tilde{\theta}_{k+1,k+1}$ for $k = 0, \ldots, n-1$. Combining, we obtain $\theta_{k,k} = \tilde{\theta}_{k,k}$ for all k, and then $\theta = \tilde{\theta}$. Q.E.D.)

At this point, we make the following notational convention: For a positive (1, 1) form

$$\Omega = \frac{\sqrt{-1}}{2\pi}\, h\, d\zeta \wedge d\bar{\zeta} = \frac{\sqrt{-1}}{2\pi}\, \theta \wedge \bar{\theta}$$

where $\theta = e^{\sqrt{-1}\,\psi}\sqrt{h}\,d\zeta$, the connection form ϕ is characterized by

$$d\theta = \phi \wedge \theta$$
$$\phi + \bar{\phi} = 0 ,$$

and we define the <u>Ricci form</u>*

$$\operatorname{Ric} \Omega = \frac{\sqrt{-1}}{2\pi}\, d\phi .$$

Since $\operatorname{Ric} \Omega$ is a constant times the curvature form, it depends only on Ω.

From the first equation in (5.18), the connection form for Ω_k is $\theta_{k,k} - \theta_{k+1,k+1}$. By the second equation there,

(5.19) $$\operatorname{Ric} \Omega_k = -2\Omega_k + \Omega_{k-1} + \Omega_{k+1} .$$

This beautiful relation, which is due originally to H. and J. Weyl, has many applications, especially to "non-compact algebraic geometry." A first one is:

(5.20) Ω_0 uniquely determines $\Omega_1, \ldots, \Omega_{n-1}$.

This follows from $\operatorname{Ric} \Omega_0 = -2\Omega_0 + \Omega_1$, $\operatorname{Ric} \Omega_1 = -2\Omega_1 + \Omega_0 + \Omega_2$, etc. In geometric terms, (5.20) states:

For holomorphic curves, the curvature, torsion, ... are all functions of arclength alone. In particular, if two such curves osculate to first order, they are congruent (theorem of Calabi).

Before giving the second application we need some preliminary remarks on the Wirtinger theorem. Suppose that S is a relatively compact open set in a larger Riemann surface S' on which Z is defined (an extreme

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* This terminology will be justified in the next lecture.

case is when $S = S'$ is compact). For each hyperplane $H \in \mathbb{P}^{n*}$, the projective space dual to $\mathbb{P}^n$, the number of points of intersection $n(S, H)$ of $Z(S)$ with H is finite. Crofton's formula from integral geometry is the relation

$$\int_S \Omega_0 = \int_{\mathbb{P}^{n*}} n(S, H)\, dH \tag{5.21}$$

expressing the area of $Z(S)$ as the average number of intersections of $Z(S)$ with a hyperplane. A proof of (5.21) using frames is given in the appendix to this lecture.

In case $S = S'$ is compact, $n(S, H)$ is independent of the hyperplane H since, by the Cauchy integral formula, a meromorphic function has the same number of zeroes as poles. The integer $n(S, H)$ is called the degree of the algebraic curve, and (5.21) is the Wirtinger theorem

$$\text{area}(S) = \text{degree}(S) . \tag{5.22}$$

For non-compact S, say for definiteness that $S' = \mathbb{C}$ and $S = \Delta_R = \{\zeta \in \mathbb{C} : |\zeta| < R\}$,* the proof that $n(S, H)$ is independent of H leads to the following significant analytical generalization of the Wirtinger theorem (5.22), illustrating quite well, we think, the principle of non-compact algebraic geometry: Setting $n(r, H) = n(\Delta_r, H)$ and $n(0, H) = \lim_{r \to 0} n(r, H)$, the Nevanlinna inequality, also to be proved in the appendix

$$\int_0^R \{n(r, H) - n(o, H)\} \frac{dr}{r} \leq \int_0^R \{\int_{\Delta_r} \Omega_0\} \frac{dr}{r} + O(1) , \tag{5.23}$$

bounding the growth of $n(r, H)$ by the growth of the area, is valid.

To give our second application of (5.19), we define the mean degree of the k^{th} osculating curve by

$$\delta_k(S) = \int_S \Omega_k .$$

For plane curves $(n = 2)$, $\delta_0(S)$ is the average number of intersections of $Z(S)$ with a line, and $\delta_1(S)$ is the average number of tangent lines to $Z(S)$ passing through a point in $\mathbb{P}^2$ (mean class).

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* $Z : \mathbb{C} \to \mathbb{P}^n$ may be called an entire holomorphic curve.

When $S = S'$ is compact, we may apply the Gauss-Bonnet theorem for singular metrics to (5.19) to obtain a formula

$$\chi(S) + N_k(S) + 2\delta_k(S) = \delta_{k-1}(S) + \delta_{k+1}(S) . \tag{5.24}$$

Here, $\chi(S) = 2 - 2g$ is the Euler characteristic of S, and $N_k(S)$ measures the number and type of singular points on the k^{th} osculating curve. The relations (5.24) are the general Plücker formulae of an algebraic curve.

When S is non-compact, e.g. in the study of an entire holomorphic curve, the same Gauss-Bonnet method applies to give the Plücker estimates, which roughly state that

$$\int_0^R \{\int_{\Delta_r} \Omega_{k-1} + \Omega_{k+1}\} \frac{dr}{r} \leq (2 + \varepsilon) \int_0^R \{\int_{\Delta_r} \Omega_k\} \frac{dr}{r} ,$$

and which serve to relate the orders of growth of the various associated curves. These inequalities are of fundamental importance in the transcendental theory of holomorphic curves, and serve to illustrate once again the principle of non-compact algebraic geometry.

APPENDIX TO LECTURE 5:
PROOF OF CROFTON'S FORMULA AND THE NEVANLINNA INEQUALITY

A hyperplane H in $\mathbb{P}^n$ is spanned by n orthonormal vectors $W_0, \ldots, W_{n-1}$; i.e.

$$H = W_0 \wedge \cdots \wedge W_{n-1} . \tag{A.5.1}$$

We consider the manifold $F(\mathbb{P}^n)$ of unitary frames $\{W_0, \ldots, W_n\}$ in $\mathbb{C}^{n+1}$ and fibering $F(\mathbb{P}^n) \to \mathbb{P}^{n*}$ given by $\{W_0, \ldots, W_n\} \to W_0 \wedge \cdots \wedge W_{n-1}$. Writing, as in (5.2),

$$dW_i = \sum_{j=0}^{n} \phi_{ij} W_j$$

$$\phi_{ij} + \bar{\phi}_{ji} = 0 ,$$

we have that

$$d(W_0 \wedge \cdots \wedge W_{n-1}) = (\sum_{i=0}^{n-1} \phi_{ii}) W_0 \wedge \cdots \wedge W_{n-1} + \sum_{\mu=0}^{n-1} \phi_{\mu n} W_0 \wedge \cdots \wedge W_n \wedge \cdots \wedge W_{n-1} .$$

It follows from (5.5) that the Kähler form Ω^* on $\mathbb{P}^{n*}$, pulled back to $F(\mathbb{P}^n)$, is given by

$$\Omega^* = \frac{\sqrt{-1}}{2}\{\sum_{\mu=0}^{n-1} \phi_{\mu n} \wedge \bar{\phi}_{\mu n}\}.$$

The invariant measure on $\mathbb{P}^{n*}$ is thus

$$\text{(A.5.2)} \qquad dH = (\Omega^*)^n = n!\left(\frac{\sqrt{-1}}{2}\right)^n \{\sum_{\mu=0}^{n-1} \phi_{\mu n} \wedge \bar{\phi}_{\mu n}\}^n.$$

In $S \times \mathbb{P}^{n*}$ we consider the incidence divisor I of all points (ζ, H) such that $Z(\zeta) \in H$. Clearly,

$$\int_{\mathbb{P}^{n*}} n(S, H) = \int_I dH.$$

On I, we shall rewrite dH as

$$dH(\zeta, H) = \Omega_0(\zeta) \times \Psi(\zeta, H)$$

where $\Psi(\zeta, H)$ is the measure on the set of hyperplanes passing through $Z(\zeta)$, and then apply the Fubini theorem to conclude (5.21).

Choose a Frênet frame field $\{Z_0(\zeta), \ldots, Z_n(\zeta)\}$. Then all hyperplanes (A.5.1) passing through $Z(\zeta)$ are given by frames $\{W_0, W_1, \ldots, W_n\}$ where

$$W_0 = Z_0(\zeta)$$
$$W_\alpha = \sum_{\beta=1}^{n} A_{\alpha\beta} Z_\beta(\zeta)$$

and $A = (A_{\alpha\beta})$ is an arbitrary unitary matrix. Using ${}^t\bar{A} = A^{-1}$ and the Frênet equation (5.13),

$$dW_0 = \theta_{00} Z_0 + \theta_{01}\left(\sum_{\alpha=1}^{n} \bar{A}_{\alpha 1} W_\alpha\right)$$
$$dW_\alpha \equiv \sum_{\beta,\gamma=1}^{n} dA_{\alpha\beta}\bar{A}_{\alpha\beta} W_\gamma \qquad (d\zeta, d\bar{\zeta}),$$

which implies that, on the incidence divisor I,

$$\phi_{0n} = \bar{A}_{n1}\theta_{01}$$
$$\phi_{\alpha n} \equiv \sum_{\beta=1}^{n} dA_{\alpha\beta}\bar{A}_{n\beta} \qquad (d\zeta, d\bar{\zeta}).$$

From (A.5.2) it follows that

$$dH = \Omega_0 \wedge \Psi$$

where Ψ is a differential form involving the matrix entries in A and dA. Integrating Ψ over all hyperplanes containing $Z(\zeta)$ gives a constant C independent of $Z(\zeta)$, and thus, by the Fubini theorem,

$$\int_I dH = C \int_S \Omega_0.$$

Taking S to be a line in $\mathbb{P}^n$ gives $C = 1$. Q.E.D.

Now to the proof of (5.23). Let $A \in (\mathbb{C}^{n+1})^*$ be a unit vector such that the hyperplane H is defined by

$$\langle A, Z \rangle = 0.$$

On $\mathbb{C}$ we define the potential function

$$u(\zeta) = \log \frac{|\langle A, Z(\zeta)\rangle|^2}{\|Z(\zeta)\|^2}.$$

This function has logarithmic singularities on the divisor D_H where the holomorphic curve meets the hyperplane H, and

$$dd^c u = -\Omega_0$$

on $\mathbb{C} - D_H$. Applying Stokes' theorem to $d^c u$ and taking into account the singularities of u gives

$$n(r, H) = \int_{|\zeta|=r} d^c u + \int_{\Delta_r} \Omega_0.$$

In polar coordinates, $d^c = \frac{1}{2\pi} r \frac{d}{dr} \otimes d\theta + \frac{1}{2\pi}\frac{d}{d\theta} \otimes dr$, and thus

$$n(r, H) = r\frac{d}{dr}\cdot\left(\frac{1}{2\pi}\int_{|\zeta|=r} u\, d\theta\right) + \int_{\Delta_r} \Omega_0.$$

Assuming for simplicity that $\{0\} \notin D_H$, we may integrate this equation and obtain the <u>First Main Theorem</u>

$$\text{(A.5.3)} \qquad \int_0^R n(r, H)\frac{dr}{r} = \frac{1}{2\pi}\int_{|\zeta|=R} u\, d\theta + \int_0^R \left(\int_{\Delta_r} \Omega_0\right)\frac{dr}{r}.$$

In particular, since $u \leq 0$, we find the Nevanlinna inequality (5.33).

