Propagation of Sobolev regularity for a class of random kinetic models on the real line

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Abstract. A class of Kac-like kinetic equations on the real line is considered, with general smoothing transforms as collisional kernels. These equations have been introduced recently e.g. in the context of econophysics [8] or as models for granular gases with a background heat bath [6]. We show that the stationary solutions to these equations are not smooth in general, and we characterize their (finite) Sobolev regularity in dependence of the properties of the collisional kernel. Moreover, we prove that any initial Sobolev regularity below a well-defined threshold is uniformly propagated in time by the transient weak solutions, implying their strong convergence to the steady state. The applied techniques differ from the classical ones developed for the Kac equation as the models at hand do neither dissipate the entropy nor the Fisher information. Instead, the proof relies on direct estimates on the collisional operator.

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1. Introduction

1.1. The model

In this article, we are concerned with the regularity and strong convergence of solutions to the spatially homogeneous kinetic equation

$$\partial_t f = Q_+(f,f) - f, \quad f(0) = f_0, \quad t \geq 0, \tag{1}$$

which caricatures the Boltzmann equation in one spatial dimension. The solution $f = f(t;v)$ is a time-dependent probability density on $\mathbb{R}$, describing the distribution of particle velocities v in a homogeneous gas. The gain operator Q_+ models velocity changes due to binary particle collisions. Our fundamental assumption is that the frequency of collisions is constant, and that the exchange of momentum is linear. The latter means that the post-collisional particle velocities v' and v'_* are related to the pre-collisional velocities v and v_* by

$$v' = pv + qv_*, \quad v'_* = p_*v_* + q_*v, \tag{2}$$

where (p,q) and (p_*,q_*) are i.i.d. random vectors of prescribed distribution $\mathbb{P}$ on $\mathbb{R}^2_+$. The gain operator in (1) thus takes the convolution form

$$\begin{aligned} Q_+(f,f)(v) &= \big\langle p^{-1}f\big(p^{-1}\,\cdot\,\big) \star q^{-1}f\big(q^{-1}\,\cdot\,\big)\big\rangle(v) \\ &= \int_{\mathbb{R}^2_+} \left(\frac{1}{pq}\int_{\mathbb{R}} f\Big(\frac{v-w}{p}\Big) f\Big(\frac{w}{q}\Big)\,dw\right) d\mathbb{P}(p,q). \end{aligned} \tag{3}$$

In (3), the expectation value $\langle\cdot\rangle$ with respect to $\mathbb{P}$ replaces — morally — the integral with respect to the scattering angle in the classical Boltzmann equation.

The first model of the type (1)&(3) has been introduced by Kac [15], with the collisional parameters $p = \sin\theta$ and $q = \cos\theta$ for a uniformly distributed random angle $\theta \in [0, 2\pi]$. The dynamics describes a gas in which the colliding molecules exchange a random fraction of their kinetic energies. This idea has been extended to gases with inelastically colliding molecules, which in addition lose a random part of their energy in each interaction, by Pulvirenti and the second author [19]. The inelastic Kac equation corresponds to (1)&(3) with $p = |\sin\theta|^\kappa \sin\theta$ and $q = |\cos\theta|^\kappa\cos\theta$, with $\kappa > 0$ being the parameter of in-elasticity. Recently, more general versions of (1)&(3) have been considered: their applications range from gases under the influence of a background heat bath [6] to models for the redistribution of wealth in simple market economies [8, 18].

Here, we consider the class of models which satisfies the following additional hypothesis on $\mathbb{P}$,

$$\langle p^\alpha + q^\alpha\rangle = 1, \tag{4}$$

with either $\alpha = 2$, referred to as *energy conserving case* in the following, or $\alpha = 1$, referred to as the *momentum conserving case*. These notions reflect that fact that the second moment (total energy) or the first moment (total linear momentum), respectively, of the solution $f(t)$ are preserved in time, see e.g. [1]. We remark that the Kac equation falls into the energy conserving case since $\sin^2\theta + \cos^2\theta = 1$. Examples for momentum conserving equations are provided by the wealth distribution models from [18].

In a recent article by Bassetti, Ladelli and the first author [1], the weak long-time convergence of solutions $f(t)$ to (1)&(3) under the hypothesis (4) has been proven under very general assumptions. The relevant results are reviewed in Theorem 2.1 below. Here, we show that this weak convergence is actually strong, under some mild additional hypotheses on the distribution $\mathbb{P}$ of p and q.

Our results are stated for solutions of the Boltzmann equation (1). We just mention that they could equivalently be rephrased in more probabilistic terms: the operator Q_+ defined in (3) and satisfying (4) constitutes a so-called *smoothing transform*, which is related to branched random walks, see e.g. [11, 17]. Solutions to (1)&(3) are directly related to iterations of the transformation Q_+. In particular, the steady states for (1), whose regularity is described in Theorem 1.1 below, are fixed points of Q_+.

1.2. Statement of the main result

In addition to (4), we introduce the following hypotheses on the probability measure $\mathbb{P}$.

(H1) For some $\beta > \alpha$, one has $\langle p^\beta + q^\beta \rangle < 1$, and it is *not* true that $p + q = 1$ a.s.

(H2) For some $\vartheta > 0$, one has $\langle p^{-\vartheta} + q^{-\vartheta}\rangle < +\infty$, and moreover

$$\Theta := \sup\left\{\theta \ge 0 \,\middle|\, \langle \max(p,q)^{-\theta}\rangle < \infty\right\} > \frac{1}{2}. \tag{5}$$

The role of hypothesis (H1) is to guarantee the existence of a stationary state f_∞ for (1)&(3) that is *not* concentrated in a point. The quantity Θ defined in hypothesis (H2) determines (a lower bound on) the Sobolev-regularity of f_∞. Our main result reads as follows:

Theorem 1.1. *Assume* (4) *with* $\alpha = 1$ *or* $\alpha = 2$, *and let hypotheses (H1) and (H2) be satisfied. Then* (1) *possesses a (unique up to rescaling) steady state* $f_\infty \in H^{(\Theta-1/2)-}(\mathbb{R})$.

Let an initial condition f_0 *be given, which is supported on* $\overline{\mathbb{R}_+}$ *and has unit first moment if* $\alpha = 1$, *or is centered with unit second moment if* $\alpha = 2$. *Assume further that*‡ $f_0 \in H^m(\mathbb{R})$ *with* $0 \le m < \Theta - 1/2$, *and additionally either* $f_0 \in H^2(\mathbb{R})$, *or* $\sqrt{f_0} \in H^s(\mathbb{R})$ *for some* $s > 0$.

Then the transient solution $f(t)$ *to* (1) *remains bounded in* $H^m(\mathbb{R})$, *uniformly in* $t \ge 0$, *and converges strongly in* $H^{m-}(\mathbb{R})$ *to the stationary solution* f_∞ *as* $t \to \infty$.

We emphasize that we admit very general initial conditions in view of moments requirements: we only assume that f_0 has finite first or second moment, respectively, if $\alpha = 1$ or $\alpha = 2$. This is the minimal hypothesis under which convergence of the transient solution to a steady state can be expected.

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‡ The definition of the Sobolev spaces $H^m(\mathbb{R})$ and $H^{m-}(\mathbb{R})$ is reviewed in Section 2.3 below.

A further comment is in place about the assumption $\sqrt{f_0} \in H^s(\mathbb{R})$: it is clearly sufficient that the Fisher information, defined by

$$\mathcal{F}_*[f] = \int_{\mathbb{R}} \bigl(\sqrt{f(v)}\bigr)_v^2 \, dv, \tag{6}$$

of f_0 is finite. However, we require much less than finite Fisher information, namely only an arbitrarily small amount of Sobolev-regularity for $\sqrt{f_0}$. The advantage of our weaker requirement is illustrated by the profile $f_0(v) = |v|e^{-v^2} \in H^1(\mathbb{R})$, which does *not* have finite Fisher information, whereas $\sqrt{f_0} \in H^s(\mathbb{R})$ for any $s < 1$.

Finally, we stress that the condition $m < \Theta - 1/2$ in Theorem 1.1 is *sharp* in the sense that we can provide examples — both for the energy and in the momentum conservative case — for which $f_\infty \notin H^{\Theta-1/2}(\mathbb{R})$; see Section 6.3. On the other hand, for a variety of models, like the Kac equation, one finds $\Theta = +\infty$, implying propagation of regularity in arbitrary Sobolev spaces, and hence also $f_\infty \in C^\infty(\mathbb{R})$.

1.3. Related results from the literature

An extensive amount of literature is available on the smoothness of solutions to kinetic equations. Here we mention a small selection of results on equations similar to (1)&(3); for further information, we refer the reader to the references in the quoted papers.

In [5], uniform propagation of Sobolev-regularity and strong convergence (at nearly optimal rate) have been shown for the Kac equation as a by-product of a general investigation on multi-dimensional Maxwell molecules. An analogue of Theorem 1.1 has been proven, without hypotheses on the regularity of $\sqrt{f_0}$, but under additional assumptions on the moments of f_0. The proof uses several particular properties of the Kac equation, most notably that the entropy is a Lyapunov functional — which is in general not true in the situation considered here.

A somewhat different approach to the proof of strong convergence in the Kac equation has been taken in [4]. Instead of proving the propagation of initial Sobolev-regularity to later times, the McKean construction has been used to decompose the transient solution into a smooth part and a non-smooth remainder, which becomes negligible in $L^1(\mathbb{R})$ as $t \to \infty$. The proof heavily uses that the initial condition has finite Fisher information (6), and that the latter is a Lyapunov functional for the Kac equation. Developing these ideas further, and employing heavy machinery from probability theory, a proof of strong convergence in the Kac equation at the truly optimal rate was achieved in [10]. Existence of the fourth moment of f_0 is required here, as well as a (rather opaque) regularity hypothesis.

