

Finite speed of propagation in porous media by mass transportation methods[★]

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Abstract

In this note we make use of mass transportation techniques to give a simple proof of the finite speed of propagation of the solution to the one-dimensional porous medium equation. The result follows by showing that the difference of support of any two solutions corresponding to different compactly supported initial data is a bounded in time function of a suitable Monge–Kantorovich related metric. *To cite this article: J. A. Carrillo, M. P. Gualdani, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Dans cette note nous utilisons des techniques de transport de masse pour donner une preuve élémentaire de la finitude de la vitesse de propagation des solutions de l'équation mono-dimensionnelle des milieux poreux. Le résultat repose sur la preuve de la propriété suivante : la différence du support entre deux solutions quelconques correspondant à des données initiales à support compact différentes est une fonction, bornée en temps, d'une métrique de Monge–Kantorovitch appropriée. *Pour citer cet article : J. A. Carrillo, M. P. Gualdani, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

1. Introduction

We consider the problem

$$u_t = (u^m)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad m > 1, \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

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where $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_0 \geq 0$ and u_0 is compactly supported.

Much is already known for problem (1)-(2): see [1,2,3,4,5] and the references therein for existence, uniqueness and asymptotic behaviour results of the porous media equation. It also known that the degeneracy at level $u = 0$ of the diffusivity $D(u) = mu^{m-1}$ causes the phenomenon called *finite speed of propagation*. This means that the support of the solution $u(\cdot, t)$ to (1)-(2) is a bounded set for all $t \geq 0$. In fact it can be proved that the solution $u(x, t)$ as $t \rightarrow +\infty$ converges to the Barenblatt *source-type* solution $U(x, t, C)$ with the same mass as the initial data.

In this paper we want to give a simple proof of the *finite propagation* property using mass transportation techniques. Precisely, we prove that the difference of support of two solutions of (1)-(2) with different compactly supported initial conditions is a bounded in time function of a suitable Monge-Kantorovich related metric.

Theorem 1.1 *Let $u_1(x, t)$ and $u_2(x, t)$ be strong solutions of (1)-(2) with initial conditions $u_{01}(x)$ and $u_{02}(x)$ respectively, where $u_{0i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_{0i} \geq 0$ and u_{0i} is compactly supported, $i = 1, 2$, and let $\Omega_i = \{(x, t) \in \mathbb{R} \times [0, +\infty) / u_i(x, t) > 0\}$, $i = 1, 2$.*

Let $\xi_i(t) = \inf_{x \in \mathbb{R}} \Omega_i$, $\Xi_i(t) = \sup_{x \in \mathbb{R}} \Omega_i$, for $t \geq 0$, $i = 1, 2$. Then

$$\max \{|\xi_1(t) - \xi_2(t)|, |\Xi_1(t) - \Xi_2(t)|\} \leq W_\infty(u_{01}, u_{02}), \quad \forall t \in [0, +\infty), \quad (3)$$

where $W_\infty(u_{01}, u_{02})$ is a constant, which depends only on the initial data u_{01}, u_{02} and is defined in (16).

The finite speed of propagation property follows by just taking as one of the solutions a time translation of the explicit Barenblatt solution which is known to have compact support expanding at the rate $t^{1/(m+1)}$.

2. Proof

Consider a sequence of functions $u_n \in C^\infty([0, +\infty) \times \mathbb{R})$, which are strong solutions (see [3]) of the problems P_n

$$u_t = (u^m)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad m > 1, \quad (4)$$

$$u(x, 0) = u_{0n}(x), \quad x \in \mathbb{R}, \quad (5)$$

where $u_{0n}(x)$, $n \in \mathbb{N}$, is a sequence of bounded, integrable and strictly positive C^∞ -smooth functions such that all their derivatives are bounded in \mathbb{R} , the condition $(m-1)(u_{0n}^m)_{xx} \geq -au_{0n}$ holds for some constant $a > 0$, and $u_{0n} \rightarrow u_0$ in $L^1(\mathbb{R})$ if $n \rightarrow +\infty$. We may always do it in such a way that $\|u_{0n}\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}$ and $\|u_{0n}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$. From the L^1 -contraction property it follows that $u_n \rightarrow u$ in $C([0, +\infty) : L^1(\mathbb{R}))$ if $n \rightarrow +\infty$, where u is a strong solution of (1)-(2) (see [3], chapt. III).