REFERENCES FOR LECTURE 5

The classic reference for moving frames is

E. Cartan, Groupes finis et continus et la géométrie différentielle, Gauthier-Villars, Paris (1935).

A recent account is given in

P. Griffiths, On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, to appear in Duke Math. J.

The theory of holomorphic curves is discussed in

M. Cowen and P. Griffiths, Holomorphic curves and metrics of negative curvature, to appear in Jour. d'analyse.

Integral geometry is treated in

L. Santalo, Introduction to integral geometry, Hermann, Paris (1953).

6. CURVATURE AND HOLOMORPHIC MAPPINGS

The goal of this lecture is to provide an introduction to hyperbolic complex analysis, which can be described as the study of how negative curvature conditions influence holomorphic mappings. Together with Lie group theory, this provides the basic tools for dealing with general variations of Hodge structure: examples of applications will be given in the next lecture. On the other hand, hyperbolic complex analysis applies to Picard-type theorems and their beautiful quantitative refinement, the value distribution theory of R. Nevanlinna. It is mainly this aspect that we will be discussing today.

In particular, we will be giving applications of the Ahlfors lemma, a simple but extremely powerful generalization of the classical Schwarz lemma. This result alone, for example, provides the estimates leading to the results on variations of Hodge structure discussed in the fourth lecture. Before formally stating the Ahlfors lemma, we will give some examples of hermitian manifolds to which it applies.

A volume form Ψ on a complex manifold M of dimension m is a smooth, positive (m, m) form. In local coordinates we may write

$$\Psi = h\Phi$$

where

$$\Phi = \prod_{j=1}^{m} \left(\frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j \right)$$

is the euclidean volume form and h is a positive function. We will also be using pseudo-volume forms, these being non-negative, smooth (m, m) forms Ψ that can be written locally as

$$\Psi = |f|^2 h\Phi$$

where h is a positive function and f is a holomorphic function that is not identically zero. Pseudo-volume forms naturally arise by pulling back volume forms under non-degenerate holomorphic mappings. The Ricci form of Ψ is the global, smooth, real $(1, 1)$-form given locally by

$$\mathrm{Ric}\,\Psi = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h \, .$$

Volume forms which satisfy the curvature estimates:

$$\begin{aligned} \mathrm{Ric}\,\Psi &\geqq 0 \\ (\mathrm{Ric}\,\Psi)^m &\geqq \Psi \end{aligned} \tag{6.1}$$

play a crucial role in the theory. The estimates (6.1) are obviously invariant under non-degenerate holomorphic mappings. Here are a few examples.

a) When $m = 1$, there is a natural correspondence between hermitian metrics and volume forms, given by:

$$ds^2 = h\, dz d\bar{z} \longleftrightarrow \frac{\sqrt{-1}}{2} h\, dz \wedge d\bar{z} = \Psi \, .$$

For such a Ψ,

$$\mathrm{Ric}\,\Psi = -K\Psi$$

where

$$K = -\frac{1}{\pi}\frac{1}{h}\frac{\partial^2 \log h}{\partial z \partial \bar{z}}$$

is the Gaussian curvature of ds^2. Hence, in this case (6.1) just says that ds^2 has Gaussian curvature bounded above by -1. From this point of view (6.1) appears to be a generalization to higher dimension of the condition of having Gaussian curvature bounded above by a negative constant.

In particular, on the disk:

$$\Delta(R) = \{z \in \mathbb{C} : |z| < R\}$$

the Poincaré metric

$$\eta(R) = \frac{\sqrt{-1}}{\pi} \frac{R^2 dz \wedge d\bar{z}}{(R^2 - |z|^2)^2}$$

is the unique invariant metric such that

$$\mathrm{Ric}\ \eta(R) = \eta(R) .$$

We write $\eta(1) = \eta$, $\Delta(1) = \Delta$. The holomorphic mapping

$$w \to (\sqrt{-1}\, w + 1)/(-\sqrt{-1}\, w + 1)$$

gives a conformal equivalence between Δ and the upper half-plane:

$$H = \{w \in \mathbb{C} \mid \mathrm{Im}\, w > 0\} .$$

Under this equivalence, η pulls back to

$$\frac{\sqrt{-1}}{4\pi} \frac{dw \wedge d\bar{w}}{(\mathrm{Im}\, w)^2} .$$

This metric will also be denoted by η and referred to as the Poincaré metric. The same will apply to the metric

$$\frac{\sqrt{-1}}{\pi} \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|^2 (\log |\zeta|^2)^2}$$

on the punctured disk

$$\Delta^* = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}$$

which corresponds to the Poincaré metric on H via the covering map

$$\zeta = e^{2\pi\sqrt{-1}\, w} .$$

b) On the polycylinder

$$\Delta^m(R) = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_i| < R , \quad i = 1, \ldots, m\}$$

the product of the Poincaré metrics induces a volume form $\eta_m(R)$ such that

$$(6.2) \qquad \begin{aligned} \mathrm{Ric}\ \eta_m(R) &> 0 \\ (\mathrm{Ric}\ \eta_m(R))^m &= \eta_m(R) . \end{aligned}$$

As above, η_m induces volume forms, which will be denoted by the same symbol, on the punctured polycylinders

$$\Delta^*_{k,m} \doteq \Delta^{*k} \times \Delta^{m-k} .$$

The Bergmann volume form on the ball $\{z \in \mathbb{C}^m \mid \|z\| < R\}$ also satisfies the inequalities (6.1).

c) A volume form Ψ on M is the same thing as a metric for the dual of the canonical bundle K_M, and $\mathrm{Ric}\ \Psi$ is the same as the first Chern form of K_M relative to this metric. In case M is compact it follows that, after adjusting constants, we may find a volume form on M satisfying (6.1) exactly when K_M is positive.

This suggests that, for a general compact M, we look for such "negatively curved" volume forms on $M - D$, where D is an effective divisor such that $K_M \otimes [D]$ is positive. Some restrictions on the singularities of D are necessary, and we will assume that D has simple normal crossings, i.e. that

$$D = D_1 + \cdots + D_N$$

where the D_i are distinct smooth divisors meeting transversally. In a suitable neighborhood of each point $p \in D$, $M - D$ looks like a punctured polycylinder and this suggests the following global version of the Poincaré volume form. Choose a smooth volume form Ψ_M on M and metrics in the bundles $[D_i]$ such that the inequality of Chern forms:

$$(6.3) \qquad c_1([D]) + c_1(K_M) > 0$$

holds, and for each i let σ_i be a section of $[D_i]$ which defines D_i. Then for a suitable choice of the constants α_i, the volume form:

$$(6.4) \qquad \Psi = \Psi_M \Big/ \prod_{i=1}^{N} |\sigma_i|^2 (\log(\alpha_i |\sigma_i|^2))^2$$

on $M - D$ satisfies the curvature conditions (6.1).

The simplest special case is when M is projective m-space and the D_i are hyperplanes. Recalling that the canonical bundle of $\mathbb{P}^m$ is H^{-m-1}, where H is the hyperplane bundle, (6.3) translates into $N > m + 1$. When $m = 1$, this means that D must consist of at least three points.

We now state and prove the ubiquitous

AHLFORS LEMMA: If Ψ is a pseudo-volume form on $\Delta^m(R)$ such that (6.1) holds then $\Psi \leq \eta_m(R)$.

PROOF: It obviously suffices to prove that $\Psi \leq \eta_m(r)$ for $r < R$. On $\Delta^m(r)$ Ψ is bounded, whereas $\eta_m(r)$ goes to infinity at the boundary. Therefore, if we write

$$\Psi = u\eta_m(r)$$

u has an interior maximum at some z_0. It follows that, at z_0,

$$0 \geq \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log u = \text{Ric } \Psi - \text{Ric } \eta_m(r) .$$

Taking m^{th} exterior powers and using (6.1) and (6.2) gives

$$\Psi \leq (\text{Ric } \Psi)^m \leq (\text{Ric } \eta_m(r))^m = \eta_m(r)$$

i.e. $u(z_0) \leq 1$, and therefore $u \leq 1$ everywhere as was to be proved.

A COROLLARY to the Ahlfors lemma is that if we write

$$\Psi = h\Phi$$

where Φ is the euclidean metric and Ψ is a metric on $\Delta^m(R)$ satisfying (6.1), then

(6.5) $$R \leq C_m h(0)^{-1/2m}$$

for some universal constant C_m.

We now give an immediate application of the Ahlfors lemma. Let

$$f : \Delta^m(R) \to N$$

be a holomorphic mapping into an m-dimensional complex manifold N with a volume form Ψ. Write

$$f^*\Psi = |Jf|^2\Phi$$

where Φ is the euclidean metric. If Ψ satisfies (6.1) then (6.5) implies that

(6.6) $$R \leq C_m|Jf(0)|^{-1/m} .$$

Applying this to the volume form Ψ constructed in example c) gives the

GENERALIZED PICARD THEOREM IN FINITE FORM: Let M be an m-dimensional compact, complex manifold, and let D be a divisor with simple normal crossings whose Chern class satisfies (6.3). Then for any non-degenerate holomorphic mapping:

$$f : \Delta^m(R) \to M - D$$

the estimate (6.6) holds. In particular, an entire holomorphic mapping:

$$f : \mathbb{C}^m \to M - D$$

is degenerate.

When $M = \mathbb{P}^1$ and D consists of three distinct points, this implies the usual Picard theorem. However, the above result gives more: restricting to the case of an entire meromorphic mapping

$$f : \mathbb{C} \to \mathbb{P}^1$$

it says that, for any three points z_1, z_2, z_3 in $\mathbb{P}^1$ and any point $\zeta_0 \in \mathbb{C}$ such that $f'(\zeta_0) \neq 0$, any disc around ζ_0 of radius $\geq R(f'(\zeta_0))$ will meet $f^{-1}(\{z_1, z_2, z_3\})$. In principle this gives a lower bound on the "size" of $f^{-1}(\{z_1, z_2, z_3\})$ which, when made precise, leads to the beautiful defect relations of R. Nevanlinna.

We will now describe another, closely related, application of the Ahlfors lemma, which leads to the basic estimates for studying variations of Hodge structure. To give this we must first discuss holomorphic sectional curvature.

Let M be a complex manifold, and suppose a hermitian metric with associated exterior form ω is given on its tangent bundle. In the first lecture we have defined the curvature form $\Theta(\xi)$ attached to such a metric. For any holomorphic tangent vector ξ, the holomorphic sectional curvature in the ξ-direction, $K(\xi)$, is defined as follows:

$$K(\xi) = \frac{2\sqrt{-1} < \Theta(\xi), \xi \wedge \bar{\xi} >}{\|\xi\|^4} .$$

It is clear from the definition that $K(\xi)$ depends only on the direction of ξ. When $m = 1$, $K(\xi)$ is just the Gaussian curvature of M. We say that M is negatively curved when

$$K(\xi) \leq -A < 0$$

for some positive A and every ξ. Multiplying the metric by $1/A$, we may assume that $A = 1$.