Further, there are several recent results on the self-similarity asymptotics for the Boltzmann equation for inelastic Maxwell molecules. Although this class of models does not fit into the framework (1)&(3), the obtained results and applied techniques are in the same spirit. In [12], strong convergence to the homogeneous cooling state (which is Gevrey regular) for rescaled solutions to the one-dimensional inelastic Maxwell model has been shown. Finiteness of some moment above the second is assumed initially. The developed techniques extend to multiple dimensions, see [13]. A different approach has been taken in [3], which is based on a balance between the exponential-in-time growth of the Fisher information and the exponential convergence to equilibrium in weak metrics.

Finally, a result on strong convergence for (1)&(3) in the energy conserving case has already been given in [1]. Finiteness of f_0's Fisher information was required, and instead of (H2), the stricter assumption $p^r + q^r > 1$ a.s. for some $r > 0$ has been made. The latter induces that f_∞ is smooth.

To our knowledge, the current result is the first which provides propagation of regularity for kinetic equations of type (1)&(3) with non-smooth steady states, and under minimal assumptions on the moments of the initial condition.

1.4. Plan of the proof

In its core, the proof of Theorem 1.1 follows the by now classical strategy developed in [5]. Equation (1)&(3) is rewritten in terms of the Fourier transform $\widehat{f}$, and a pointwise bound on the *high-frequency* tail is proven,

$$\big|\widehat{f}(t;\xi)\big| \leq M_0|\xi|^{-\gamma} \qquad (\xi \in \mathbb{R} \setminus \{0\}), \tag{7}$$

with positive constants M_0 and γ that do not depend on time; see Lemma 4.3. Once the very weak regularity estimate (7) on $f(t)$ is established, it follows that also the $H^m(\mathbb{R})$-regularity of $f(t)$ is uniformly propagated in time.

The proof of (7) relies on direct estimates on the (Fourier transformed) gain operator $\widehat{Q}_+$, which show that *if* (7) holds at time $T \geq 0$, *then* it continues to hold at all later times $t \geq T$. This part is quite technical; partly, we follow a procedure that has originally been developed to show propagation of Gevrey regularity by the Kac equation [9], and has since then been modified and adapted to various other applications, see e.g. [12, 18]. The difficulty in the situation at hand is that very little is known about the behavior of the coefficients p and q, and all estimates need to be based entirely on property (4), and on hypotheses (H1) and (H2).

To conclude propagation of (7) from $t = T$ to later times $t \geq T$, we need in addition an a priori estimate on the *low frequencies*,

$$\big|\,\widehat{f}(t;\xi)\big| \leq 1 - \kappa|\xi|^2 \quad \text{for all} \quad |\xi| < \rho, \tag{8}$$

with time-independent positive constants κ and ρ. In Lemma 4.1, it is proven that (8) indeed holds for sufficiently large times $t \geq T$ since then $f(t)$ lies “close enough” (in a weak sense) to the stationary state f_∞. However, with our minimal assumptions on the moments of the initial condition f_0, estimate (8) might simply not extend with uniform constants to $0 < t < T$. Hence, a separate proof is needed to show that the $H^m(\mathbb{R})$-norm of $f(t)$ does not blow up for $t < T$, and that (7) can be established at $t = T$. (This issue makes our proof considerably more difficult than that of the corresponding result in [12].)

Finite-time boundedness of $f(t)$ in $H^m(\mathbb{R})$ is rather straight-forwardly to obtain; see Lemma 3.1. In order to conclude (7) at $t = T$, it suffices to have $\sqrt{f(T)} \in H^m(\mathbb{R})$, see Corollary 3.2. This, in turn, is proven by deriving an exponentially growing bound on the generalized Fisher information $\mathcal{F}_s[f(t)]$ in Corollary 3.1. The functional $\mathcal{F}_s$ is defined by

$$\mathcal{F}_s[f] := \iint_{\mathbb{R}^2} \Big(w^{-(1+2s)} \big| \sqrt{f(v+w)} - \sqrt{f(v)} \big| \Big)\, dv\, dw \tag{9}$$

and is equivalent to the $H^s(\mathbb{R})$-norm of $\sqrt{f(t)}$. Our proof of the boundedness of $\mathcal{F}_s[f(t)]$ is of some interest in itself: we show in Proposition 3.1 that $\mathcal{F}_s$ satisfies the estimate (22), which is similar to (although weaker than) the *Blachman-Stam-inequality* [2],

$$\mathcal{F}_*[f \star g] \le \Lambda^2 \mathcal{F}_*[f] + (1-\Lambda^2)\mathcal{F}_*[g].$$

Inequality (22) allows to estimate Q_+ in its convolution form.

1.5. *Further results*

In Section 6, we collect some auxiliary results related to Theorem 1.1. In Corollary 6.1 we show that $f(t)$ converges strongly to f_∞ in various Sobolev norms *at an exponential rate*, provided that the initial condition f_0 possesses a finite moment above the α's. Corollary 6.2 states that in the energy conserving case $\alpha = 2$, any non-smoothness of f_∞ must be located at the origin. Finally, the examples in Section 6.3 demonstrate that the conclusion $f_\infty \in H^{(\Theta-1/2)-}(\mathbb{R})$ from Theorem 1.1 cannot be improved without further restrictions on $\mathbb{P}$.

2. Preliminaries and Hypotheses

2.1. *Weak solutions and weak convergence*

Define the set $\mathcal{P}$ of probability densities on $\mathbb{R}$, equipped with the weak-$*$ topology. Denote by $\mathcal{P}_2$ the subset consisting of densities with unit linear momentum, and by $\mathcal{P}_2$ the one with centered densities of unit second moment,

$$\begin{aligned}
\mathcal{P}_1 &= \Big\{ f \in L^1(\mathbb{R}) \,\Big|\, f \ge 0,\ f(v) = 0 \text{ for } v \le 0, \text{ and } \textstyle\int_{\mathbb{R}} f(v)dv = \int_{\mathbb{R}} v\, f(v)dv = 1 \Big\}, \\
\mathcal{P}_2 &= \Big\{ f \in L^1(\mathbb{R}) \,\Big|\, f \ge 0,\ \textstyle\int_{\mathbb{R}} f(v)dv = \int_{\mathbb{R}} v^2\, f(v)dv = 1 \text{ and } \int_{\mathbb{R}} v\, f(v)dv = 0 \Big\}.
\end{aligned}$$

Notice that the hypotheses of Theorem 1.1 require that $f_0 \in \mathcal{P}_\alpha$.

We shall use various representations of the Boltzmann equation (1) and its solutions. The *weak formulation* asserts that

$$\frac{d}{dt} \int_{\mathbb{R}} \phi(v)\, f(v)dv + \int_{\mathbb{R}} \phi(v)\, f(v)dv = \iint_{\mathbb{R}^2} \big\langle \phi(pv + qv_*) \big\rangle\, f(v)dv\, f(v_*)dv_* \tag{10}$$

holds for all bounded and continuous test functions $\phi \in C^0_b(\mathbb{R})$. Equivalently, $f(t)$'s Fourier transform

$$\widehat{f}(t;\xi) = \int_{\mathbb{R}} e^{iv\xi} f(t;v)\,dv$$

satisfies the equation

$$\partial_t \widehat{f}(\xi) + \widehat{f}(\xi) = \widehat{Q}_+(f,f)(\xi) = \big\langle \widehat{f}(p\xi)\widehat{f}(q\xi)\big\rangle. \tag{11}$$

Moreover, from the weak formulation (10), the *mild formulation*

$$f(t) = e^{-t}\Big(f_0 + \int_0^t e^{\tau} Q_+\big(f(\tau), f(\tau)\big)d\tau\Big) \tag{12}$$

is obtained by the "variation of constant" formula. The following theorem summarizes some basic facts on weak solutions in the sense (10).

Theorem 2.1. *Assume* (4) *with* $\alpha = 1$ *or* $\alpha = 2$, *and that hypothesis (H1) holds. Let an initial condition* $f_0 \in \mathcal{P}_\alpha$ *be given. Then there exists a unique and global-in-time weak solution* $f(t) \in \mathcal{P}_\alpha$ *to* (1)*&*(3)*. As* $t \to \infty$, *this solution converges weakly-*$\star$ *to the unique stationary solution* $f_\infty \in \mathcal{P}_\alpha$.

For a proof, see [1] or the references therein.

2.2. An tailored version of the Gronwall lemma

The Gronwall inequality plays a key role in our proofs. For the sake of completeness, we state and prove the particular version that is tailored to our needs.

Lemma 2.1. *Let a weak solution* $f : [0,+\infty) \to \mathcal{P}$ *to the Boltzmann equation* (1) *be given, which is such that* $f(t)$ *belongs to some closed convex set* $K \subset \mathcal{P}$ *for all* $t \geq T$. *Further, let* $\Psi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ *be a convex function, and assume that the collisional gain operator* Q_+ *can be estimated (uniformly in* $f \in K$*) as follows,*

$$\Psi\big(Q_+(f,f)\big) \leq \lambda\Psi(f) + M, \tag{13}$$

where either $0 \leq \lambda < 1$ *and* $M \geq 0$, or $\lambda \geq 1$ *and* $M = 0$. *Then, the solution* f *is uniformly bounded for all* $t \geq T$:

$$\Psi\big(f(t)\big) \leq \Psi\big(f(T)\big) + \frac{M}{1-\lambda}. \tag{14}$$

In particular, (13) *with* $\lambda = 1$ *and* $M = 0$ *implies* $\Psi(f(t)) \leq \Psi(f(T))$.

Proof. Without loss of generality, we may assume that $T = 0$, and replace $f(T)$ by f_0. In fact, observe that if f satisfies (12), then the time-translated solution $\tilde{f}$ defined by $\tilde{f}(t) = f(t+T)$ satisfies (12) as well, with $\tilde{f}_0 = f(T)$.