This sequence of regularized solutions can be further approximated by a sequence of initial boundary value problems. We introduce a cutoff sequence $\theta_k \in C^\infty(\mathbb{R})$, $1 < k \in \mathbb{N}$, with the following properties:

$$\theta_k(x) = 1 \quad \text{for } |x| < k-1, \quad \theta_k(x) = 0 \quad \text{for } |x| \geq k, \quad 0 < \theta_k < 1 \quad \text{for } k-1 < |x| < k. \quad (6)$$

The initial boundary value problem P_{nk}

$$u_t = (u^m)_{xx}, \quad x \in (-k, k), \quad t > 0, \quad (7)$$

$$u(x, 0) = u_{0nk}(x) := \frac{u_{0n}(x)\theta_k(x)}{\|u_{0n}(x)\theta_k(x)\|_{L^1}}, \quad (8)$$

$$u(x, t) = 0 \quad \text{for } |x| = k, \quad t \geq 0, \quad (9)$$

is mass preserving and has a unique solution $u_{nk}(x, t) \in C^\infty((0, +\infty) \times [-k, k]) \cap C([0, +\infty) \times [-k, k])$, strictly positive for $x \in (-k, k)$ and zero at the boundary (see [3], prop.6, chapt.II). Because $u_{0nk} \rightarrow u_{0n}$ as $k \rightarrow +\infty$, for all $n \in \mathbb{N}$, $u_{nk} \rightarrow u_n$ in $C([0, +\infty) : L^1(\mathbb{R}))$ if $k \rightarrow +\infty$, where u_n is solution of the problem P_n .

Thanks to estimates independent of k for the moments of the solutions of the P_{nk} problems and passing to the limit in the corresponding inequalities, it can be easily shown that the solution $u_n(x, t)$ of (4)-(5) enjoys an important property. It holds

$$\int_{\mathbb{R}} |x|^p u_n(x, t) dx < +\infty, \quad \forall t \geq 0, \quad \forall p \in [1, +\infty). \quad (10)$$

We shall denote by $\mathbb{P}_p(\mathbb{R})$, with $p \in [1, +\infty)$, the set of all probability measures on \mathbb{R} with finite moments of order p . Let $\Pi(\mu, \nu)$ be the set of all probability measures on \mathbb{R}^2 having $\mu, \nu \in \mathbb{P}_p(\mathbb{R})$ as marginal distributions (see [6]). The Wasserstein p -distance between two probability measures $\mu, \nu \in \mathbb{P}_p(\mathbb{R})$ is defined as

$$W_p(\mu, \nu)^p := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^2} |x - y|^p d\pi(x, y), \quad \forall p \in [1, +\infty). \quad (11)$$

W_p defines a metric on $\mathbb{P}_p(\mathbb{R})$ (see [6]). Bound (10) then shows that the Wasserstein p -distance between any two solutions which is initially finite, remains finite at any subsequent time.

Any probability measure μ on the real line can be described in terms of its *cumulative distribution function* $F(x) = \mu((-\infty, x])$ which is a right-continuous and non-decreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Then, the *generalized inverse* of F defined by $F^{-1}(\eta) = \inf\{x \in \mathbb{R} / F(x) > \eta\}$ is also a right-continuous and non-decreasing function on $[0, 1]$.

Let $\mu, \nu \in \mathbb{P}_p(\mathbb{R})$ be probability measures and let $F(x), G(x)$ be the respective distribution functions. On the real line (see [6]), the value of the Wasserstein p -distance $W_p(\mu, \nu)$ can be explicitly written in terms of the generalized inverse of the distribution functions,

$$W_p(\mu, \nu)^p = \int_0^1 |F^{-1}(\eta) - G^{-1}(\eta)|^p d\eta, \quad \forall p \in [1, +\infty). \quad (12)$$

Let $u_1(x, t), u_2(x, t)$ be strong solutions of (1)-(2) corresponding to initial conditions $u_{01}(x)$ and $u_{02}(x)$ respectively. We denote by $u_{1n}(x, t)$ and $u_{2n}(x, t)$ the strong solutions of (4)-(5) with initial conditions $u_{01n}(x)$ and $u_{02n}(x)$ respectively, where $u_{0in} \rightarrow u_{0i}$ in $L^1(\mathbb{R})$ for $i = 1, 2$. Analogously, we consider the solutions $u_{1nk}(x, t)$ and $u_{2nk}(x, t)$ of the problems P_{nk} converging towards $u_{in}(x, t)$ for $i = 1, 2$ in $C([0, +\infty) : L^1(\mathbb{R}))$ as $k \rightarrow \infty$.