PROPOSITION (GENERALIZED SCHWARZ LEMMA): Let $f : \Delta \to M$ be a holomorphic mapping of the unit disc into a negatively curved complex manifold. Then f is distance decreasing, in the sense that

$$f^*\omega \leq \eta$$

where η is the Poincaré metric on Δ.

PROOF: To apply the Ahlfors lemma it suffices to show that $f^*\omega$ satisfies the inequality:

(6.7) $$\text{Ric}\,(f^*\omega) \geq f^*\omega$$

it points where $f' \neq 0$. If U is a sufficiently small neighborhood of such a point, $f(U)$ is a submanifold of M, and has an induced hermitian metric $\omega|f(U)$. (6.7) now follows from the curvature assumptions on M and from the principle that curvature decreases on submanifolds (cf. lecture 1).

REMARK: The same proof as given above applies to the case when we only know that $K(\xi) \leq -1$ for all vectors ξ which are tangent to $f(\Delta)$. This will be the case in applications to variation of Hodge structure.

When applied to M = unit disc and ω = Poincaré metric, the above proposition gives the invariant form of the Schwarz lemma due to Pick

$$\frac{|f'(z)|^2}{(1 - |f(z)|^2)^2} \leq \frac{1}{(1 - |z|^2)^2} .$$

The usual statement

$$|f'(0)| \leq 1$$

follows by assuming that $f(0) = 0$ and setting $z = 0$ in the above inequality.

REFERENCES FOR LECTURE 6

The basic reference for hyperbolic complex analysis is

S. Kobayashi, Hyperbolic manifolds and holomorphic mappings, Marcel Dekker, New York (1970).

For value distribution theory, the classic is

R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris (1929).

A modern treatment is given in the monograph

P. Griffiths, Entire holomorphic mappings in one and several complex variables, to appear as an Annals of Math. Studies, Princeton Univ. Press.

7. PROOF OF SOME RESULTS ON VARIATIONS OF HODGE STRUCTURE

We first establish some notational conventions, which will be valid for the rest of this lecture. Let $V = \{S, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$ be a variation of Hodge structure of weight m. The second Hodge-Riemann bilinear relation says that, on $E^{p,m-p}$, the hermitian form

$$(-1)^p \sqrt{-1}^{\,-m}\, Q(\xi, \bar{\eta})$$

is positive definite. To simplify notations in the following we shall write Q for $\sqrt{-1}^{\,-m} Q$, so that the Hodge hermitian metric $(\, , \,)$ is related to $Q(\xi, \bar{\eta})$ by

(7.1) $$(\alpha, \beta) = \sum (\alpha_p, \beta_p) = \sum (-1)^p Q(\alpha_p, \bar{\beta}_p)$$

where α_p and β_p are the $(p, m-p)$ components of α and β, respectively. It will be convenient to consider E^{pq} and E/F^p as C^∞ subbundles of E; for example E/F^p will be identified to the orthogonal complement of F^p in E, and so on.

In lecture one the metric connection and curvature of a hermitian metric were defined. That discussion carries over verbatim to non-degenerate, but possibly indefinite, hermitian metrics. From this point of view the Gauss-Manin connection ∇ appears as the $(1, 0)$ part of the metric connection for E relative to the indefinite metric $Q(\xi, \bar{\eta})$. The metric connection on F^p is related to ∇ by:

$$D'_{F^p} + \psi_p = \nabla$$

where ψ_p is the second fundamental form of F^p in E (Kodaira-Spencer map). The $(1, 0)$ part of the metric connection D_{E/F^p} on E/F^p, on the other hand, agrees with ∇, whereas the $(0, 1)$ part is

$$\bar{\partial} \pm {}^t\psi_p$$

where $\bar{\partial}$ is the Cauchy-Riemann operator for the complex structure of E and ${}^t\psi_p$ is the adjoint of ψ_p. With these notations we may now prove the

a) CURVATURE PROPERTIES OF HODGE BUNDLES. We will show that the following formula holds for the curvature Θ_p of $E^{p,m-p}$:

(7.2) $$(\Theta_p e, e') = (\psi_p e, \psi_p e') + ({}^t\psi_{p+1} e, {}^t\psi_{p+1} e') .$$

To prove (7.2) we have to describe how curvature behaves when one goes to subbundles or quotient bundles. To some extent, this has already been done in the first of these lectures. Suppose we are given an exact sequence of holomorphic vector bundles

$$0 \to H' \to H \to H'' \to 0$$

and that H has a (possibly indefinite) hermitian metric which induces non-degenerate metrics on H' and H''. As usual, identify H'', as a C^∞ bundle, with the orthogonal complement of H' in H. Let σ be the second fundamental form of H' in H. Then the curvatures $\Theta_{H'}$, $\Theta_{H''}$ are given by:

(7.3) $$(\Theta_{H'} e, e') = (\Theta_H e, e') - (\sigma e, \sigma e')$$
$$(\Theta_{H''} e, e') = (\Theta_H e, e') - ({}^t\sigma e, {}^t\sigma e') .$$

The two equations (7.3) express the principle that curvature decreases on subbundles and increases on quotient bundles (notice that ${}^t\sigma$ has type $(0, 1)$). Applying (7.3) to the exact sequences:

$$0 \to F_p \to E \to E/F^p \to 0$$
$$0 \to F_{p+1} \to F_p \to E^{p,m-p} \to 0$$

and taking into account the alternation of signs (7.1) gives formula (7.2).

b) APPLICATIONS TO RIGIDITY THEOREMS AND RELATED MATTERS WHEN THE BASE IS COMPACT. Let $V = \{S, E, E_{\mathbb{Z}}, Q, \nabla, \{F^p\}\}$ be a variation of Hodge structure with compact base S. Let ξ be a global holomorphic section of F^p. Suppose that ξ is quasi-horizontal, i.e. suppose that $\nabla\xi$ is a section of $\Omega^1(F^p)$. In other terms this means that $\psi_p \xi = 0$. Let ξ_p be the $(p, m-p)$-component of ξ. Then, as follows from the results of the first lecture,

$$\partial\bar{\partial}\|\xi_p\|^2 = (D'_p\xi_p, D'_p\xi_p) - (\Theta_p\xi_p, \xi_p)$$

where D_p is the metric connection of $E^{p,m-p}$. Taking into account formula (7.2) and the quasi-horizontality of ξ, it follows that $\|\xi_p\|^2$ is plurisubharmonic, hence constant, due to the compactness of the base, therefore:

$$D'_p\xi_p = {}^t\psi_{p+1}\xi_p = \psi_p\xi_p = 0 .$$

This means that ξ_p is holomorphic, as a section of E, and horizontal. If we assume that ξ is horizontal, we may apply the same procedure to $\xi - \xi_p$ and inductively obtain the following statement:

THEOREM OF THE FIXED PART: Let $V = \{S, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$ be a variation of Hodge structure with compact base S. Let ξ be a flat global holomorphic section of E. Then each of the (p, q)-components of ξ is holomorphic and flat. In particular, if ξ has pure (p, q)-type at one point, then it has pure (p, q)-type everywhere.

A straightforward application of the above theorem is a proof of the <u>rigidity theorem</u> when the base is compact. Let V and V' be variations of Hodge structure of the same weight m. A $\pi_1(S, s_0)$-equivariant isomorphism of Hodge structures

$$\phi_{s_0} : E_{s_0} \to E'_{s_0}$$

extends by parallel translation to a flat section ϕ of $\mathrm{Hom}(E, E')$, which has type $(0, 0)$ at s_0. Applying the theorem of the fixed part to the variation of Hodge structure $\mathrm{Hom}(V, V')$ gives that ϕ has type $(0, 0)$ everywhere, which is just the rigidity theorem.

Another application is the <u>complete reducibility theorem</u> of Deligne, always when the base S is compact. The full argument is too long to be given here and we will content ourselves with the weaker statement:

Let V be a variation of Hodge structure with compact base S and monodromy group Γ. Then Q is non-degenerate on the space E^Γ of Γ-invariants. In particular,

$$E \cong E^\Gamma \oplus (E^\Gamma)^\perp .$$

To conclude we would like to remark that, since a bounded plurisubharmonic function on an algebraic (possibly non-compact) variety is constant, the above proofs of the theorem of the fixed part and its corollaries would go through for an arbitrary algebraic base S if we knew that a flat section of $E \to S$ has <u>bounded length</u>. This, and much more, follows from the results of Schmid that we will discuss in a short while.

c) THE DISTANCE-DECREASING PROPERTY AND SOME APPLICATIONS. When studying variations of Hodge structure on a non-complete algebraic curve, one is naturally led, by localizing near points at infinity, to study variations of Hodge structure on a punctured disk Δ^*. In the language of classifying spaces for Hodge structures we have a diagram

$$\begin{array}{ccc} H & \xrightarrow{\tilde\Phi} & D \\ {\scriptstyle e^{2\pi\sqrt{-1}\,w}}\downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\Phi} & \Gamma\backslash D \end{array}$$

The monodromy group Γ consists of multiples of the Picard-Lefschetz transformation T and $\tilde\Phi$ satisfies

$$T\tilde\Phi(w) = \tilde\Phi(w+1) .$$

Moreover $\tilde\Phi$ is <u>horizontal</u> in the following sense. The tangent bundle of D has a distinguished holomorphic subbundle, the horizontal subbundle $TH(D)$, consisting of all tangent vectors X such that

$$XF^p \subset F^{p-1}/F^p$$

(recall that a tangent vector at a point $\{F^p\}$ of D can be identified with a collection of homomorphisms from F^p to E/F^p, for each p). Saying that a mapping of a complex manifold into D is horizontal means that it is tangent to $TH(D)$. Mappings arising from variations of Hodge structure are horizontal by definition (cf. (5.9)).

A consequence of the curvature properties of Hodge bundles is that, for any suitably normalized $G_{\mathbb{R}}$-invariant metric, the holomorphic sectional curvature of D satisfies an inequality

$$K(\xi) \leq -1$$

<u>whenever ξ lies in the horizontal subbundle</u> $TH(D)$. Hence the Ahlfors lemma applies to $\tilde\Phi$ and says that $\tilde\Phi$ is <u>distance decreasing</u>, in other terms:

$$\rho_D(\tilde\Phi(w), \tilde\Phi(w')) \leq \rho_H(w, w') \qquad (7.4)$$

where ρ_D is the invariant distance on D and ρ_H is the Poincaré distance on H.