Moreover, we specialize on the situation $0 \le \lambda < 1$; the other — similar but simpler — case is left to the reader. The right-hand side of (12) constitutes a convex combination of probability measures in $\mathcal{P}$, hence the convexity of Ψ implies

$$\Psi\big(f(t)\big) \le e^{-t}\Big[\Psi\big(f_0\big) + \int_0^t e^{\tau}\Psi\big(Q_+(f(\tau),f(\tau))\big)\,d\tau\Big].$$

Multiply both sides by e^t and use assumption (13) to conclude

$$e^t\Psi\big(f(t)\big) \le \Psi\big(f_0\big) + \int_0^t e^{\tau}\big[\lambda\Psi\big(f(\tau)\big) + M\big]\,d\tau. \tag{15}$$

Call the integral expression $G(t)$. Then, using (15),

$$G'(t) = e^t\big[\lambda\Psi\big(f(t)\big) + M\big] \le \lambda\Psi\big(f_0\big) + \lambda J(t) + e^t M. \tag{16}$$

Now it is sufficient to observe that the function

$$\overline{G}(t) = \Psi(f_0)(e^{\lambda t} - 1) + \frac{M}{1-\lambda}(e^t - e^{\lambda t}).$$

achieves equality in (16), and also satisfies $0 = G(0) = \overline{G}(0)$. Standard ODE theory implies $G(t) \le \overline{G}(t)$ for all $t \ge 0$. Insert this into (15) above to conclude (14). □

2.3. Sobolev norms

Finally, we need to fix notations for the involved Sobolev spaces and their norms. The spaces $H^m(\mathbb{R})$ with $m \ge 0$ consist of those functions $f \in H^0(\mathbb{R}) := L^2(\mathbb{R})$ which satisfy

$$\|f\|_{H^m} := \Big(\int_{\mathbb{R}} (1+\xi^2)^m |\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2} < \infty. \tag{17}$$

Note that $\|f\|_{H^k} \le \|f\|_{H^m}$ and hence $H^m(\mathbb{R}) \subset H^k(\mathbb{R})$ whenever $0 \le k \le m$. For integer $m \ge 1$, the space $H^m(\mathbb{R})$ is naturally isomorphic to the standard Sobolev space $W^{m,2}(\mathbb{R})$. Further, introduce for $m > 0$ the locally convex spaces

$$H^{m-}(\mathbb{R}) := \bigcap_{0<k<m} H^k(\mathbb{R})$$

whose topology is induced by the collection of norms $\|\cdot\|_{H^k}$ with $0 < k < m$. Analogously, introduce $H^\infty(\mathbb{R}) = \bigcap_{k>0} H^k(\mathbb{R}) \subset C^\infty(\mathbb{R})$.

Moreover, a natural semi-norm on each $H^m(\mathbb{R})$ is given by

$$\|f\|_m = \Big(\int_{\mathbb{R}} |\xi|^{2m} |\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2}. \tag{18}$$

On the intersection $H^m(\mathbb{R}) \cap \mathcal{P}$, this semi-norm is actually equivalent to the full $H^m(\mathbb{R})$-norm. Indeed, $f \in \mathcal{P}$ implies that $|\widehat{f}(\xi)| \le 1$ for all $\xi \in \mathbb{R}$, since obviously $(1+\xi^2)^m \le 2^{m-1}(1+\xi^{2m})$, it follows that

$$\begin{aligned}
\|f\|_{H^m} &\le 2^{(m-1)/2}\big(\|f\|_{H^0} + \|f\|_m\big)\\
&\le 2^{(m-1)/2}\Big(\Big(\int_{|\xi|\le 1} |\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2} + \Big(\int_{|\xi|\ge 1} |\xi|^{2m}|\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2} + \|f\|_m\Big)\\
&\le 2^{(m+1)/2}\big(1 + \|f\|_m\big).
\end{aligned} \tag{19}$$

3. Exponentially growing bounds

The aim of this section is to obtain an exponentially growing upper bound on the smoothness of the solution. We shall estimate both the $H^m(\mathbb{R})$-semi-norm of $f(t)$ and the $H^s(\mathbb{R})$-semi-norm of $\sqrt{f(t)}$. The obtained estimates are very rough, but their only purpose is to guarantee non-explosion of these norms in finite time.

Lemma 3.1. *Under the hypotheses of Theorem 1.1,*

$$\|f(t)\|_m \leq \|f_0\|_m \exp\big(\langle \max(p,q)^{-(m+1/2)}\rangle t\big) \tag{20}$$

for all $t \geq 0$*, which is finite by hypothesis (H2).*

Proof. We wish to apply the Gronwall Lemma 2.1 with the semi-norm $\Psi(f) := \|f\|_m$, see (18), which is clearly a convex functional on $K := \mathcal{P}$. By the Fourier representation of the gain operator Q_+ in (11), and since $|\widehat{f}(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$, we obtain

$$\begin{aligned}
\|Q_+(f,f)\|_m^2 &\leq \int_{\mathbb{R}} |\xi|^{2m} \bigg[\int_{\mathbb{R}_+^2} |\widehat{f}(p\xi)||\widehat{f}(q\xi)|\, d\mathbb{P}(p,q) \bigg]^2 d\xi \\
&= \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} |\xi|^{2m} \big(|\widehat{f}(p\xi)||\widehat{f}(q\xi)|\big) \big(|\widehat{f}(p_*\xi)|\ |\widehat{f}(q_*\xi)|\big)\, d\xi\, d\mathbb{P}(p,q)\, d\mathbb{P}(p_*,q_*) \\
&\leq \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} \big(|\sigma\xi|^m |\widehat{f}(\sigma\xi)| \sigma^{1/2}\big) \big(|\sigma\xi|^m |\widehat{f}(\sigma_*\xi)| \sigma_*^{1/2}\big)\, d\xi\, \frac{d\mathbb{P}(p,q)}{\sigma^{m+1/2}}\, \frac{d\mathbb{P}(p_*,q_*)}{\sigma_*^{m+1/2}}.
\end{aligned} \tag{21}$$

Above, we have denoted $\sigma = \max(p,q)$ and $\sigma_* = \max(p_*,q_*)$. Use Hölder's inequality and substitute $\eta = \sigma\xi$ or $\eta = \sigma_*\xi$, respectively, under the integral to conclude

$$\begin{aligned}
&\int_{\mathbb{R}} \big(|\sigma\xi|^m |\widehat{f}(\sigma\xi)| \sigma^{1/2}\big) \big(|\sigma_*\xi|^m |\widehat{f}(\sigma_*\xi)| \sigma_*^{1/2}\big)\, d\xi \\
&\qquad \leq \bigg(\int_{\mathbb{R}} |\sigma\xi|^{2m} |\widehat{f}(\sigma\xi)|^2\, \sigma\, d\xi \bigg)^{1/2} \bigg(\int_{\mathbb{R}} |\sigma_*\xi|^{2m} |\widehat{f}(\sigma_*\xi)|^2\, \sigma_*\, d\xi \bigg)^{1/2} \\
&\qquad = \int_{\mathbb{R}} |\eta|^{2m} |\widehat{f}(\eta)|^2\, d\eta = \Psi(f)^2.
\end{aligned}$$

Evaluating the remaining integrals in (21) and taking the square root, it follows that

$$\Psi\big(Q_+(f,f)\big) \leq \langle \max(p,q)^{-(m+1/2)} \rangle \Psi(f).$$

The corresponding Gronwall estimate (14) yields (20). □

The regularity estimates on the square root $\sqrt{f}$ are more complicated to obtain. We recall that the usual Kac model (and Maxwell molecules etc.) dissipates the Fisher information, so that the $H^1(\mathbb{R})$-norm of $\sqrt{f}$ is non-increasing in time. No property like that seems to be available for the general class of models considered here. On the other hand, we show in Corollary 3.1 below that exponential-in-time growing upper bounds are available for all $H^s(\mathbb{R})$-norms of $\sqrt{f}$ with sufficiently small $s > 0$.

Remark 3.1. *In [3], the authors are able to prove strong long-time convergence for the rescaled velocity distribution of inelastic Maxwell molecules by compensating the exponential-in-time growth of the Fisher information by a sufficiently rapid exponential convergence of $f(t)$ to the homogeneous cooling state in a weak metric. Unfortunately, this idea apparently does not carry over to the situation at hand: the exponential rate for explosion obtained in the proof of Corollary 3.1 below always exceeds one, even after possible optimizations, and thus it is larger than the best known exponential convergence rate in Fourier metrics [18] or Wasserstein distance [1].*

Instead of working with the norms $\|\sqrt{f(t)}\|_s$ directly, we prefer to use the equivalent functionals $\mathcal{F}_s$ introduced in (9). The latter representation is more convenient for subsequent proofs.

Lemma 3.2. *For each $s \in (0,1)$, there exists a constant $N_s > 0$ such that $\mathcal{F}_s[f] = 2N_s\|\sqrt{f(t)}\|_s^2$ for all $f \in \mathcal{P}$.*

A particular consequence of this equality is the convexity of the functionals $\mathcal{F}_s$ for $0 < s < 1$.

Proof. Let $g := \sqrt{f} \in L^2(\mathbb{R})$ for short, and define for $w \in \mathbb{R}$ its translate $\mathrm{T}_w g$ by $(\mathrm{T}_w g)(v) = g(v+w)$ for all $v \in \mathbb{R}$. Invoking Plancherel's identity,

$$\begin{aligned}
\mathcal{F}_s[f] &= 2\int_0^\infty w^{-(1+2s)}\left(\int_{\mathbb{R}} |\mathrm{T}_w g(v) - g(v)|^2\,dv\right)dw \\
&= 2\int_0^\infty w^{-(1+2s)}\left(\int_{\mathbb{R}} \left|\widehat{\mathrm{T}_w g}(\zeta) - \widehat{g}(\zeta)\right|^2 d\zeta\right)dw \\
&= 2\int_{\mathbb{R}} |\widehat{g}(\zeta)|^2 \left(\int_0^\infty w^{-(1+2s)}\left|e^{-iw\zeta} - 1\right|^2 dw\right)d\zeta.
\end{aligned}$$

Here we have employed the classical relation $\widehat{\mathrm{T}_w g}(\zeta) = e^{-iw\zeta}\widehat{g}(\zeta)$. Now observe that a change of variables $w = z|\zeta|$ yields

$$\int_0^\infty w^{-(1+2s)}\left|e^{-iw\zeta} - 1\right|^2 dw = N_s|\zeta|^{2s}, \quad \text{where} \quad N_s := \int_0^\infty z^{-(1+2s)}\left|e^{-iz} - 1\right|^2 dz.$$

N_s is finite for $0 < s < 1$, since $|e^{-iz} - 1|^2 \leq \min(1, z^2)$ for $z \geq 0$; the claim follows. □

The crucial property of $\mathcal{F}_s$ that enters into the proof of the bound on its exponential growth is the following weak analogue of the Blachman-Stam-inequality.