Let $F_{ink}(x, t)$ be the distribution functions of u_{ink} for $i = 1, 2$. A direct computation shows that $F_{ink}^{-1}(\eta, t)$ solves the following equation

$$\frac{\partial F_{ink}^{-1}}{\partial t} = -\frac{\partial}{\partial \eta} \left(\left(\frac{\partial F_{ink}^{-1}}{\partial \eta} \right)^{-m} \right), \quad i = 1, 2, \quad (13)$$

for $t > 0$ and $\eta \in [0, 1]$. Making use of equation (13), it is easy to prove that the Wasserstein p -distance

$$W_p(u_{1nk}, u_{2nk})(t) = \left\{ \int_0^1 |F_{1nk}^{-1}(\eta, t) - F_{2nk}^{-1}(\eta, t)|^p d\eta \right\}^{\frac{1}{p}}, \quad \forall p \in [1, +\infty), \quad (14)$$

is a non-increasing in time function. In fact, for any given $p \geq 1$, integrating by parts one obtains

$$\begin{aligned} \frac{d}{dt} \int_0^1 |F_{1nk}^{-1}(\eta, t) - F_{2nk}^{-1}(\eta, t)|^p d\eta &= p(p-1) \int_0^1 |F_{1nk}^{-1}(\eta, t) - F_{2nk}^{-1}(\eta, t)|^{p-2} \\ &\times (F_{1nk}^{-1}(\eta, t)_\eta - F_{2nk}^{-1}(\eta, t)_\eta) \left[(F_{1nk}^{-1}(\eta, t)_\eta)^{-m} - (F_{2nk}^{-1}(\eta, t)_\eta)^{-m} \right] d\eta \leq 0 \end{aligned}$$

since the function x^{-m} , $m \geq 1$, is decreasing. Note that the boundary terms vanish due to the compact support of the solutions, which implies

$$\lim_{\eta \rightarrow 0^+} (F_{ink}^{-1}(\eta, t)_\eta)^{-1} = \lim_{\eta \rightarrow 1^-} (F_{ink}^{-1}(\eta, t)_\eta)^{-1} = 0 \quad i = 1, 2.$$

On the other hand, for all $p \in [1, +\infty)$,

$$W_p(u_{1nk}, u_{2nk}) \rightarrow W_p(u_{1n}, u_{2n}), \quad k \rightarrow +\infty, \quad W_p(u_{1n}, u_{2n}) \rightarrow W_p(u_1, u_2), \quad n \rightarrow +\infty. \quad (15)$$

This implies that $W_p(u_1, u_2) \leq W_p(u_{01}, u_{02})$, $\forall p \in [1, +\infty)$. Since the function $W_p(u_1, u_2)$ is increasing with respect to p , we can define the quantity

$$W_\infty(u_1, u_2) := \lim_{p \uparrow +\infty} W_p(u_1, u_2) = \sup_{\eta \in (0,1)} \text{ess} |F_1^{-1}(\eta, t) - F_2^{-1}(\eta, t)|. \quad (16)$$

Since $W_\infty(u_{01}, u_{02})$ is finite, we deduce easily that $W_\infty(u_1, u_2)$ is also a non-increasing in time function. Note that the inverse function $F^{-1}(\eta)$ of a distribution $F(x) = \int_{-\infty}^x u(s)ds$, where $u(s)$ is a integrable compactly supported function, is continuous at the point $\eta = 0$ and $\eta = 1$. Thus we can justify the inequality

$$\begin{aligned} W_\infty(u_1, u_2) &= \sup_{\eta \in (0,1)} \text{ess} |F_1^{-1}(\eta, t) - F_2^{-1}(\eta, t)| \geq \\ &\max \{ |F_1^{-1}(0, t) - F_2^{-1}(0, t)|, |F_1^{-1}(1, t) - F_2^{-1}(1, t)| \} \geq \max \{ |\xi_1(t) - \xi_2(t)|, |\Xi_1(t) - \Xi_2(t)| \}. \end{aligned} \quad (17)$$

We remark that the above arguments only hold in one space dimension due to the fact that only in this case one can express the p -Wasserstein distance in terms of pseudo-inverse distribution functions, as given in (12).

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