A very elegant application of the distance decreasing property in this form is <u>Borel's proof of the quasi-unipotency of the Picard-Lefschetz transformation</u>. Since T is integral, by a theorem of Kronecker (an algebraic integer all of whose conjugates have absolute value one is a root of unity) it will suffice to prove that the eigenvalues of T have absolute value one. Choose a sequence $\{w_n\}$ of points of H whose imaginary part goes to infinity with n and write

$$\tilde{\Phi}(w_n) = g_n p_0$$

where p_0 is a reference Hodge filtration and g_n belongs to $G_{\mathbb{R}}$. Then

$$\rho_D(\tilde{\Phi}(w_n + 1), \tilde{\Phi}(w_n)) = \rho_D(Tg_np_0, g_np_0) = \rho_D(g_n^{-1}Tg_np_0, p_0) .$$

On the other hand

$$\rho_H(w_n + 1, w_n) = O(\frac{1}{\operatorname{Im} w_n})$$

so that $\rho_D(g_n^{-1}Tg_np_0, p_0)$ geos to zero as n tends to infinity. Since D is the quotient of $G_{\mathbb{R}}$ by a compact subgroup H, this says that a sequence of conjugates of T tends to an element of H, and therefore all eigenvalues of T must have absolute value one.

Another way of looking at the distance decreasing property for a variation of Hodge structure over Δ, H or Δ^* is the following:

For every p and every section e of F^p

(7.5) $$\frac{\sqrt{-1}}{2\pi}(\psi_p e, \psi_p e) \leqq C(e, e)\eta$$

where η is the Poincaré metric and C a suitable positive constant.

This is seen to be equivalent to (7.4) by recalling the explicit description of the tangent bundle to a Grassmannian (lecture 1).

Formula (7.5) can be viewed as giving an estimate

(7.6) $$-C\eta \leqq \frac{(\Theta_p e, e)}{\|e\|^2} \leqq C\eta$$

on the curvature of the Hodge bundles $E^{p,m-p}$. The inequality (7.6) has important consequences. The first one is the algebrization theorem for E.

The basic tool is the following general result.

(7.7) Let $V \to A$ be a holomorphic, hermitian vector bundle over an affine variety A. Choose a smooth compactification $\bar{A}$ of A. If the curvature form of V satisfies estimates

$$-C\eta \leqq \frac{(\Theta e, e)}{\|e\|^2} \leqq C\eta$$

where η is the Poincaré metric in the punctured polycylinders at infinity, then V has an algebraic structure whose global sections e satisfy estimates

$$\max_{\bar{A}-T_\varepsilon} (\log \|e\|) = O(\log (\tfrac{1}{\varepsilon}))$$

where T_ε is an ε-tube around $\bar{A} - A$.

This result immediately gives an algebraic structure on $E^{p,m-p}$, for each p. In addition to (7.7) one has a description of the algebraic cohomology of E, this being given by closed modulo exact forms with appropriate growth conditions at infinity.

To give the algebraic structure on E, we proceed by steps. Suppose F^{p+1} has been algebraicized. Then it follows by the above description of cohomology and (again!) the distance decreasing property that the extension class of $E^{p,m-p}$ by F^{p+1} is algebraic, which allows to algebraicize F^p, and so on.

The following local version of (7.7) also holds.

(7.8) Let $V = \{\Delta^*, E, E_{\mathbb{Z}}, \nabla, Q, \{F^p\}\}$ be a variation of Hodge structures over the punctured disk $\Delta^* = \{\zeta \in C \mid 0 < |\zeta| < 1\}$. Then each of the F^p is generated by a finite number of sections e such that

(7.9) $$\max_{|\zeta| \geq r} (\log \|e\|) = O(\log (\tfrac{1}{r})) .$$

Rather than going into the details of the proof of (7.8) or of its global counterpart we wish to show how one can deduce from the distance decreasing property that the Gauss-Manin connection is algebraic and has regular singular points, when the base is an algebraic curve $S = \bar{S} - \{s_1, \ldots, s_N\}$, $\bar{S}$ being smooth and complete.

By localizing at infinity we may work on a punctured disk $\Delta^* = \{\zeta \mid 0 < |\zeta| < 1\}$. Let e be a (multi-valued) holomorphic flat section of E and write $\zeta = re^{i\theta}$. We decompose e into $(p, m-p)$-components

$$e = \sum e_p \qquad e_p \in E^{p,m-p}.$$

The condition that e be flat means that, for any p

$$D'_p e_p = -\psi_{p+1} e_{p+1}$$

and to say that e is holomorphic means that

$$D''_p e_p = \pm\, {}^t\psi_{p-1} e_{p-1}.$$

Now we take the radial derivative of the Hodge length of e.

$$\begin{aligned}\frac{\partial}{\partial r}(e, e) &= \sum \frac{\partial}{\partial r}(e_p, e_p)\\ &= \sum 2\,\mathrm{Re}\big(\tfrac{\partial}{\partial r} \lrcorner (D_p e_p, e_p)\big)\\ &= -\sum 2\,\mathrm{Re}\big(\tfrac{\partial}{\partial r} \lrcorner (\psi_{p-1} e_{p-1}, e_p)\big) \pm \sum 2\,\mathrm{Re}\big(\tfrac{\partial}{\partial r} \lrcorner ({}^t\psi_{p+1} e_{p+1}, e_p)\big)\end{aligned}$$

Taking absolute values and using the Schwarz inequality plus the distance decreasing property and the explicit form of the Poincaré metric on Δ^* one gets

$$(7.10) \qquad \left| \frac{\partial}{\partial r}(e, e) \right| \le \frac{2(e, e)}{r \log \frac{1}{r}}.$$

By integrating the differential inequality (7.10) we obtain inequalities

$$(7.11) \qquad C_1\left(\log \frac{1}{|\zeta|}\right)^{-k} \le \|e\|^2 \le C_2\left(\log \frac{1}{|\zeta|}\right)^{k}$$

which hold uniformly on any angular sector, for suitable C_1, C_2, k.

Now let T be the Picard-Lefschetz transformation. For the sake of simplicity, here, and in the following, we will always assume that the Picard-Lefschetz transformations involved are unipotent. Write

$$N = \log T = \sum_{q=0}^{\infty} \frac{(I-T)^{q+1}}{q+1}.$$

Due to the unipotency of T, the sum on the right hand side is a finite sum. Then for every flat section e of E,

$$(7.12) \qquad \exp\left(-\frac{\log N}{2\pi\sqrt{-1}}\right) e$$

is a single-valued never vanishing holomorphic section of E. These sections give a privileged extension of E across the puncture of Δ^*, so that, returning to the global situation, we have a privileged extension of E to a vector bundle E' over $\bar{S}$. By Serre's G.A.G.A. this has an algebraic structure: the upper bound in (7.11) together with (7.12) tells us that this agrees with the intrinsic algebraic structure of E. Also

$$\nabla \exp\left(-\frac{\log N}{2\pi\sqrt{-1}}\right) e = -\frac{N}{2\pi\sqrt{-1}} \exp\left(-\frac{\log N}{2\pi\sqrt{-1}}\right) e\, \frac{d\zeta}{\zeta}$$

which shows that ∇ is algebraic and has regular singular points.

d) THE NILPOTENT ORBIT THEOREM. Let V be a variation of Hodge structure of weight m on Δ^*. Another straightforward consequence of (7.11) is the following metric comparison lemma. We denote by E' the privileged extension of E to Δ constructed at the end of c) and let $\langle\, ,\, \rangle$ be any C^∞ inner product on E'. Then, on every compact subset of Δ and for every section e of E,

$$(7.13) \qquad A\left(\log\left(\frac{1}{|\zeta|}\right)\right)^{-k} \|e\|^2 \le \langle e, e \rangle \le B\left(\log\left(\frac{1}{|\zeta|}\right)\right)^{k} \|e\|^2$$

where A, B, k are suitable constants.

Now let f be a section of F^p which satisfies the estimate (7.9). The metric comparison lemma tells us that f is a linear combination with meromorphic coefficients of sections of the form (7.12). Since the base is one-dimensional, this and (7.8) imply that the filtration $\{F^p\}$ extends to a filtration $\{F'^p\}$ on E'.

What we have done here is essentially deducing from (7.8) the first part of W. Schmid's nilpotent orbit theorem. To formally state this we have to define the compact dual $\check{D}$ of D. Recall that D can be viewed as the set of all filtrations (with appropriate Hodge numbers) on a fixed vector space which satisfy the two Hodge-Riemann bilinear relations. $\check{D}$ is defined to be the set of all filtrations satisfying the first Hodge-Riemann bilinear relation, but not necessarily the second. $\check{D}$ is easily seen to be a projective homogeneous algebraic variety: D is an open subset of $\check{D}$. With these notations what we have shown, in the language of classifying spaces for Hodge filtrations, reads as follows.

EXTENSION LEMMA. Let

$$\begin{array}{ccc} H & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\Phi} & \Gamma\backslash D \end{array}$$

be a variation of Hodge structure over Δ^*. Let T be the Picard-Lefschetz transformation (which, for simplicity, we will assume to be unipotent) and set $N = \log T$. Then the mapping

$$w \rightarrow \exp(-wN)\tilde{\Phi}(w)$$

descends to a mapping

$$\Psi : \Delta^* \rightarrow D$$

which extends across the puncture.

The remaining part of the nilpotent orbit theorem, which follows fairly easily from (7.13), says that, if we write $p_0 = \Psi(0)$,

(i) $\exp(wN)p_0$ is horizontal;

(ii) there is a non-negative number α such that, if $\mathrm{Im}(w) > \alpha$, then $\exp(wN)p_0$ belongs to D;

(iii) $\exp(wN)p_0$ is strongly asymptotic to $\tilde{\Phi}(w)$ in the sense that

$$\rho_D(\tilde{\Phi}(w), \exp(wN)p_0) \leq (\mathrm{Im}\ w)^{\beta} e^{-2\pi\ \mathrm{Im}\ w}$$

for some $\beta \geq 0$ and large enough $\mathrm{Im}\ w$.

We now give a sketch of the proof of (i), (ii), (iii). In the Hodge bundle framework, the nilpotent orbit $\exp(wN)p_0$ corresponds to a filtration $\{\tilde{F}^p\}$ of E' which agrees with $\{F'^p\}$ at 0 and is constant, relative to the sections of the type:

$$\exp\left(-\frac{\log \zeta N}{2\pi\sqrt{-1}}\right)e$$

where e is flat. N can be viewed as a flat endomorphism of E which extends across the puncture. (i) just says that $\{\tilde{F}^p\}$ satisfies the

infinitesimal period relations (4.11). This follows from the fact that

$$NF'^p \subset F'^{p-1}$$

at the origin, which in turn is a consequence of the infinitesimal period relations for $\{F^p\}$.