Proposition 3.1. *For any $0 < s < 1$ and every $\Lambda \in [0,1]$,*

$$\mathcal{F}_s[f \star g] \leq 2\Lambda\mathcal{F}_s[f] + 2(1-\Lambda)\mathcal{F}_s[g] \tag{22}$$

for all densities $f, g \in \mathcal{P}$.

Proof. Let $f,g\in\mathcal{P}$ be given and introduce $h=f\star g$. First note that from the elementary inequality $x^2+y^2\le(x+y)^2$ for real numbers $x,y\ge 0$, it follows

$$\begin{aligned}\mathcal{F}_s[h]&=\iint_{\mathbb{R}^2}|w|^{-(1+2s)}\frac{|h(v+w)-h(v)|^2}{|\sqrt{h(v+w)}+\sqrt{h(v)}|^2}dv\,dw\\&\le\iint_{\mathbb{R}^2}|w|^{-(1+2s)}\frac{|h(v+w)-h(v)|^2}{h(v+w)+h(v)}dv\,dw.\end{aligned}\tag{23}$$

The quotient inside the last integral is now estimated pointwise. Based on the identity

$$h(v+w)\pm h(v)=\int_{\mathbb{R}}\big(f(w+z)\pm f(z)\big)g(v-z)\,dz$$

and the elementary inequality $(x+y)^2\le 2(x^2+y^2)$, we obtain that

$$\begin{aligned}&\big(h(v+w)-h(v)\big)^2=\Big(\int_{\mathbb{R}}\big(f(z+w)-f(z)\big)g(v-z)\,dz\Big)^2\\&\le\int_{\mathbb{R}}\big(f(z+w)+f(z)\big)g(v-z)\,dz\int_{\mathbb{R}}\frac{\big(f(z+w)-f(z)\big)^2}{f(z+w)+f(z)}g(v-z)\,dy\\&\le 2\big(h(v+w)+h(v)\big)\int_{\mathbb{R}}\frac{\big(f(z+w)-f(z)\big)^2}{\big(\sqrt{f(z+w)}+\sqrt{f(z)}\big)^2}g(v-z)\,dz\\&=2\big(h(v+w)+h(v)\big)\int_{\mathbb{R}}\big(\sqrt{f(w+z)}-\sqrt{f(z)}\big)^2g(v-z)\,dz.\end{aligned}$$

Hence, taking into account that g is a probability density,

$$\int_{\mathbb{R}}\frac{\big(h(v+w)-h(v)\big)^2}{h(v+w)+h(v)}\,dv\le 2\int_{\mathbb{R}}\big(\sqrt{f(z+w)}-\sqrt{f(z)}\big)^2\,dz.\tag{24}$$

In a completely analogous manner, one derives from

$$h(v+w)\pm h(v)=\int_{\mathbb{R}}f(v-z)\big(g(z+w)\pm g(z)\big)\,dz$$

the inequality

$$\int_{\mathbb{R}}\frac{\big(h(v+w)-h(v)\big)^2}{h(v+w)+h(w)}\,dv\le 2\int_{\mathbb{R}}\big(\sqrt{g(z+w)}-\sqrt{g(z)}\big)^2\,dz.\tag{25}$$

Add Λ times inequality (24) and $1-\Lambda$ times (25), and substitute them into (23). This yields the claim (22). □

Proposition 3.1 enables us to estimate the collisional gain operator Q_+ in its convolution representation (3). The induced Gronwall estimate yields the exponentially growing upper bound of $\mathcal{F}_s[f(t)]$.

Corollary 3.1. *Under the hypotheses of Theorem 1.1, and provided that* $0<s<\min(1,\Theta)/2$, *it follows that* $\mathcal{F}_s[f_0]<\infty$, *and that the weak solution* $f(t)$ *satisfies*

$$\mathcal{F}_s\big[f(t)\big]\le\mathcal{F}_s\big[f_0\big]\exp\big((2K_s-1)t\big),\qquad K_s=\big\langle\max(p,q)^{-2s}\big\rangle\tag{26}$$

at any time $t\ge 0$.

Proof. If $\sqrt{f_0} \in H^s(\mathbb{R})$, then clearly $\mathcal{F}_s[f_0] < \infty$ by Lemma 3.2. Next, we show that also $f_0 \in \mathcal{P} \cap H^2(\mathbb{R})$ is sufficient for $\mathcal{F}_s[f_0] < \infty$. As an intermediate step, we prove that $f_{0,v} \in L^1(\mathbb{R})$; notice that $f_0 \in H^2(\mathbb{R})$ only provides *local* integrability of $f_{0,v}$. To conclude *global* integrability, first observe that

$$\int_{\mathbb{R}} f_0(v)^{2/3}\, dv \leq \left(\int_{\mathbb{R}} \big(1 + |v|\ \big) f_0(v)\, dv \right)^{2/3} \left(\int_{\mathbb{R}} \big(1 + |v|\big)^{-2} dv \right)^{1/3} \leq 2,$$

since the first moment of $f_0 \in \mathcal{P}_\alpha$ is less than (if $\alpha = 2$) or equal to (if $\alpha = 1$) unity. Thus, applying Hölder's inequality again,

$$\int_{\mathbb{R}} |f_{0,v}|\, dv \leq \left(\int_{\mathbb{R}} \frac{f_{0,v}^4}{f_0^2}\, dv \right)^{1/4} \left(\int_{\mathbb{R}} f_0^{2/3}\, dv \right)^{3/4} = 2^{3/4} \big\| \big(\sqrt{f_0}\big)_v \big\|_{L^4}.$$

By the main result of [16], the $L^4(\mathbb{R})$-norm of $\big(\sqrt{f_0}\big)_v$ is estimated in terms of the $L^2(\mathbb{R})$-norm of $f_{0,vv}$, which is finite by hypothesis. Altogether, it follows $f_{0,v} \in L^1(\mathbb{R})$. To conclude $\mathcal{F}_s[f_0] < \infty$ from here, we employ the elementary inequality $|x - y|^2 \leq |x^2 - y^2|$ for arbitrary $x, y \geq 0$ to find

$$\begin{aligned}
\mathcal{F}_s[f_0] &\leq 2 \int_0^\infty w^{-(1+2s)} \left(\int_{\mathbb{R}} \big| f_0(v + w) - f_0(v) \big|\, dv \right) dw \\
&\leq 2 \int_0^1 w^{-(1+2s)} \left(\int_{\mathbb{R}} \int_0^w \big| f_{0,v}(v + z) \big|\, dz\, dv \right) dw \\
&\qquad + 2 \int_1^\infty w^{-(1+2s)} \left(\int_{\mathbb{R}} \big(f_0(v + w) + f_0(v) \big)\, dv \right) dw \\
&\leq 2 \big\| f_{0,v} \big\|_{L^1} \int_0^1 w^{-2s}\, dw + 4 \big\| f_0 \big\|_{L^1} \int_1^\infty w^{-(1+2s)}\, dw.
\end{aligned}$$

Now recall that $0 < s < 1/2$, so that both integrals in the last line are finite.

Next, we are going to conclude (26) by means of applying the Gronwall Lemma 2.1 with $\Psi = \mathcal{F}_s$. With Proposition 3.1 at hand, it only remains to be shown that (22) implies (13). By the definition of Q_+ in (3) and convexity of $\mathcal{F}_s$, we have

$$\mathcal{F}_s\big[Q_+(f, f)\big] \leq \big\langle \mathcal{F}_s\big[p^{-1} f(p^{-1} \cdot) \star q^{-1} f(q^{-1} \cdot)\big] \big\rangle. \tag{27}$$

On the other hand, the change of variables $x = v/p$ and $y = w/p$ in the definition of $\mathcal{F}_s$ yield

$$\begin{aligned}
\mathcal{F}_s\big[p^{-1} f(p^{-1} \cdot)\big] &= p^{-1} \iint_{\mathbb{R}^2} |w|^{-(1+2s)} \big| \sqrt{f((v + w)/p)} - \sqrt{f(v/p)} \big|^2\, dv\, dw & (28) \\
&= p^{-2s} \iint |\eta|^{-(1+2s)} \big| \sqrt{f(x + y)} - \sqrt{f(x)} \big|^2\, dx\, dy = p^{-2s} \mathcal{F}_s[f]. & (29)
\end{aligned}$$

An application of (22) with $\Lambda := p^{2s}/(p^{2s} + q^{2s})$ gives

$$\mathcal{F}_s\big[p^{-1} f(p^{-1} \cdot) \star q^{-1} f(q^{-1} \cdot)\big] \leq \frac{2}{p^{2s} + q^{2s}} \mathcal{F}_s[f] \leq 2 \max(p, q)^{-2s} \mathcal{F}_s[f].$$

Thus, indeed, (27) implies the desired estimate

$$\mathcal{F}_s\big[Q_+(f,f)\big] \leq 2K_s\mathcal{F}_s[f],$$

which is (13) with $\lambda = 2K_s$ and $M = 0$. Estimate (26) results from (14). □

The significance of the estimate (26) is that is provides, at arbitrary times $t \geq 0$, a pointwise bound on the high frequency tail of $\widehat{f}(t)$.

Corollary 3.2. *Under the hypotheses of Corollary 3.1, there exists a constant C such that*

$$|\widehat{f}(t;\xi)| \leq C \exp\big((K_s - 1/2)t\big)|\xi|^{-s} \tag{30}$$

for all $t > 0$, with K_s as defined above.

Proof. Corollary 3.1 implies that $\mathcal{F}_s[f(t)]$ grows at most exponentially in t with rate $2K_s - 1$, so it is always finite. Let $g = \sqrt{f}$ with Fourier transform $\widehat{g}$. Notice that $\widehat{f} = \widehat{g} \star \widehat{g}$ since $f = g^2$. Hence, for all $\xi \in \mathbb{R}$, using $|\xi|^s \leq |\xi - \eta|^s + |\eta|^s$,

$$\begin{aligned}
|\xi|^r|\widehat{f}(\xi)| &= \int_{\mathbb{R}} |\xi|^s|\widehat{g}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta \\
&\leq \int_{\mathbb{R}} |\xi-\eta|^s|\widehat{g}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta + \int_{\mathbb{R}} |\eta|^s|\widehat{g}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta \\
&\leq 2\Big(\int_{\mathbb{R}} |\widehat{g}(\zeta)|^2\,d\zeta\Big)^{1/2}\Big(\int_{\mathbb{R}} |\zeta|^{2s}|\widehat{g}(\zeta)|^2\,d\zeta\Big)^{1/2}.
\end{aligned}$$

By Plancherel's identity, the value of the first integral equals the mass of $f \in \mathcal{P}$, which is one. The second integral constitutes the H^s-semi-norm of $\sqrt{f}$, which is related to $\mathcal{F}_s[f]$ by Lemma 3.2. It follows that

$$|\widehat{f}(t;\xi)| \leq \sqrt{2/N_s}\,\mathcal{F}_s\big[f(t)\big]^{1/2}|\xi|^{-s}. \tag{31}$$

Now apply estimate (26) to conclude (30). □

4. A time-uniform bound on the high frequencies

Before turning to estimate the high-frequency tail of $\widehat{f}(t)$ in Lemmata 4.2 and 4.3 below, we start with a time-uniform bound on the *low*-frequency part of $\widehat{f}(t)$, which enters crucially in the proofs below. Once and for all, let $\kappa > 0$ be chosen so that

$$2\kappa < \int_{\mathbb{R}} v^2 f_\infty(v)dv - \Big(\int_{\mathbb{R}} v\, f_\infty(v)dv\Big)^2. \tag{32}$$

Such a $\kappa > 0$ exists since $f_\infty \in \mathcal{P}_\alpha$ has finite first moment and is *not* a Dirac distribution (and thus has positive — possibly infinite — variance). In particular, κ can be chosen arbitrarily if $\alpha = 1$ and the second moment of f_∞ is infinite.

Lemma 4.1. *Under the hypotheses of Theorem 1.1, there exist a radius $\rho > 0$ and a time $T \geq 0$ such that the weak solution $f(t)$ satisfies*

$$|\widehat{f}(t;\xi)| \leq 1 - \kappa\xi^2 \quad \textit{for all } |\xi| \leq \rho \tag{33}$$

for all $t \geq T$, with the κ chosen in (32).

Proof. Decompose $\widehat{f}(t;\xi) = (1 - A) + iB$ into its real and imaginary part,

$$A(t) = \int_{\mathbb{R}} \big(1 - \cos(\xi v)\big) f(t;v)\, dv, \quad B(t) = \int_{\mathbb{R}} \sin(\xi v) f(t;v)\, dv.$$

We shall derive a *lower* bound on $A(t)$, that is uniform in $t \geq T$, for some sufficiently large T. Choose numbers $\epsilon > 0$ and $\delta > 0$ so that

$$1 + 2\kappa + 3\delta < \int_{\mathbb{R}} v^2\, f_\infty(v) dv \quad \text{and} \quad (1-\epsilon)(1 + 2\kappa + 2\delta) \geq 1 + 2\kappa + \delta. \tag{34}$$

Observe that there exists a positive $x_\epsilon \in (0, \pi)$ such that

$$1 - \cos x \geq \frac{1-\epsilon}{2} x^2 \quad \text{for all} \quad |x| \leq x_\epsilon.$$

In consequence, for all $|\xi| \leq \rho$ with some $\rho > 0$ to be determined,

$$A(t) \geq \frac{1-\epsilon}{2} \int_{|\xi v| \leq x_\epsilon} (\xi v)^2 f(t;v)\, dv \geq \frac{1-\epsilon}{2} \xi^2 \int_{|v| \leq x_\epsilon/\rho} v^2 f(t;v)\, dv.$$

Let $\varphi_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be an even, non-negative and continuous function with the following properties: $\varphi_1(v) = v^2$ for $|v| \leq 1/2$; $\varphi_1(v)$ is non-increasing for $1/2 \leq v \leq 1$; and $\varphi_1(v) = 0$ for $|v| \geq 1$. Defining $\varphi_R(v) = R^2 \varphi_1(v/R)$, it follows that

$$\Phi(R) := \int_{\mathbb{R}} \varphi_R(v) f_\infty(v) dv \to \int_{\mathbb{R}} v^2\, f_\infty(v) dv$$

as $R \to \infty$ by the monotone convergence theorem; recall that the second moment of f_∞ might be infinite, in which case $\Phi(R) \to \infty$. Choose $\rho > 0$ small enough, so that $\Phi(x_\epsilon/\rho) > 1 + 2\kappa + 3\delta$, which is possible by (34).

Next, due to the weak-$\star$ convergence of $f(t)$ to f_∞, see Theorem 2.1, there exists a $T \geq 0$ such that

$$\left| \Phi(x_\epsilon/\rho) - \int_{\mathbb{R}} \varphi_{x_\epsilon/\rho}(v) f(t;v)\, dv \right| \leq \delta$$

for all $t \geq T$. Altogether, for $|\xi| \leq \rho$ and $t \geq T$,

$$A(t) \geq \frac{1-\epsilon}{2} \xi^2 \int_{\mathbb{R}} \varphi_{x_\epsilon/\rho}(v) f(t;v)\, dv \geq \frac{1-\epsilon}{2} \xi^2 (\Phi(x_\epsilon/\rho) - \delta) \geq \frac{1 + 2\kappa + \delta}{2} \xi^2,$$

where the last inequality follows from (34). This lower bound on $A(t)$ implies further that

$$\big(1-A(t)\big)^2 \le \big(1-\frac{1+2\kappa+\delta}{2}\xi^2\big)^2 = 1-(1+2\kappa+\delta)\xi^2+c\xi^4,$$

with $c=(1+2\kappa+\delta)^2/4$.

Next, we shall estimate $B(t)$ from above, uniformly in t. For this, we distinguish the cases $\alpha=1$ and $\alpha=2$. If $\alpha=1$, then $f(t)\in\mathcal{P}_1$ has unit first moment and is supported on $\overline{\mathbb{R}_+}$, hence

$$|B(t)| \le \int_{\mathbb{R}_+} |\sin(\xi v)|\, f(t;v)dv \le \int_{\mathbb{R}} |\xi| v\, f(t;v)dv = |\xi|.$$

If $\alpha=2$, then $\int_{\mathbb{R}} v\, f(t;v)dv = 0$ and thus

$$|B(t)| = \left|\int_{\mathbb{R}} \sin(\xi v)\, f(t;v)dv - \xi\int_{\mathbb{R}} v\, f(t;v)dv\right| \le |\xi|\int_{\mathbb{R}}\left|1-\frac{\sin(\xi v)}{\xi v}\right||v|\, f(t;v)dv.$$

It is easily seen that $|1-\sin z/z|\le|z|$ for all $z\in\mathbb{R}\setminus\{0\}$, which further implies that

$$|B(t)| \le |\xi|^2\int_{\mathbb{R}} v^2\, f(t;v)dv \le |\xi|$$

for all $|\xi|\le 1$ since $f(t)\in\mathcal{P}_2$ has unit second moment.

In both cases, the modulus of the Fourier transform can be estimated as follows,

$$\begin{aligned}|\widehat{f}(\xi)|^2 &= \big(1-A(t)\big)^2 + B(t)^2\\ &\le 1-(1+2\kappa+\delta)\xi^2+\xi^2+c\xi^4 \le \big(1-\kappa\xi^2\big)^2\end{aligned}$$

for all $|\xi|\le\rho<1$ small enough, uniformly in $t\ge T$. This proves (33). □

Lemma 4.2. *Under the hypotheses of Theorem 1.1, there exists some* $\sigma_*<1$ *such that* $|\widehat{f}(t;\xi)|\le\sigma_*$ *for all* $|\xi|\ge\rho$ *and all* $t\ge T$*, with the quantities* $\rho>0$ *and* $T\ge0$ *defined in Lemma 4.1.*

Proof. The claim will be proven by means of the Gronwall estimate in Lemma 2.1. Define the convex subset $K_1\subset\mathcal{P}$ by

$$K_1 = \big\{\ f\in\mathcal{P}_\alpha \big|\ |\widehat{f}(\xi)|\le 1-\kappa\xi^2 \text{ for } |\xi|\le\rho\big\}. \tag{35}$$

By Lemma 4.1, $f(t)\in K_1$ for all $t\ge T$.

Given a positive $\sigma<1$, define $\rho_\sigma := \sqrt{(1-\sigma)/\kappa}$. Then any $f\in K_1$ satisfies

$$|\widehat{f}(\xi)|\ \le 1-\kappa|\xi|^2 \le 1-\kappa\rho_\sigma^2 = \sigma \tag{36}$$

whenever $\rho_\sigma\le|\xi|\le\rho$. Clearly, $\rho_\sigma\to0$ as $\sigma\uparrow1$. Further, let P_σ be the probability that $\min(p,q)\rho\le\rho_\sigma$. By Markov's inequality, hypothesis (H2) assures that $P_\sigma \leq \langle p^{-\vartheta} + q^{-\vartheta}\rangle(\rho_\sigma/\rho)^\vartheta \downarrow 0$ as $\sigma \uparrow 1$. Define $\bar\sigma \in (1/2, 1)$ such that $\rho_\sigma \leq \rho$ and $P_\sigma \leq 1/3$ for all $\sigma \geq \bar\sigma$.