To prove (ii) we must show that, for any non vanishing section e of E'

$$\sum (-1)^p Q(e_p', \overline{e_p'}) > 0$$

in a neighborhood of the origin, where $e = \sum e_p'$ is the decomposition of e into $(p, m-p)$ type relative to the filtration $\{\tilde{F}^p\}$. Assume, inductively, that this has already been proved for sections of $\tilde{F}^{p+1}$ and let e be a section of $\tilde{F}^p$ whose projection into $\tilde{F}^p/\tilde{F}^{p+1}$ does not vanish at the origin. What we have to show is that

$$(-1)^p Q(e_p', \overline{e_p'}) > 0$$

near the origin. Notice that the length of e_p', $< e_p', e_p' >^{1/2}$, relative to any C^∞ metric on E', is bounded both above and below. We may write

$$e_p' = \alpha + \beta + \gamma$$

where α belongs to $E^{p,m-p}$, β belongs to F^{p+1}, γ belongs to $\overline{F^{m-p+1}}$. The metric comparison lemma plus the fact that $\{F'^p\}$ and $\{\tilde{F}^p\}$ agree at the origin imply the estimates

$$(7.14)\qquad \begin{array}{l} (\alpha, \alpha) \geq C_1\left(\log \frac{1}{|\zeta|}\right)^{-k} \\ (\beta, \beta) \leq C_2|\zeta|^2\left(\log \frac{1}{|\zeta|}\right)^k \geq (\gamma, \gamma) \end{array}$$

in a neighborhood of the origin. It follows that

$$(-1)^p Q(e_p', \overline{e_p'}) \geq (\alpha, \alpha) - (\beta, \beta) - (\gamma, \gamma) > 0$$

near 0, as was to be proved.

When suitably interpreted, the estimates (7.14) also give part (iii) of the nilpotent orbit theorem.

In a way the nilpotent orbit theorem enables one to reduce most questions about a general variation of Hodge structure on Δ^* to questions about the approximating nilpotent orbit. However, studying nilpotent orbits seems to be a deeper matter than the nilpotent orbit theorem itself. Very detailed information about nilpotent orbits is given by W. Schmid's SL_2-orbit theorem which will be discussed in the appendix to this lecture. This theorem will not be formally stated here. We will just say that, roughly, it enables us to construct a mapping

$$\Xi : H \to D$$

which lifts to a homomorphism of Lie groups

$$\tilde{\Xi} : SL_2(\mathbb{R}) \to G_{\mathbb{R}}$$

and is asymptotic to the nilpotent orbit $\exp(wN)p_0$ (in a much weaker sense than above). Moreover the theorem gives very detailed information on the way Ξ is related to $\exp(wN)p_0$, and this is crucial in most applications.

Instead of insisting on the SL_2-orbit theorem we will give some of its consequences. To do so we have to go back to the original situation, when we have a variation of Hodge structure of weight m over Δ^*: as usual the Picard-Lefschetz transformation T will be assumed to be unipotent. The monodromy theorem in strong form, as follows from the SL_2-orbit theorem, says that the index of unipotency of T is at most m, provided the Hodge numbers h^{pq} are zero if $p < 0$ or $q > m$. We have constructed extensions of E and F^p to Δ. We will denote these with the same symbols as their restrictions to Δ^*. N may be viewed as a flat endomorphism of E on all of Δ. The weight filtration of E is the unique filtration

$$0 \subset W_0 \subset \cdots \subset W_{2m} = E$$

which satisfies the following conditions:

$$NW_i \subset W_{i-2}$$

$$N^\ell : W_{m+\ell}/W_{m+\ell-1} \to W_{m-\ell}/W_{m-\ell-1}$$
is an isomorphism for every $\ell \geq 0$

$\{W_i\}$ can be constructed as follows. It is clear that $W_{2m-1} = \ker N^m$ and $W_0 = N^m(E)$. Then we consider the linear mapping induced by N on

W_{2m-1}/W_0 and work our way down by induction on m.

Now denote by $\{\tilde{F}^p\}$ an approximating nilpotent orbit to $\{F^p\}$. One of the consequences of the SL_2-orbit theorem is that

(7.15) $\{W_i\}$, $\{\tilde{F}^p\}$, $E_{\mathbb{Z}}$ induce a mixed Hodge structure on each of the fibres of $E|\Delta^*$. Relative to this, N is a morphism of type $(-1, -1)$.

Let us remark that the last assertion follows immediately from the infinitesimal period relation for the nilpotent orbit $\{\tilde{F}^p\}$ and from the definition of $\{W_i\}$.

What (7.15) tells us is that the Hodge structures on the fibres of E asymptotically approach a mixed Hodge structure as we go into the puncture.

Another consequence is a concrete description of the weight filtration. This is a refinement of the estimates (7.11). Let e be a flat section of E. Saying that e belongs to W_ℓ (at each point) means that the Hodge length $\|e\|$ satisfies an estimate

$$\|e\| = O\left(\log\left(\frac{1}{|\zeta|}\right)^{\frac{\ell-m}{2}}\right)$$

uniformly on each angular sector. In particular, since obviously

$$\ker N \subset W_m$$

then every invariant e ($Te = e$, or, which is the same, $Ne = 0$) is bounded near the puncture. This is the statement needed to prove the theorem of the fixed part and its corollaries without completeness assumptions on the base.

REFERENCES FOR LECTURE 7

The curvature properties of Hodge bundles and subsequent applications to variation of Hodge structure over a compact base were given in

P. Griffiths, Periods of integrals on algebraic manifolds (III), Publ. Math. I.H.E.S., vol. 38 (1970), 125-180.

The algebraization theorem 7.7 is proved in

M. Cornalba and P. Griffiths, Analytic cycles and vector bundles on non-compact algebraic varieties. To appear in Inventiones Math.

The theory of regular singular points is discussed in

P. Deligne, Equations différentielles à points singuliers réguliers, Lecture notes in math. no. 163, Springer-Verlag (1970).

Finally, the basic results of Schmid appear in

W. Schmid, The singularities of the period mapping, Inventiones Math., vol. 22 (1973), 211-319.

APPENDIX TO LECTURE 7: ON SCHMID'S SECOND THEOREM

This appendix is intended as an aid in understanding Schmid's proof of his basic technical theorem, the $s\ell_2$-orbit theorem, which appears in his paper listed in the references to lecture 7. It is not our purpose to give either the precise statement or complete proof, but rather to extract the essential aspects of the argument and put them in a more differential-geometric and less Lie-theoretic setting.

a) HEURISTIC REASONING. We consider a variation of Hodge structure

$$\Phi : \Delta^* \to \{T^n\}\backslash D$$

over the punctured disc. By passing to a finite covering of Δ^*, we may assume that the Picard-Lefschetz transformation T is unipotent with logarithm

$$N = \log T = \sum_q (-1)^q \frac{(T-I)^q}{q} .$$

Lifting up to the universal covering $H \to \Delta^*$, there is an induced variation of Hodge structure, still denoted by Φ,

$$\Phi : H \to D \quad \text{satisfying}$$
$$\Phi(z+1) = T\Phi(z) .$$

Schmid's results allow us to approximate any such Φ by an equivariant variation of Hodge structure

$$\Sigma : H \to D$$

satisfying the asymptotic estimate $(z = x + \sqrt{-1}\, y)$

$$\rho_D(\Phi(z), \Sigma(z)) = O(y^{-1})$$

for $y \geq C$. Σ is induced by a representation

$$\eta : SL_2(\mathbb{R}) \to G_{\mathbb{R}}$$

having very special properties, and most aspects of the behavior of $\Phi(z)$ as $\operatorname{Im} z \to \infty$ are the same as for $\Sigma(z)$ and can therefore be deduced by Lie algebra calculations.

Here is the heuristic reasoning behind Schmid's theorem. Let $F(z)$ be a Hodge frame field for $\Phi(z)$ with Maurer-Cartan matrix ω defined by

$$\text{(A.7.1)} \qquad \begin{aligned} dF &= \omega\cdot F , \quad \text{where} \\ d\omega &= \omega \wedge \omega . \end{aligned}$$

We adopt the viewpoint that Φ is completely determined by ω, and thus we should try to select the Hodge frame field $F(z)$ such that the asymptotic behavior of $\omega(z)$ as $\operatorname{Im} z \to \infty$ becomes most transparent. As in lecture 5 we write

$$\omega = \phi + \psi + {}^t\bar{\psi}$$

where

$$\psi = \begin{pmatrix} 0 & \psi_n & \cdot & 0 \\ \cdot & & & \cdot \\ \cdot & & & \psi_1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}$$

is a matrix of $(1, 0)$ forms giving the Kodaira-Spencer class for the variation of Hodge structure, and where

$$\phi = \begin{pmatrix} \phi_n & \cdot & \cdot & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & \cdot & \cdot & \phi_0 \end{pmatrix}$$

gives the connection matrices in the various Hodge bundles. The pull-back of the $G_{\mathbb{R}}$-invariant metric on D is

$$\sum_q \text{Trace}\,(\psi_p \wedge {}^t\bar{\psi}_p)\ ,$$

so that if we write $\psi = Adz$, the Ahlfors lemma from lecture 6 gives the basic estimate

$$(\text{A.7.2}) \qquad A(z) = O(y^{-1}) \qquad (z = x + iy)$$

which underlies the whole development. What is suggested is that we try to choose $\underline{F}(z)$ such that ϕ can also be estimated.

The integrability condition (A.7.1) is now

$$(\text{A.7.3}) \qquad \begin{aligned} d\phi - \phi \wedge \phi &= \psi \wedge {}^t\bar{\psi} + {}^t\bar{\psi} \wedge \psi \\ d\psi &= \phi \wedge \psi + \psi \wedge \phi\ . \end{aligned}$$

Writing $\phi = Bdx + Cdy$, the first equation gives

$$(\text{A.7.4}) \qquad \frac{\partial C}{\partial x} - \frac{\partial B}{\partial y} - [B, C] = -2\sqrt{-1}\,[A, {}^t\bar{A}] = O(y^{-2})\ .$$

This fails to yield an estimate on B or C, unless one or the other is zero. The form of (A.7.4) suggests that we try to select our frame field such that $C = 0$. Then $\frac{\partial B}{\partial y} = O(y^{-2})$ so that

$$(\text{A.7.5}) \qquad B = O(y^{-1})\ .$$

Frame fields with the property that $\phi \equiv O(dx)$ are characterized geometrically by the condition that $\underline{F}(z)$ remains parallel to itself along the vertical lines $x = \text{constant}$.* We shall call these geodesic frame fields; they are easy to construct by prescribing the initial values $\underline{F}(x + \sqrt{-1})$ and then making parallel displacement up and down vertical lines. Geodesic frame fields are the analogue of normal coordinates in Riemannian geometry, and allow one to most easily recover properties of the connection matrix from the curvature.

EXAMPLE: Over the upper half plane H we consider the universal family of Hodge structures of weight one and genus one. The vector space E is $\mathbb{C}^2$, the quadratic form $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and each point $z \in H$ gives the polarized Hodge decomposition

* Here, parallel displacement is relative to the Hodge connection ϕ and not the Gauss-Manin connection.

$$\mathbb{C}^2 = [\mathbb{C}\cdot(z, 1)] \oplus [\mathbb{C}\cdot(\bar{z}, 1)]\ .$$

Since $\sqrt{-1}\,Q((z, 1), \overline{(z, 1)}) = 2y$, the manifold $F(H)$ of Hodge frames consists of all pairs

$$(\text{A.7.6}) \qquad \{e, \bar{e}\} \quad \text{where} \quad e = \frac{\exp \sqrt{-1}\,\sigma}{\sqrt{2y}}\,(z, 1)\ .$$

The Maurer-Cartan matrix

$$(\text{A.7.7}) \qquad \omega_H = \begin{pmatrix} \sqrt{-1}\,d\sigma + \frac{\sqrt{-1}\,dx}{2y} & \exp 2\sqrt{-1}\,\sigma\,\frac{dz}{2y} \\ \exp(-2\sqrt{-1}\,\sigma)\,\frac{d\bar{z}}{2y} & -\sqrt{-1}\,d\sigma - \frac{\sqrt{-1}\,dx}{2y} \end{pmatrix}$$

Geodesic frame fields are those frame fields $\{e(z), \overline{e(z)}\} \in F(H)$ for which $\sigma = \sigma(x)$ is a function of x alone. In particular, $\sigma = \text{constant}$ defines one such. This example shows that the estimates (A.7.2) and (A.7.5) are sharp.