Let $f \in K_1$ satisfy $|\widehat{f}(\xi)| \leq \sigma$ for $|\xi| > \rho$, with some $\sigma \in [\bar\sigma, 1]$. Recalling (36), it follows that actually $|\widehat{f}(\xi)| \leq \sigma$ even for $|\xi| \geq \rho_\sigma$. Hence, for any given $\xi \in \mathbb{R}$ with $|\xi| > \rho$, one has

$$\begin{aligned} |\widehat{Q}_+(f,f)(\xi)| &\leq \big\langle |\widehat{f}(p\xi)||\widehat{f}(q\xi)|\big\rangle \\ &\leq P_\sigma \sup_{|\eta|<\rho_\sigma} |\widehat{f}(\eta)| + (1-P_\sigma)\big(\sup_{|\eta|\geq\rho_\sigma} |\widehat{f}(\eta)|\big)^2 \leq P_\sigma + (1-P_\sigma)\sigma^2. \end{aligned}$$

However, this last expression is less or equal to σ, since $1/2 \leq \sigma \leq 1$ and $P_\sigma \leq 1/3$.

In order to apply Lemma 2.1, define the convex functional Ψ_1 on K_1 by

$$\Psi_1(f) := \max\big(\bar\sigma, \sup_{|\xi|>\rho} |\widehat{f}(\xi)|\big).$$

Then, by the preceding calculations, inequality (13) is satisfied with Ψ for the parameters $\lambda = 1$ and $M = 0$. Consequently, $\Psi_1(f(t)) \leq \Psi_1(f(T))$ for all $t \geq T$. To finish the proof, recall that $f(T)$ is a genuine probability density, and hence its Fourier transform satisfies $|\widehat{f}(T;\xi)| \leq \sigma_*$ for all $|\xi| > \rho$, with an appropriate $\sigma_* \in (\bar\sigma, 1)$. ☐

Lemma 4.3. *Under the hypotheses of Theorem 1.1, there exists some constants* M_0 *and* $\gamma > 0$ *so that* $|\widehat{f}(t;\xi)| \leq M_0|\xi|^{-\gamma}$ *for all* $\xi \neq 0$ *and all* $t \geq T$, *with* $T \geq 0$ *from Lemma 4.1.*

The importance of this lemma is clearly that M_0 and γ do not depend on time.

Proof. As in the previous proof, the claim will be obtained in application of a Gronwall estimate. This time, we choose

$$K_2 = K_1 \cap \big\{\, f \in \mathcal{P}_\alpha \big|\, |\widehat{f}(\xi)| \leq \sigma_* \text{ for all } |\xi| > \rho\big\}$$

as convex subset of $\mathcal{P}$, with the bound $\sigma_* < 1$ obtained in Lemma 4.2, and K_1 defined above in (35). In fact, Lemma 4.1 and Lemma 4.2 show that $f(t) \in K_2$ for all $t \geq T$. For the convex function $\Psi_2 : \mathcal{P} \to \mathbb{R}$, we shall use

$$\Psi_2(f) = \max\big(\rho^\gamma, \sup_{\xi\in\mathbb{R}} |\xi|^\gamma |\widehat{f}(\xi)|\big);$$

the appropriate exponent γ is defined below.

The goal is to show that Ψ_2 satisfies (13) with $\lambda = 1$ and $M = 0$. Several quantities need to be introduced: First, $C := \langle p^{-\vartheta} + q^{-\vartheta}\rangle$ is finite by hypothesis (H2), and Markov's inequality implies

$$\mathbb{P}[\min(p,q) \leq \epsilon] \leq \langle \min(p,q)^{-\vartheta}\rangle \epsilon^\vartheta \leq C\epsilon^{-\vartheta}, \tag{37}$$

for arbitrary $\epsilon > 0$. Second, with $\gamma > 0$ (yet to be determined), define $\bar a > 0$ by

$$\bar a^{\vartheta+2\gamma} = \frac{2\gamma}{\vartheta C}\sigma_*^{-(\vartheta-2\gamma)/2\gamma},$$

which is the minimizer in

$$\begin{aligned} U_\gamma &:= \min_{a>0} \left(C\sigma_*^{(\vartheta-2\gamma)/(2\gamma)} a^\vartheta + a^{-2\gamma} \right) \\ &= (\vartheta + 2\gamma)\vartheta^{-\vartheta/(\vartheta+2\gamma)} C^{2\gamma/(\vartheta+2\gamma)} \sigma_*^{(\vartheta-2\gamma)/(\vartheta+2\gamma)}. \end{aligned}$$

Third, since obviously $U_\gamma \to \sigma_* < 1$ as $\gamma \downarrow 0$, we can define $\gamma > 0$ so that $U_\gamma \leq 1$, and moreover $2\gamma \leq \vartheta$ and $\gamma \leq s$, where $s > 0$ is such that $\mathcal{F}_s[f(T)] < \infty$.

Let $f \in K_2$ be arbitrary function with $\Psi_2(f) < \infty$. Since $K_1 \subset K_2$, the estimates of the proof of Lemma 4.2 apply and provide

$$|\widehat{Q}_+(f,f)(\xi)| \leq \sigma_*$$

for all $|\xi| > \rho$. Defining $\rho_* := (\Psi_2(f)/\sigma_*)^{1/\gamma} > \rho$, one readily verifies that

$$\begin{aligned} \sup_{|\xi|\leq\rho_*} |\xi|^\gamma |\widehat{Q}_+(f,f)(\xi)| &\leq \max\left(\rho^\gamma \sup_{|\xi|\leq\rho} |\hat{Q}_+(f,f)(\xi)|,\ \sup_{\rho\leq|\xi|\leq\rho_*} |\xi|^\gamma |\hat{Q}_+(f,f)(\xi)|\right) \\ &\leq \max(\rho^\gamma, \rho_*^\gamma \sigma) \leq \Psi_2(f). \end{aligned} \tag{38}$$

Now let $\xi \in \mathbb{R}$ with $|\xi| > \rho_*$ be given. Define $\eta := \bar{a}\Psi_2(f)^{1/(2\gamma)}\sqrt{|\xi|}$ and the associated probability

$$\mathbb{P}_\xi = \mathbb{P}[\min(p,q)|\xi| \leq \eta] \leq C(\eta/|\xi|)^\vartheta = C\Psi_2(f)^{\vartheta/(2\gamma)}\bar{a}|\xi|^{-\vartheta/2},$$

where the estimate is a consequence of (37). Now

$$\begin{aligned} |\xi|^\gamma |\widehat{Q}_+(f,f)(\xi)| &\leq |\xi|^\gamma \langle |\widehat{f}(p\xi)||\widehat{f}(q\xi)| \rangle \\ &\leq |\xi|^\gamma \left(\mathbb{P}_\xi \sup_{\xi'\in\mathbb{R}} |\widehat{f}(\xi')|^2 + \sup_{\xi'\geq\eta} |\widehat{f}(\xi')|^2 \right) \\ &\leq |\xi|^\gamma \mathbb{P}_\xi + \left(\eta/\sqrt{|\xi|} \right)^{-2\gamma} \Psi_2(f)^2 \\ &\leq C\bar{a}^\vartheta \Psi_2(f)^{\vartheta/(2\gamma)} |\xi|^{\gamma-\vartheta/2} + \bar{a}^{-2\vartheta}\Psi_2(f) \\ &\leq \left(C\bar{a}^\vartheta \left(|\xi|^{-\gamma}\Psi_2(f) \right)^{(\vartheta-2\gamma)/(2\gamma)} + \bar{a}^{-2\vartheta} \right) \Psi_2(f). \end{aligned}$$

Since $|\xi| \geq \rho_*$, it follows that $|\xi|^{-\gamma}\Psi_2(f) \leq \rho_*^{-\gamma}\Psi_2(f) = \sigma_*$. Recalling the definition of $\bar{a}$ in (37) and that $U_\gamma \leq 1$, we conclude

$$|\xi|^\gamma |\widehat{Q}_+[f,f](\xi)| \leq \left(C\bar{a}^\lambda \sigma_*^{(\lambda-2\gamma)/(2\gamma)} + \bar{a}^{-2\gamma} \right) \Psi_2(f) \leq U_\gamma \Psi_2(f) \leq \Psi_2(f).$$

In combination with (38), we obtain estimate (13) for Ψ_2, with $\lambda = 1$ and $M = 0$.

By Corollary 3.2, it follows that for some $M_0 \geq 1$

$$|\widehat{f}(T;\xi)| \ \leq M_0 |\xi|^{-s}. \tag{39}$$

It is easily seen (using the trivial estimate $|\widehat{f}| \leq 1$) that (39) is true also with $\gamma < s$ in place of s, and with the same constant M_0. Hence $\Psi_2(f(T)) = M_0 < \infty$, and Lemma 2.1 applies. □

5. Proof of Theorem 1.1

In summary of the previous sections, we have obtained a pointwise bound on the high-frequency tail of $\widehat{f}(t)$, which is uniform in $t \geq T$, in Lemma 4.3, and we know that $f(T) \in H^m(\mathbb{R})$ from Lemma 3.1. We shall now combine these elements — adapting the classical strategy from [5] to our needs — to finish the proof of Theorem 1.1.