Returning to the general case, we let $\underline{F}(z)$ be a geodesic frame field lying over a variation of Hodge structure $\Phi(z)$. On the basis of (A.7.2), (A.7.5) and the above example, we might hope to have an expansion

$$\omega = \omega_{-1} + \tilde{\omega}$$

where ω_{-1} is a linear combination of $\frac{dz}{y}$ and $\frac{d\bar{z}}{y}$ whose coefficient matrices are constant, and where $\tilde{\omega}(z)$ is of lower order as $\text{Im}\, z \to \infty$. The second equation in (A.7.1) should imply that

$$d\omega_{-1} = \omega_{-1} \wedge \omega_{-1}\ .$$

In this case, we may define a mapping

$$\Sigma : H \to D$$

by solving the differential equation

$$(\text{A.7.9}) \qquad d\underline{G}(z) = \omega_{-1}(z)\underline{G}(z)$$

to find a Hodge frame field $\underline{G}$ for Σ. By comparing the form $\omega = \phi + \psi + {}^t\bar{\psi}$ of ω with the form (A.7.7) of the Maurer-Cartan matrix for $SL_2(\mathbb{R}) \cong F(H)$, and taking into account that ω_{-1} is a constant linear combination of $\frac{dz}{y}$ and $\frac{d\bar{z}}{y}$, Σ as defined by (A.7.9) is an equivariant variation of Hodge structure induced from a homomorphism $\eta : SL_2(\mathbb{R}) \to G_{\mathbb{R}}$ whose induced Lie algebra mapping is just ω_{-1} . Moreover, it is plausible that Σ , viewed as the "principal part" of Φ as $\operatorname{Im} z \to \infty$, should share its essential properties. Making this argument precise constitutes Schmid's theorem.

To carry all this out, one essentially needs to know that for the original variation of Hodge structure $\Phi(x + \sqrt{-1}\, y)$ looked at along lines parallel to the imaginary axis, the Maurer-Cartan matrix $\omega(x + \sqrt{-1}\, y)$ for a geodesic frame field has a Laurent series expansion in powers of y^{-1} as $y \to \infty$. In this case, the decomposition (A.7.6) can be analyzed by using series expansions in the structure equations (A.7.3). However, it is not at all clear that, for our given variation of Hodge structure $\Phi(z)$, there is some Hodge frame field $\underline{F}(z)$ (much less a geodesic one) whose Maurer-Cartan matrix has such a Laurent series. Thus, the proof consists of first replacing $\Phi(z)$ by a nilpotent orbit $\Psi(z)$ where it is easy to find a (non-geodesic) frame field with this property, and then proving that rotating into a geodesic frame field still yields a Maurer-Cartan matrix having the desired Laurent series.

b) FRAMING THE NILPOTENT ORBIT. Given a variation of Hodge structure $\Phi : H \to D$ satisfying $\Phi(z + 1) = T\Phi(z)$ where T is unipotent with logarithm N , we consider nilpotent orbits

$$\Psi(z) = \exp(zN)\cdot\Psi_0 \qquad (\Psi_0 \in \check{D}) .$$

Any such nilpotent orbit is the restriction to $H \in \mathbb{P}^1$ of a polynomial mapping into the dual classifying space $\check{D}$ of all filtrations $\{F^p\}$ on E satisfying the first bilinear relation $Q(F^p, F^{n-p-1}) = 0$. In general, however, $\Psi(z)$ is neither horizontal nor maps H into D . Schmid's first theorem asserts that, for a suitable choice of reference point Ψ_0 , the nilpotent orbit gives a variation of Hodge structure which is strongly asymptotic to $\Phi(z)$ in the sense that

$$\rho_D(\Phi(z), \Psi(z)) = O(y^{\alpha}e^{-y})$$

for $z = x + \sqrt{-1}\, y \in H$, $y \geq C$. Henceforth, we shall restrict our attention to $\Psi(z)$.

We first note that

$$\exp(zN) = \exp(xN)\exp(\sqrt{-1}\, yN)$$

where $\exp(xN) \in G_{\mathbb{R}}$. Thus, the $G_{\mathbb{R}}$-invariant properties of $\Psi(z)$ are all described by looking at $\Psi(\sqrt{-1}\, y)$.

In order to frame $\Psi(z)$, we fix a reference Hodge frame $\underline{F}_0$ lying over Ψ_0 . Then the frames

$$\underline{\tilde{F}}(z) = \exp(zN)\underline{F}_0$$

give a field of frames lying over $\Psi(z)$ which depend in a polynomial fashion on z . However, these frames are not Hodge frames, but are only frames adapted to the filtration $\Psi(z)$. To obtain a Hodge frame field, it is natural to apply the Gram-Schmidt process, which we now review.

Given a vector space V with Hermitian form $(\ ,\ )$, a subspace W such that $(\ ,\ )$ is definite on W and on $W^{\perp}$, and a basis $\{w_1, \ldots, w_d; v_1, \ldots, v_e\}$ for V such that $w_1, \ldots, w_d$ is a basis for W , the Gram-Schmidt process without normalization consists of the following three steps:

(i) a transformation

$$\begin{aligned} w_1 &\to w_1 \\ w_2 &\to w_2 - \frac{(w_1, w_2)}{(w_1, w_1)} w_1 \\ &\vdots \end{aligned}$$

converts $w_1, \ldots, w_d$ into an orthogonal basis for W ;

(ii) assuming (i), the transformation

$$v_\alpha \to v_\alpha - \sum_i \frac{(w_i, v_\alpha)}{(w_i, w_i)} w_i$$

where the v_α's orthogonal to the w_i's ; and

(iii) assuming (i) and (ii), we apply (i) to the v_α's .

In an obvious way, we may apply this process to the vectors in $\underline{F}(z)$ relative to the Hodge filtration $\Psi(z)$ and obtain an orthogonal frame $\underline{F}^{\#}(z)$ in which the vectors are rational functions of z and $\bar{z}$. Finally, we may normalize to make the vectors in $\underline{F}^{\#}(z)$ have unit length to obtain a Hodge frame field $\underline{F}(z)$. Since, if $a(y)$ is a rational function of y with $a(y) > 0$ for $y \geq C$ then $\sqrt{a(y)}$ has a convergent Laurent series expansion in $y^{-1/2}$ for $y \geq C'$, our <u>rational frame field</u> $\underline{F}(z)$ has the properties:

$\underline{F}(z)$ is a Hodge frame field for $\Psi(z)$;

$\underline{F}(z + x) = \exp(xN)\underline{F}(z)$ $\quad (\exp(xN) \in G_{\mathbb{R}})$; and

$\underline{F}(\sqrt{-1}\,y)$ has a convergent Laurent series expansion in $y^{-1/2}$ for $y \geq C$.

Moreover, if the Maurer-Cartan matrix $\omega_{\underline{F}}(z)$ is defined by $d\underline{F} = \omega_{\underline{F}}\underline{F}$, then

$$\omega_{\underline{F}}(z + x) = \omega_{\underline{F}}(z)$$

since ω is $G_{\mathbb{R}}$-invariant. Writing as usual

$$\begin{aligned} \omega_{\underline{F}} &= \phi_{\underline{F}} + \psi_{\underline{F}} + {}^t\bar{\psi}_{\underline{F}} \\ \psi_{\underline{F}} &= a(y)\,dz \\ \phi_{\underline{F}} &= b(y)\,dx + c(y)\,dy, \end{aligned}$$

the Ahlfors lemma gives

$$a(y) = \sum_{n=0}^{\infty} a_n y^{-(n+2)/2}.$$

As discussed above, this does not seem to easily yield information on the Laurent series expansions of $b(y)$ and $c(y)$.

Therefore, for the reasons discussed previously, we are led to consider a rotation

$$\underline{G}(z) = h(z)\underline{F}(z) \qquad (h(z) \in H)$$

taking $\underline{F}(z)$ into a geodesic frame field $\underline{G}(z)$. We may obviously assume that $h(x + \sqrt{-1}\,y) = h(y)$ is independent of x. The relation between Maurer-Cartan matrices is

$$\mathrm{Ad}h\,\omega_{\underline{G}} = h^{-1}dh + \omega_{\underline{F}}.$$

Writing $\omega_{\underline{G}} = \phi_{\underline{G}} + \psi_{\underline{G}} + {}^t\bar{\psi}_{\underline{G}}$ where

(A.7.10) $$\begin{aligned} \psi_{\underline{G}}(z) &= A(y)\,dz \\ \phi_{\underline{G}}(z) &= B(y)\,dx \qquad \text{(since } \underline{G} \text{ is geodesic)}, \end{aligned}$$

it follows that $h(y)$ is defined by the O.D.E.

(A.7.11) $$h'(y) + h(y)c(y) = 0.$$

Our major step will be to prove that (A.7.11) has a regular singular point at $y = \infty$, which, when taken together with certain addition properties of the solution matrix $h(y)$, will lead to the desired Laurent series expansion. Writing

$$c(y) = \sum_{n=-N}^{\infty} c_n y^{-(n+2)/2} = c_{-N}y^{N/2-1} + (\cdots) \qquad (c_{-N} \neq 0),$$

we want to show that $N = 0$. The idea is to use the O.D.E. (A.7.11) to obtain certain growth estimates on the derivatives $h^{(k)}(y)$, and then to compare these with estimates coming from the Ahlfors lemma and the structure equations (A.7.3).

LEMMA: $h^{(k)}(y) = O(y^{(kN/2)-k})$, and no better estimate is possible.

PROOF: Differentiating (A.7.11) leads to

$$\begin{aligned} h'' &= hc^2 + hc' \\ &\vdots \\ h^{(k)} &= (-1)^k hc^k + hd_k \end{aligned}$$

where d_k has a Laurent series beginning with $y^{(kN/2)-(k+1)}$. This follows inductively from

$$d_{k+1} = (-1)^k c'c^k - cd_k + d_{k+1}.$$

Since H is a compact matrix group, $h(y) = O(1)$ and $c_{-N} + {}^t\bar{c}_{-N} = 0$. Thus c_{-N} can be diagonalized, and in particular $(c_{-N})^k \neq 0$ for all k. Consequently

$$h^{(k)} = O((c_{-N})^k y^{(kN/2)-k}) = O(y^{(kN/2)-k})$$

and no better estimate is possible. Q.E.D.

Next, referring to (A.7.10), we shall prove the

LEMMA: $A^{(k)}(y)$ and $B^{(k)}(y)$ are $O(y^{-k-1})$.