Lemma 5.1. *Under the assumptions of Theorem 1.1, there exists a finite constant A such that $\|f(t)\|_{H^m(\mathbb{R})} \leq A$ for all $t \geq 0$.*

Proof. Let $T \geq 0$ be the time defined in Lemma 4.1. By Lemma 3.1,

$$\sup_{t \leq T} \|f(t)\|_m \leq A_1 := \|f_0\|_m \exp\big(\langle \max(p,q)^{-(m+1/2)} \rangle T\big) < \infty. \tag{40}$$

It remains to prove uniform boundedness of $f(t)$ in $H^m(\mathbb{R})$ for $t \geq T$. To begin with, choose $\nu \in (0,1)$ so that $m + 1/2 < (1-\nu)\Theta$, which is possible since $m < \Theta - 1/2$. Let $\gamma > 0$ and M_0 be defined as in Lemma 4.3, and define further $\delta := \nu\vartheta > 0$. The $H^m(\mathbb{R})$-semi-norm of the collisional gain operator is estimated as follows,

$$\begin{aligned}
&\|Q_+[f,f]\|_m^2 \leq \int_{\mathbb{R}} |\xi|^{2m} \left[\int_{\mathbb{R}_+^2} |\widehat{f}(p\xi)||\widehat{f}(q\xi)| \, d\mathbb{P}(p,q) \right]^2 d\xi \\
&\leq M_0^2 \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} |\xi|^{2(m-\delta)} |\widehat{f}(\max(p,q)\xi)||\widehat{f}(\max(p',q')\xi)| \, d\xi \, \frac{d\mathbb{P}(p,q)}{\min(p,q)^\delta} \, \frac{d\mathbb{P}(p_*,q_*)}{\min(p_*,q_*)^\delta} \\
&= M_0^2 \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} \Big(|\sigma\xi|^{m-\delta} |\widehat{f}(\sigma\xi)| \sigma^{1/2} \Big) \Big(|\sigma_*\xi|^{m-\delta} |\widehat{f}(\sigma_*\xi)| \sigma_*^{1/2} \Big) \, d\xi \, \frac{d\mathbb{P}(p,q)}{\tau^\delta \sigma^{m+1/2-\delta}} \, \frac{d\mathbb{P}(p_*,q_*)}{\tau_*^\delta \sigma_*^{m+1/2-\delta}}.
\end{aligned}$$

Here we have abbreviated by σ, σ_* and by τ, τ_*, respectively, the maxima and minima of p, q and of p_*, q_*. At this point, the ξ-integral is further estimated by Hölder's inequality, similarly as in the proof of Lemma 3.1. This results in

$$\|Q_+[f,f]\|_m \leq M_0 \big\langle \min(p,q)^{-\delta} \max(p,q)^{-(m+1/2-\delta)} \big\rangle \bigg(\int_{\mathbb{R}} |\eta|^{2(m-\delta)} |\widehat{f}(\eta)|^2 \, d\eta \bigg)^{1/2}.$$

Above, the expectation value is finite and can be estimated invoking Hölder's inequality once again,

$$\langle \min(p,q)^{-\delta} \max(p,q)^{-(m+1/2-\delta)} \rangle \leq \langle \min(p,q)^{-\delta/\nu} \rangle \langle \max(p,q)^{-(m+1/2-\delta)/(1-\nu)} \rangle.$$

Both terms in the product are finite by hypothesis (H2), recall that ν has been chosen so that $\delta/\nu = \vartheta$ and $(m+1/2)/(1-\nu) < \Theta$. On the other hand, for an arbitrary $R > 0$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} |\eta|^{2(m-\delta)} |\widehat{f}(\eta)|^2 \, d\eta &\leq \int_{|\eta| \leq R} |\eta|^{2(m-\delta)} \, d\eta + R^{-2\delta} \int_{|\eta| > R} |\eta|^{2m} |\widehat{f}(\eta)|^2 \, d\eta \\
&\leq \frac{2R^{2(m-\delta)+1}}{2(m-\delta)+1} + R^{-2\delta} \|f\|_m^2,
\end{aligned}$$

where we have used once again that $|\widehat{f}(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$. So R can be made sufficiently large to achieve

$$\lambda := M_0 R^{-\delta} \langle \min(p,q)^{-\delta} \max(p,q)^{-(m+1/2-\delta)} \rangle < 1.$$

Thus Lemma 2.1 applies, with λ as above, with

$$M := M_0 \Big(\frac{2R^{2(m-\delta)+1}}{2(m-\delta)+1} \Big)^{1/2} \langle \min(p,q)^{-\delta} \max(p,q)^{-(m+1/2-\delta)} \rangle,$$

and with the convex functional $\Psi(f) = \|f\|_m$ defined on

$$K := \big\{ \ f \in \mathcal{P}_\alpha \big| |\widehat{f}(\xi)| \leq M_0 |\xi|^{-\gamma} \text{ for all } \xi \neq 0. \big\} \ .$$

This proves

$$\sup_{t \geq T} \|f(t)\|_m \leq A_2 := \|f(T)\|_m + \frac{M}{1-\lambda} < \infty. \tag{41}$$

In combination, the estimates (40) and (41) show that $\|f(t)\|_m \leq \max(A_1, A_2)$ for all $t \geq 0$. By the equivalence of the semi-norm $\|\cdot\|_m$ to the full $H^m(\mathbb{R})$-norm stated in (19), the claim follows with $A := 2^{(m+1)/2}(1 + max(A_1, A_2))$. □

Lemma 5.2. *Under the hypotheses of Theorem 1.1,* $f_\infty \in H^{(\Theta-1/2)-}(\mathbb{R})$.

Proof. Define an initial condition $f_0 \in \mathcal{P}_\alpha$ of regularity $f_0 \in H^\infty(\mathbb{R})$; for instance, one may take $f_0(v) = (2\pi)^{-1/2} \exp(-v^2/2)$ if $\alpha = 2$, or $f_0(v) = a \exp(-v/b - b/v)$ if $\alpha = 1$, with appropriate positive constants a and b. By Theorem 2.1, the corresponding solution $f(t)$ converges weakly-$\star$ to f_∞. On the other hand, Lemma 5.1 above yields that $\sup_{t \geq 0} \|f(t)\|_{H^m} \leq A_m < \infty$ for each $m < \Theta - 1/2$. The $H^m(\mathbb{R})$-norms are clearly convex functionals on $\mathcal{P}$, and thus also weakly-$\star$ lower semi-continuous. In conclusion, $\|f_\infty\|_{H^m} \leq A_m$ as desired. □

Lemma 5.3. *Under the hypotheses of Theorem 1.1, the function $f(t)$ converges strongly to f_∞ in $H^{m-}(\mathbb{R})$.*

Proof. Let an $\epsilon > 0$ and some $k < m$ be given. For arbitrary $R > 1$, one finds

$$\begin{aligned}
\|f(t) - f_\infty\|_{H^k} \leq \Big(\int_{|\xi| \leq R} & (1+\xi^2)^k |\widehat{f}(t;\xi) - \widehat{f}_\infty(\xi)|^2 \, d\xi \Big)^{1/2} \\
& + (1+R^2)^{-(m-k)/2} \Big(\int_{|\xi| > R} (1+\xi^2)^m \big(|\widehat{f}(t;\xi)| + |\widehat{f}_\infty(\xi)| \big)^2 \, d\xi \Big)^{1/2} \\
& \leq (2R)^{k+1/2} \sup_{|\xi| \leq R} |\widehat{f}(t;\xi) - \widehat{f}_\infty(\xi)| + R^{-(m-k)} \big(\sup_{t \geq 0} \|f(t)\|_{H^m} + \|f_\infty\|_{H^m} \big).
\end{aligned}$$

In view of Lemmata 5.1 and 5.2, the second term above can be made smaller than $\epsilon/2$ by choosing $R \geq R_\epsilon$ large enough. For any fixed R, the first term above converges to zero as $t \to \infty$, since the weak-$\star$ convergence of $f(t)$ to f_∞ implies locally uniform (in ξ) convergence of $\widehat{f}(t)$ to $\widehat{f}_\infty$. Hence, the first term is smaller than $\epsilon/2$ provided $t \geq t_\epsilon$ is large enough. Thus, $\|f(t) - f_\infty\|_{H^k} < \epsilon$. □

6. Examples and extensions

6.1. Exponential rates of convergence

Under additional hypotheses on the moments of the initial condition, the strong convergence $f(t) \to f_\infty$ is actually exponentially fast.

Corollary 6.1. *In addition to the hypotheses of Theorem 1.1, assume that f_0 possesses a finite moment of order $\nu_0 > \alpha$. For each $0 \le k < m$, there exist constants C and $r > 0$ such that*

$$\|f(t) - f_\infty\|_{H^k} \le Ce^{-rt} \quad \textit{for all } t \ge 0. \tag{42}$$

The key element of the proof is that the evolution by (1)&(3) is contractive in the so-called Fourier distances introduced in [14].

Definition 6.1. *For $\nu > 0$, the Fourier-ν-distance of f, $g \in \mathcal{P}$ is defined by*

$$d_\nu(f, g) = \sup_{\xi \in \mathbb{R}\setminus\{0\}} \frac{\big|\widehat{f}(\xi) - \widehat{g}(\xi)\big|}{|\xi|^\nu}. \tag{43}$$

Strictly speaking, the d_ν are not genuine distances on $\mathcal{P}$ since $d_\nu(f, g) < \infty$ only if f and g meet additional requirements. For instance, a sufficient criterion for finite ν-distance is that f, $g \in \mathcal{P}_\alpha$, that $\nu \le \alpha + 1$, and that both f and g have finite moment of order s, i.e.,

$$\int_{\mathbb{R}} |v|^s f(v)\, dv < +\infty \quad \text{and} \quad \int_{\mathbb{R}} |v|^s g(v)\, dv < +\infty.$$

A recent review of the properties of Fourier metrics is provided in [7].

Lemma 6.1. *Under the hypotheses of Corollary 6.1, and for any $\nu > 0$ with $\alpha < \nu \le \min(\nu_0, \alpha + 1, \beta)$, one has $d_\nu(f_0, f_\infty) < +\infty$ and $r_\nu := 1 - \langle p^\nu + q^\nu \rangle > 0$. Moreover, the associated solution $f(t)$ satisfies*

$$d_\nu\big(f(t), f_\infty\big) \le e^{-r_\nu t} d_\nu(f_0, f_\infty). \tag{44}$$

Proof. First observe that $\mathbf{S} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\mathbf{S}(\nu) = \langle p^\nu + q^\nu \rangle$ is convex and satisfies $\mathbf{S}(\alpha) = 1$ by hypothesis (4) and $\mathbf{S}(\beta) < 1$ by hypothesis (H1). Consequently, $\mathbf{S}(\nu) < 1$ if $\alpha < \nu \le \beta$. Hence, choosing $\nu > 0$ with $\alpha < \nu \le \min(\nu_0, \alpha + 1, \beta)$, it follows that $r_\nu = 1 - \mathbf{S}(\nu) > 0$.

By assumption, f_0's moment of order $\nu \le \nu_0$ is finite. Also f_∞ has finite moment of order ν, because $\mathbf{S}(\nu) < 1$; see [1]. Since f_0, $f_\infty \in \mathcal{P}_\alpha$ and moreover $\nu \le \alpha + 1$, it follows that $d_\nu(f_0, f_\infty)$ is finite; see Section 2.3.