PROOF: The structure equations (A.7.3) give

$$B'(y) = [A(y), {}^t\bar{A}(y)]$$
$$A'(y) = [B(y), A(y)] .$$

The Ahlfors lemma gives the estimate

$$A(y) = O(y^{-1}) .$$

It follows from the first equation that $B'(y) = O(y^{-2})$ so that $B(y) = O(y^{-1})$. The second equation then implies that $A'(y) = O(y^{-2})$. Differentiating the first equation then yields $B''(y) = O(y^{-3})$, and doing the same in the second equation gives $A''(y) = O(y^{-3})$. Continuing this process gives the lemma.

LEMMA: $h^{(k)}(y) = O(y^{\gamma-k})$ for some fixed γ .

PROOF: We write $\underline{G}(y) = \underline{G}(\sqrt{-1}\,y)$, $\underline{F}(y) = \underline{F}(\sqrt{-1}\,y)$ and identify the Hodge frame manifold with the matrix group $G_{\mathbb{R}}$. Then

(A.7.12) $$h(y) = \underline{G}(y)\underline{F}^{-1}(y) .$$

Moreover, $\underline{F}^{-1}(y)$ has a Laurent series and so

$$\underline{F}^{-1}(y)^{(k)} = O(y^{\alpha-k})$$

for some fixed α . If we can prove that

(A.7.13) $$\underline{G}(y)^{(k)} = O(y^{\beta-k}) ,$$

then the lemma will follow by differentiating (A.7.12) and taking $\gamma = \alpha + \beta$.

Since $h(y) = O(1)$, $\underline{G}(y) = h(y)\underline{F}(y)$ is $O(y^{\beta+1})$. Moreover, the structure equation $d\underline{G} = \omega_{\underline{G}}\cdot\underline{G}$ and previous lemma give

$$\underline{G}'(y) = \omega(y)\underline{G}(y)$$

where $\omega(y)^{(k)} = O(y^{-k-1})$. (A.7.13) now results inductively by differentiating this equation. Q.E.D.

Comparing the first and third lemmas gives $N \leq 0$; i.e.

The O.D.E. (A.7.11) has a regular singular point at $y = \infty$.

The solution matrix to any such equation has an expansion in powers of $y^{-1/2}$ and $(\log y)^{\alpha}y^{-\sigma/2}$ where σ is an eigenvalue of c_0 . Moreover, the log terms can occur only if two eigenvalues of c_0 differ by an integer. Since $c_0 + {}^t\bar{c}_0 = 0$, the eigenvalues of c_0 are purely imaginary, and thus no log terms can appear. To insure that $h(y)$ has a Laurent series in $y^{-1/2}$, some further argument is necessary. What must be proved is that $c_0 = 0$.

For this, we consider the rational frame field $\underline{F}(z)$, which we know has a Laurent series in $y^{-1/2}$ along the imaginary axis. Since $h^{-1}h'$ and $Adh\cdot\omega_{\underline{G}}$ are both $O(y^{-1})$, the Maurer-Cartan matrix

$$\omega_{\underline{F}} = Adh\omega_{\underline{G}} - h^{-1}dh$$

is $O(y^{-1})$. Thus $b(y) = \sum_{n\geq 0} b_n y^{-(n+2)/2}$. The integrability conditions (A.7.3) give

$$b_0 + [c_0, b_0] = -2\sqrt{-1}\,[a_0, {}^t\bar{a}_0]$$
$$a_0 = \sqrt{-1}\,[b_0, a_0] - \sqrt{-1}\,[c_0, a_0]$$
$${}^t\bar{a}_0 = -\sqrt{-1}\,[b_0, {}^t\bar{a}_0] + \sqrt{-1}\,[c_0, {}^t\bar{a}_0] .$$

Under these conditions, a lemma on Lie algebras due to Deligne gives that

$$[c_0, a_0] = 0 = [c_0, {}^t\bar{a}_0] .$$

Along the imaginary axis,

$$\underline{F}'(y) = [(c_0 + a_0 + {}^t\bar{a}_0)y^{-1} + \cdots]\underline{F}(y) .$$

The matrix $a_0 + {}^t\bar{a}_0$ has real eigenvalues while those of c_0 are purely imaginary. By the commutation relation, the eigenvalues of $c_0 + a_0 + {}^t\bar{a}_0$ are real if, and only if, $c_0 = 0$. But $\underline{F}(y)$ has a Laurent series in $y^{-1/2}$, which is possible only if $c_0 + a_0 + {}^t\bar{a}_0$ has only real eigenvalues. In conclusion:

$h(y)$ has a convergent Laurent series $\sum_{n\geq 0} h_n y^{-n/2}$.

At this point, we have proved that the geodesic frame field $\underline{G}(z)$ for he nilpotent orbit has the following properties:

$\underline{G}(z+x) = \exp(xN)\underline{G}(z) \qquad (\exp(xN) \in G_{\mathbb{R}})$;

$\underline{G}(\sqrt{-1}\, y)$ has a Laurent series in $y^{-1/2}$;

the Maurer-Cartan matrix $\omega_{\underline{G}} = \phi + \psi + {}^t\bar{\psi}$ where $\phi = b(y)dx$ and $\psi = a(y)dz$ with $a(y)$, $b(y)$ having Laurent series in $y^{-1/2}$ beginning with y^{-1} .

e are now in a position to pursue further the heuristic reasoning given in a).

c) USE OF THE REPRESENTATION THEORY OF $s\ell_2$. Let $\Psi(z)$ be the nilotent orbit and $\underline{G}(z)$ the geodesic frame with Maurer-Cartan matrix $\omega_{\underline{G}}$. ccording to what was proved in (b), we may write

$$\omega_{\underline{G}} = \omega_{\underline{H}} + \tilde{\omega}$$

here $\omega_{\underline{H}}$ contains the terms in $\omega_{\underline{G}}$ involving y^{-1} and $\tilde{\omega}$ is a Laurent eries in $y^{-1/2}$ beginning with $y^{-3/2}$. The integrability condition $d\omega_{\underline{G}} = \omega_{\underline{G}} \wedge \omega_{\underline{G}}$ obviously implies that

$$d\omega_{\underline{H}} = \omega_{\underline{H}} \wedge \omega_{\underline{H}} .$$

ccording to the discussion in (a), $\omega_{\underline{H}}$ is thus the Maurer-Cartan matrix or a geodesic frame $\underline{H}(z)$ associated to an equivariant variation of Hodge tructure

$$\Sigma : H \to D .$$

To see better what is going on, we write as usual (cf. (A.7.7) for otivation of constants)

$$\omega_{\underline{G}} = \phi + \psi + {}^t\bar{\psi}$$

$$\phi = -A(y)\frac{\sqrt{-1}\,dx}{2} , \qquad \psi = B(y)\frac{dz}{2}$$

where

$$A(y) = \sum_{n\geq 0} A_n y^{-(n+2)/2} , \qquad B(y) = \sum_{m\geq 0} B_m y^{-(m+2)/2} .$$

The integrability relations (A.7.3) give

$$\text{(A.7.14)} \qquad \begin{aligned} -A'(y) &= [B(y), {}^t\overline{B(y)}] \\ -2B'(y) &= [A(y), B(y)] \\ 2\,{}^t\overline{B'(y)} &= [A(y), {}^t\overline{B(y)}] . \end{aligned}$$

When expanded out, these become

$$\text{(A.7.15)} \qquad \begin{aligned} \frac{n+2}{2}A_n &= \sum_{p+q=n} [B_p, {}^t\bar{B}_q] \\ (n+2)B_n &= \sum_{p+q=n} [A_p, B_q] \\ (n+2)\,{}^t\bar{B}_n &= \sum_{p+q=n} [A_p, {}^t\bar{B}_q] . \end{aligned}$$

For $n = 0$, we obtain

$$\text{(A.7.16)} \qquad \begin{aligned} A_0 &= [B_0, {}^t\bar{B}_0] \\ 2B_0 &= [A_0, B_0] \\ -2\,{}^t\bar{B}_0 &= [A_0, {}^t\bar{B}_0] . \end{aligned}$$

This says exactly that the assignment

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to A_0$$

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to B_0$$

$$e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to {}^t\bar{B}_0$$

gives a representation of $s\ell_2$ on $g\ell(E)$. Since

$$\omega_{\underline{H}} = \phi_0 + \psi_0 + {}^t\bar{\psi}_0 \qquad \text{where}$$

$$\phi_0 = -A_0\sqrt{-1}\,\frac{dx}{2y} , \qquad \psi_0 = B_0\,\frac{dz}{2y}$$

this is certainly consistent with Σ being induced by a representation $\eta : SL_2(\mathbb{R}) \to G_{\mathbb{R}}$.

To use this, we identify the manifold $F(D)$ of Hodge frames with the group $G_{\mathbb{R}}$ by choosing a reference frame $\underline{F}_0$. Thus $\underline{G}(z) = G(z)\cdot\underline{F}_0$ where $G(z),\ F(z) \in G_{\mathbb{R}}$, and the Maurer-Cartan forms are given by the matrices

$$\omega_{\underline{G}} = dG\cdot G^{-1}$$
$$\omega_{\underline{H}} = dH\cdot H^{-1}\ .$$

Along the imaginary axis we may explicitly describe $H(y) = H(\sqrt{-1}\ y)$ by

$$H(y) = \exp(\tfrac{1}{2}\log y\ C_0)$$
$$C_0 = \sqrt{-1}\ (B_0 - {}^t\bar{B}_0)\ .$$

(PROOF: The vector field $2y\frac{\partial}{\partial y}$ is the infinitesimal generator of the 1-parameter subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$ of $SL_2(\mathbb{R})$ acting by linear fractional transformations on $\mathcal{H}$. Setting $\Sigma_0 = \Sigma(\sqrt{-1})$,

$$\sqrt{-1}\ y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}\cdot\sqrt{-1} = \exp\left(\frac{1}{2}\log y\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)\cdot\sqrt{-1}\ ,$$

which implies that

$$\Sigma(\sqrt{-1}\ y) = \exp\left(\frac{1}{2}\log y\ \eta_*\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)\cdot\Sigma_0$$
$$= \exp[\ \frac{1}{2}\log y < \omega_{\underline{H}},\ 2y\frac{\partial}{\partial y} >]\cdot\Sigma_0$$
$$= \exp(\ \frac{1}{2}\log y\ C_0)\cdot\Sigma_0\ .)$$

We now set $J(y) = H(y)^{-1}G(y)$ so that

(A.7.17) $$dJ\cdot J^{-1} = -H^{-1}dH + \mathrm{Ad}H^{-1}(\omega_{\underline{G}})\ .$$

By what was just proved, along the imaginary axis

(A.7.18) $$-H^{-1}dH = -C_0\frac{dy}{2}\ .$$

The strategy is to first prove that the right hand side of (A.7.17) is a Laurent series in y^{-1} beginning with y^{-2}. If this has been done, $J(y)$ is regular at $y = \infty$. Then

$$\rho_D(\Sigma(\sqrt{-1}\ y),\ \Psi(\sqrt{-1}\ y)) = \rho_D(H(y)\Sigma_0,\ G(y)\Psi_0)$$
$$= \rho_D(\Sigma_0,\ H(y)^{-1}G(y)\Psi_0)$$
$$= O(1)$$

since $J(y) \in G_{\mathbb{R}}$ is regular at $y = \infty$. Choosing $\Sigma_0 = \Psi_0$ gives

$$\rho_D(\Sigma(\sqrt{-1}\ y),\ \Psi(\sqrt{-1}\ y)) = O(y^{-1})\ ,$$

from which it follows that

$$\rho_D(\Sigma(z),\ \Psi(z)) = \rho_D(\exp(xN)\Sigma(\sqrt{-1}\ y),\ \exp(xN)\Psi(\sqrt{-1}\ y))$$
$$= \rho_D(\Sigma(\sqrt{-1}\ y),\ \Psi(\sqrt{-1}\ y))$$
$$= O(y^{-1})$$

proving that the orbits are asymptotic.

To carry this out, we write $\omega_{\underline{G}} = \omega_{\underline{H}} + \tilde{\omega}$ and observe that by (A.7.18)

$$-H^{-1}dH + \mathrm{Ad}(\exp -\frac{1}{2}\log y\ C_0)\omega_{\underline{H}} = 0\ .$$

It follows that

$$dJ\cdot J^{-1} = \mathrm{Ad}(\exp -\frac{1}{2}\log y\ C_0)\tilde{\omega}$$

(A.7.19) $$= \sum_{n>0}\mathrm{Ad}(\exp -\frac{1}{2}\log y\ C_0)C_n y^{-(n+2)/2} \quad \text{where}$$

$$C_n = \sqrt{-1}\ (B_n - {}^t\bar{B}_n)\ .$$

The matrix $C_0 = {}^t\bar{C}_0$ is Hermitian and has integral eigenvalues, since $H(y) = \exp(\ \frac{1}{2}\log y\ C_0)$ is a Laurent series in $y^{-1/2}$. Thus we may write $C_n = \sum_s C_{n,s}$ where

$$[C_0,\ C_{n,s}] = sC_{n,s}\ .$$

Expanding (A.7.19) out gives

(A.7.20) $$dJ\cdot J^{-1} = \sum_{n>0,s} C_{n,s}y^{-(n+s+2)/2}\ .$$

What we must show is that $C_{n,s} = 0$ unless $n + s$ is even and positive.

The idea for proving this is to use (A.7.16) to have an action of $s\ell_2$

on $g\ell(E)$. We may decompose $g\ell(E)$ into the well-known irreducible representations, and decompose the coefficients A_n, B_n, ${}^t\overline{B}_n$ accordingly. When this information is fed into (A.7.15), many of the pieces in the resulting decomposition are forced to be zero.

There is, however, one further hitch. Referring to (A.7.19) and (A.7.20), it is desirable to have an $s\ell_2$ action in which C_0 , rather than A_0 , plays the role of h . For this we set

$$\sqrt{-1}\,(B(y) - {}^t\overline{B(y)}) = C(y) = \sum_{n\geq 0} C_n y^{-(n+2)/2}$$

$$\frac{\sqrt{-1}\,A(y)}{2} - \frac{B(y)}{2} - \frac{{}^t\overline{B}(y)}{2} = E(y) = \sum_{n\geq 0} E_n y^{-(n+2)/2}$$

$$-\frac{\sqrt{-1}\,A(y)}{2} - \frac{2B(y)}{3} - \frac{2\,{}^t\overline{B(y)}}{3} = D(y) = \sum_{n\geq 0} D_n y^{-(n+2)/2} .$$

The equations (A.7.14) then give

$$\text{(A.7.21)}\qquad \begin{aligned} -C'(y) &= [D(y), E(y)] \\ -2D'(y) &= [C(y), D(y)] \\ 2E'(y) &= [C(y), E(y)] . \end{aligned}$$

In particular, $\{C_0, D_0, E_0\}$ give an $s\ell_2$ action on $g\ell(E)$ with C_0 corresponding to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover, the relations (A.7.15) are now satisfied with C_n, D_n, E_n replacing A_n, B_n, ${}^t\overline{B}_n$.

It is well known that the irreducible $s\ell_2$-modules V_r are indexed by non-negative integers r . Moreover, V_r decomposes under h into 1-dimensional eigenspaces $V_{r,s}$ for $s = r, r-2, \ldots, -r$ on which h has eigenvalue s . Write

$$\text{(A.7.22)}\qquad g\ell(E) = \bigoplus_{r,s} g\ell(r, s)$$

where $g\ell(r) = \bigoplus_{s=-r}^{r} g\ell(r, s)$ are the copies of V_r appearing in the above $s\ell_2$ action and $g\ell(r, s)$ is the s^{th} eigenspace. (A.7.22) induces

$$C_n = \sum_{r,s} C_n^{r,s}$$

$$D_n = \sum_{r,s} D_n^{r,s}$$

$$E_n = \sum_{r,s} E_n^{r,s} .$$

'nder these conditions, the representation theory of $s\ell_2$ applied to the 'elations (A.7.15) involving the C_n, D_n, E_n's gives the following (lemma .48) in Schmid's paper):

(i) If $n < r$ or $n - r$ is not even, $C_n^{r,s} = D_n^{r,s} = E_n^{r,s} = 0$;

(ii) $C_n^{n,n} = C_n^{n,-n} = D_n^{n,-n} = D_n^{n,2-n} = E_n^{n,n} = E_n^{n,n-2} = 0$;

(iii) $C_{n-2}^{n-2,n-2} = C_{n-2}^{n-2,2-n} = D_{n-2}^{n-2,n-2} = C_{n-2}^{n-2,4-n} = E_{n-2}^{n-2,n-2} = E_{n-2}^{n-2,n-4} = 0$.

Referring to (A.7.20),

$$dJ\cdot J^{-1} = \sum_{\substack{n>0 \\ r,s}} C_n^{r,s} y^{-(n+s+2)/2} .$$

Now $C_n^{r,s} = 0$ unless $r \equiv s$ (2), so that by (i) the only non-zero $C_n^{r,s}$ are of the form

$$C_n^{n+2t,2p} \qquad (-2t \leq 2p \leq 2t) .$$

Thus $dJ\cdot J^{-1}$ has a Laurent series in y^{-1} beginning with y^{-2} as desired.

Schmid's theorems on variation of Hodge structure now follow from the preceding analysis together with further discussion about the representation theory of $s\ell_2$. Here is the idea.

Fixing a reference Hodge structure $E = \bigoplus_{p+q=n} E_0^{p,q}$, the Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes C$ has a Hodge structure of weight zero:

$$\mathfrak{g} = \bigoplus_r \mathfrak{g}^{r,-r} ,$$

where $\mathfrak{g}^{r,-r}$ are those linear transformations taking $E_0^{p,q}$ into $E_0^{p+r,q-r}$. It is visibly the case that

$$\eta_* : s\ell_2(C) \rightarrow \mathfrak{g}$$

is a morphism of Hodge structures whose image lies in $\mathfrak{g}^{0,0} + \mathfrak{g}^{1,-1} + \mathfrak{g}^{-1,1}$.

To give an application of this, we fix the reference frame $e_0 = \frac{1}{\sqrt{2}}(\sqrt{-1}, 1)$, $\bar{e}_0 = \frac{1}{\sqrt{2}}(-\sqrt{-1}, 1)$ in $F(H)$. Setting $e(t) = \frac{1}{\sqrt{2}}(\sqrt{-1} + t, 1)$, $\overline{e(t)} = \frac{1}{\sqrt{2}}(-\sqrt{-1} + t, 1)$, the logarithm L of the monodromy matrix is obviously defined by

$$\begin{pmatrix} e'(t) \\ \overline{e'(t)} \end{pmatrix}_{t=0} = L\begin{pmatrix} e_0 \\ \bar{e}_0 \end{pmatrix}$$

relative to the fixed frame. Explicitly,

$$L = \frac{1}{2}\begin{pmatrix} -\sqrt{-1} & \sqrt{-1} \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix} .$$

In $s\ell_2(C)$, L is conjugate by some element $g \in SL_2(C)$ to the standard nilpotent matrix

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

which lies in $s\ell_2^{1,-1}$. Then

$$N = \eta_*(L) = \eta_*(ge_+g^{-1}) = \eta(g)\eta_*(e_+)\eta(g)^{-1}$$

where $\eta_*(e_+) \in \mathfrak{g}^{-1,1}$ by the previous remarks. This implies the strong monodromy theorem

$$N^{n+1} = 0 .$$

Pursuing the matter further, one may completely analyze any representation

$$\eta_* : s\ell_2(\mathbb{R}) \to \mathfrak{g}_{\mathbb{R}}$$

with the above properties, and this leads to proofs of Schmid's theorems for the SL_2-orbit $\Sigma : H \to D$. If the SL_2 and nilpotent orbits were asymptotic like $y^{\alpha}e^{-y}$ as $y \to \infty$, then the general results on mixed Hodge structures and monodromy weight filtrations would easily follow. However, since the approximation is only of the order y^{-1}, one must go back to the vanishing coefficient relations (i)-(iii) above to carry out the proof. For the details together with further applications we refer to Schmid's paper, listed in the references to lecture 7.

NOTE: The approach to Schmid's theorems using Hodge frames and their structure equations was worked out with J. Carlson. Schmid's proofs are heavily based on Lie theory using Iwasawa decompositions and the like. Although the flavor of this approach is perhaps different, the three essential steps (the lemma on the regular singular points of the equation $h' + ch = 0$, the use of Deligne's lemma to show that $c_0 = 0$, and the coefficient relations (i)-(iii)) are the same.

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SOME DIRECTIONS OF RECENT PROGRESS IN COMMUTATIVE ALGEBRA

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ABSTRACT

Three recently active areas of commutative algebra are discussed, and some results from each are presented. The areas are

1) Projective modules over polynomial rings.
2) The recent work on the existence of Cohen-Macaulay modules, and its relation to the conjectures on the rigidity of Tor, and on multiplicities.
3) Ideals of low codimension; two applications of the structure theorem for perfect ideals of codimension 2.

This article contains the write-ups of three independent talks on areas of commutative algebra which have shown what seems to me striking recent progress. They are also areas which ought to go on developing -- nearly all the main problems are still unsolved.

I have not tried to merge the three talks; each even retains its own references.

I. THE SERRE PROBLEM ON PROJECTIVE MODULES

The problem, posed by Serre in 1954 [4], is: Let k be a field; is every projective $k[x_1,\ldots,x_n]$-module free? Equivalently, is every algebraic vector-bundle on affine n-space over k free?

Progress on this question was smooth, if slow, until the early sixties, thanks to the work of Serre, Seshadri, and Bass. By that time the answer to the question itself was known only for $n \leq 2$ ("Seshadri's theorem" - the answer is "yes" in this case); but there was a wealth of subsidiary information on stable freeness and cancellation, which showed, for instance, that projective $k[x_1,\ldots,x_n]$-modules of rank $\geq n+1$ are free.

Though many people continued to work on Serre's problem, little direct progress was made for the next ten years. Then, in 1973, Murthy-Towber, Swan, Roitman, and

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