The proof of (44) is another application of Lemma 2.1, with $K := \mathcal{P}_\alpha$ and $\Psi(f) := d_\nu(f, f_\infty)$. As a steady state, f_∞ clearly satisfies the stationary version of (11),

$$\widehat{f_\infty}(\xi) = \langle \widehat{f_\infty}(p\xi) \widehat{f_\infty}(q\xi) \rangle. \tag{45}$$

It follows that

$$\begin{aligned}\big|\widehat{Q}_+(f,f)(\xi)-\widehat{f}_\infty(\xi)\big| &= \big|\langle\widehat{f}(p\xi)\widehat{f}(q\xi)\rangle-\langle\widehat{f}_\infty(p\xi)\widehat{f}_\infty(q\xi)\rangle\big|\\ &\leq \big\langle|\widehat{f}(p\xi)||\widehat{f}(q\xi)-\widehat{f}_\infty(q\xi)|+|\widehat{f}(p\xi)-\widehat{f}_\infty(p\xi)||\widehat{f}_\infty(q\xi)|\ \big\rangle\\ &\leq \big\langle|q\xi|^\nu d_\nu(f,f_\infty)+|p\xi|^\nu d_\nu(f,f_\infty)\big\rangle=\mathbf{S}(\nu)|\xi|^\nu d_\nu(f,f_\infty).\end{aligned}$$

Thus, estimate (13) is satisfied with $\lambda=\mathbf{S}(\nu)<1$ and $M=0$, since

$$\Psi\big(Q_+(f,f)\big)=\sup_{\xi\neq0}|\xi|^{-\nu}\big|\widehat{Q}_+(f,f)(\xi)-\widehat{f}_\infty(\xi)\big|\leq\mathbf{S}(\nu)\Psi(f).$$

The resulting Gronwall estimate (14) yields the claim (44). □

Proof of Corollary 6.1. The proof follows by adaption of the one for Lemma 5.3. The radius $R=R_t>1$ is now chosen as a monotonically growing time-dependent quantity, and the integral for $|\xi|\leq R_t$ is estimated using (44) with, say, $\nu:=\min(\nu_0,\alpha+1,\beta)$. This gives

$$\begin{aligned}\|f(t)-f_\infty\|_{H^k}&\leq\Big(\int_{|\xi|\leq R}(1+\xi^2)^k|\xi|^{2\nu}d_\nu\big(f(t),f_\infty\big)^2\,d\xi\Big)^{1/2}\\ &\qquad+(1+R^2)^{-(m-k)/2}\Big(\int_{|\xi|>R}(1+\xi^2)^m\big(|\widehat{f}(t;\xi)|+|\widehat{f}_\infty(\xi)|\big)^2\,d\xi\Big)^{1/2}\\ &\leq e^{-r_\nu t}(2R_t)^{k+\nu+1/2}d_\nu\big(f_0,f_\infty\big)+R_t^{-(m-k)}\big(\sup_{t\geq0}\|f(t)\|_{H^m}+\|f_\infty\|_{H^m}\big).\end{aligned}$$

The radius R_t is now chosen in such a way that both terms in this sum tend to zero at the same exponential rate, i.e.,

$$e^{-r_\nu t}R_t^{k+\nu+1/2}=R_t^{-(m-k)}\quad\Longleftrightarrow\quad R_t:=\exp\Big(\frac{r_\nu}{m+\nu+1/2}t\Big).$$

Thus, the claim (42) follows with $r=(m-k)r_\nu/(m+\nu+1/2)$. □

6.2. Pointwise smoothness of the steady state

In the energy conserving case $\alpha=2$, the regularity of f_∞ can be characterized in more detail: any non-smoothness of f_∞ is localized at $v=0$.

Corollary 6.2. *If* $\alpha=2$, *then* $f_\infty\in C^\infty(\mathbb{R}\setminus\{0\})$. *More precisely,* $|\partial_v^k f_\infty(v)|\leq C_k v^{-(k+1)}$ *for all* $v\in\mathbb{R}\setminus\{0\}$, *with universal constants* C_0, C_1, $C_2\ldots$

The proof of Corollary 6.2 is a direct consequence of the following representation of f_∞ that has been derived in [1].

Theorem 6.1. *If* $\alpha=2$, *then* f_∞ *is a mixture of Gaussians, i.e.*

$$f_\infty(v)=\int_{\mathbb{R}_+}(2\pi z)^{-1/2}\exp\Big(-\frac{v^2}{2z}\Big)\,d\mu(z).\tag{46}$$

The mixing probability measure μ *on* $\mathbb{R}_+$ *is determined from* $\mathbb{P}$.

Remark 6.1. *There is an intimate relation between the steady states of the energy and the momentum conserving models. Namely, the mixing measure μ in (46) constitutes a stationary state for the impulse conserving model, which is obtained from the given energy conserving model upon replacing (p, q) by (p^2, q^2). In other words, μ satisfies $\hat{\mu}(\xi) = \langle \hat{\mu}(p^2\xi)\hat{\mu}(q^2\xi)\rangle$.*

Proof of Corollary 6.2. Recall that the kth Hermite polynomial H_k is defined by

$$\Big(\frac{d}{dx}\Big)^k e^{-x^2/2} = (-1)^k H_k(x) e^{-x^2/2}$$

and is a polynomial of degree k. Thus, for any $v \neq 0$,

$$v^{k+1}\,\partial_v^k f_\infty(v) = \frac{(-1)^k}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \Big(\frac{v}{\sqrt{z}}\Big)^{k+1} H_k\Big(\frac{v}{\sqrt{z}}\Big) \exp\Big(-\frac{1}{2}\Big(\frac{v}{\sqrt{z}}\Big)^2\Big)\, d\mu(z).$$

The integrand can be estimated (pointwise in $z \in \mathbb{R}_+$) from above and from below by

$$C'_k := \sup_{x>0} x^{k+1} |H_k(x)| e^{-x^2/2},$$

which is finite, since $x^{k+1}H_k(x)$ is a polynomial. Recalling that μ is a probability measure, the claim follows with $C_k := C'_k/\sqrt{2\pi}$. □

6.3. Sharpness of the regularity bounds

The following two examples illustrate that the regularity $f_\infty \in H^{(\Theta-1/2)-}(\mathbb{R})$ obtained in Theorem 1.1 is indeed optimal in some situations.

Let us begin with the momentum conservative case. Take p and q as independent random variables, which are uniformly distributed in the interval $[0, 1]$. For $s > -1$, one calculates

$$\langle p^s + q^s\rangle = 2\int_0^1 x^s\,dx = \frac{2}{s+1}. \tag{47}$$

Thus (4) holds with $\alpha = 1$, hypothesis (H1) is satisfied, e.g. with $\beta = 2$, and one can choose $\vartheta = 1/2$ in (H2). In order to see that $\Theta = 2$, simply observe that for any $s < 2$

$$\langle \max(p, q)^{-s}\rangle = 2\int_0^1 \Big(\int_x^1 y^{-s}\,dy\Big)\,dx = \frac{2}{2-s}. \tag{48}$$

For this particular model, the steady state f_∞ can be written is closed form, namely $f_\infty(v) = 4v\exp(-2v)$ for $v > 0$, and $f_\infty(v) = 0$ for $v \leq 0$. In fact, f_∞'s Fourier transform is simply $\widehat{f}_\infty(\xi) = (1 - i\xi/2)^{-2}$. Stationarity of f_∞ is easily verified in (45),

$$\langle \widehat{f}_\infty(p\xi)\widehat{f}_\infty(\xi)\rangle = \Big\langle \frac{1}{(1 - ip\xi/2)^2}\Big\rangle^2 = \Big(\int_0^1 \frac{dp}{(1 - ip\xi/2)^2}\Big)^2 = \Big(\frac{1}{1 - i\xi/2}\Big)^2 = \widehat{f}_\infty(\xi).$$

For any $m > 0$, one finds

$$\|f_\infty\|_m^2 = \int_{\mathbb{R}} \frac{|\xi|^{2m}d\xi}{|1-i\xi/2|^4} = \int_{\mathbb{R}} \frac{16|\xi|^{2m}d\xi}{(4+\xi^2)^2}.$$

Clearly, this integral is finite if $2m - 4 < -1$, i.e. $m < 3/2$, and diverges otherwise. Thus, $f_\infty \in H^{3/2-}(\mathbb{R})$ but $f_\infty \notin H^{3/2}(\mathbb{R})$. In other words, f_∞ has precisely the Sobolev regularity as stated in Theorem 1.1.

The example for the energy conserving case is constructed in a similar way. Again, we choose p and q to be independent, but now p^2 and q^2 are uniformly distributed on $[0,1]$. Replacing s by $s/2$ in (47) and (48), it follows that (4) holds with $\alpha = 2$, that hypothesis (H1) is satisfied with $\beta = 4$, and that hypothesis (H2) holds with $\vartheta = 1$ and $\Theta = 4$. The steady state f_∞ is explicitly given by $f_\infty(v) = \frac{1}{2}(1+2|v|)e^{-2|v|}$, and it has Fourier transform $\widehat{f}_\infty(\xi) = (1+\xi^2/4)^{-2}$. Stationarity is easily checked:

$$\langle \widehat{f}_\infty(p\xi)\widehat{f}_\infty(\xi)\rangle = \Big\langle \frac{1}{(1+p^2\xi^2/4)^2}\Big\rangle^2 = \Big(\int_0^1 \frac{dz}{(1+z\xi^2/4)^2}\Big)^2 = \Big(\frac{1}{(1+\xi^2/4)^2}\Big)^2 = \widehat{f}_\infty(\xi).$$

Concerning regularity, one needs to evaluate

$$\|f_\infty\|_m^2 = \int_{\mathbb{R}} \frac{256|\xi|^{2m}d\xi}{(4+\xi^2)^4},$$

which is finite if and only if $2m - 8 < -1$, i.e. $m < 7/2$. Consequently, $f_\infty \in H^{7/2-}(\mathbb{R})$ but $f_\infty \notin H^{7/2}(\mathbb{R})$.

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